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The phase congruence model for edge detection in multi-dimensional pictures

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Abstract. Morrone and co-workers have proposed a model for edge detection in grey-level images, based on psychophysical experiments in human vision. Assuming a one-dimensional visual signal, edges correspond to points of maximal Fourier phase congruence of the signal, and are localized at peaks of an energy function obtained as the quadratic combination of convolutions of the signal with two filters in Fourier quadrature (i.e., forming a Hilbert transform pair); such properties were explained in the case where the one-dimensional visual signal was periodic. In this report we make an in-depth theoretical study of this phase congruence model for multi-dimensional non-periodic images. We consider one-dimensional edges and features (in the sense that significant grey-level changes occur along a single direction), but whose orientation is not fixed. We extend previous work in several respects:

— The mathematical framework is improved: both the signal and the filters are multi-dimensional and non-periodic; new mathematical properties are obtained.

— We consider edges having various orientations, and characterize mathematically the joint determination of orientation and position of edges.

— We show that usual models for edge detection involving a single filter are a particular case of this phase congruence approach.

— We introduce some further filter combinations for classifying types of edges.

We discuss also types of unidirectional edges and local features that can be encountered in natural images (steps, lines, ramps, Mach bands, compound edges, etc.), and the appropriateness of the model to their detection and localization.

Key words. Edge and feature detection, edge types, symmetry, local Fourier phase, phase congruence, quadrature, directional Hilbert transform, linear and quadratic filters.

NB. This report is a revised and slightly abridged version of RR95/11 (September 1995).

Résumé. Morrone et ses collaborateurs ont proposé un modèle pour la détection d'arêtes dans les images à niveaux de gris, basé sur des expériences psychophysiques concernant la vision humaine. Supposant un signal visuel unidimensionnel, les arêtes correspondent aux points de congruence maximale des phases de Fourier du signal, et sont localisées aux sommets d'une fonction d'énergie obtenue par une combinaison quadratique du signal avec deux filtres en quadrature de Fourier (c.a.d. formant une paire de la transformée d'Hilbert); de telles propriétés furent expliquées dans le cas où le signal visuel unidimensionnel était périodique. Dans ce rapport nous faisons une étude théorique en profondeur de ce modèle de congruence de phases pour des images multidimensionnelles non péridodiques. Nous considérons des arêtes et traits unidimensionnels (dans le sens que les changements significatifs de niveaux de gris ont lieu dans une seule direction), mais dont l'orientation n'est pas fixée. Nous étendons les travaux précédents sur plusieurs points:

— Le cadre mathématique est amélioré: tant le signal que les filtres sont multidimensionnels et non-périodiques; de nouvelles propriétés mathématiques sont obtenues.
— Nous considérons des arêtes ayant des orientations variées, et caractérisons mathématiquement la détermination conjointe de l'orientation et de la position des arêtes.
— Nous montrons que les modèles usuels pour la détection d'arêtes impliquant un seul filtre sont un cas particulier de cette approche par congruence de phases.
— Nous introduisons de nouvelles combinaisons des filtres pour classer les types d'arêtes.

Nous discutons aussi des types d'arêtes et traits locaux unidirectionnels qu'on peut rencontrer dans les images naturelles (pas, lignes, bandes de Mach, arêtes composites, etc.), et de l'adéquation de ce modèle pour leur détection et leur localisation.

Mots clés. Détection d'arêtes et de traits, types d'arêtes, symétrie, phase de Fourier locale, congruence de phases, quadrature, transformée de Hilbert directionnelle, filtres linéaires et quadratiques.

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1. Introduction

Edges are visually salient features in a grey-level image, whose positions in the plane form a one-dimensional structure. They include boundaries between distinct regions of a picture, as well as discontinuities inside such regions, or line drawings in sketches [56]. They can have various grey-level profiles, such as steps, lines, roofs, and combinations of these [19]. Numerous papers have appeared on the subject of edge detection, and research carries on with this topic, for no method presented so far is completely satisfactory. One of the first problems faced by anyone attempting to devise a new edge detector is to define precisely what is an edge. We see three types of definitions:

— *Physical definition:* An image is formed as light coming from certain sources is reflected towards the viewer by surfaces in three-dimensional space. Thus edges are the optical projection of object discontinuities, such as: changes in surface reflectance or orientation, object termination (leading to boundary contours), occlusion, shadows, etc. Edge characteristics are thus determined by photometric properties of various materials. This approach has been championed by Horn [19], and recently illustrated in [24]. It can be characterized as *objective* and *materialistic*.

— *Physiological or psychophysical definition:* Edges correspond to what humans (or primates) perceive as such. This does not say what are the characteristics of an edge (such as the kind of grey-level profile in its neighbourhood), but only where an edge must be detected in a given image. Here the emphasis is put on the definition of the *edge detector* rather than that of the edge. Visual response to artificial images are measured by psychophysical or electrophysiological studies, and edge detection operators are then modeled after the behaviour of visual neurons. This approach traces back from the experiments by Hubel and Wiesel [20] on "edge and bar detectors" in the primary visual cortex of monkeys. It can be considered as *subjective*.

— *Mathematical definition:* The grey-level image is a mathematical function, and edges form the locus of points where that function satisfies certain mathematical properties, such as: step discontinuity of the function or of one of its derivatives [28], non-analyticity, zero-crossing of the convolution with Laplacian of Gaussian [36], peak in the absolute value of the convolution with gradient of Laplacian [7], Fourier phase congruence [37,38], etc. Alternatively the edge detection process can be modeled as a mathematically ill-posed problem which must be regularized [61]; for example in [62] a filter detecting step edges was considered as the regularization of differentiation. This approach can be characterized as *objective* and *idealistic*; indeed it relies often on abstract models of "ideal" edges with added noise.

Following Marr [35], there is a tendency to combine these three approaches, and to define edge detection in terms of photometry, physiology, and mathematics. In practice it is impossible to demand that an edge map computed from an image according to a mathematical algorithm, the human perception of edges in that image, and the optical projection of discontinuities in the scene giving rise to that image, should all three coincide *exactly*. However it is reasonable to ask that these three edge maps should be close to each other.

The phase congruence model for edge detection [37,38] considers that edges and features in a one-dimensional visual signal correspond to points where the cosine curves composing the signal (in a Fourier decomposition) have their phases in conjunction; this is for example the case for an ideal step or triangular line, see Figure 1. As indicated in [37], such edges can be detected as peaks (maxima) of an "energy function" obtained as the sum of squares $(I*C)^2+(I*S)^2$ of convolutions of the signal I with two constant phase filters C and S, respectively even-symmetric and odd-symmetric, which are in Fourier phase quadrature, forming in fact a Hilbert transform pair; in other words, for every positive frequency ν, the Fourier amplitudes of C and S are equal, while their Fourier phases are 0 and $-\pi/2$ respectively (we will explain this in detail in Section 4, where we investigate the model). This abstract mathematical model has been deepened in [45,51,63], and more general models using quadratic combinations of various pairs of filters have been studied [13,26,27,47]. Similar approaches have been tested [4,5] in order to model the human visual system. Authors agree that the quadratic approach allows the accurate detection of both step and line edges, something which is difficult to achieve in usual methods using a single filter, such as [7].

In fact this mathematical model arises from psychophysical experiments on human detection of edges or Mach bands in vertical gratings (images whose grey-level is constant in the vertical direction) [37,39,55]. Related psychophysical [6] and electrophysiological [48] studies suggest that the responses of simple cells in the visual cortex have Fourier phases corresponding to those of the two filters used in phase congruence model [37]. Hence this approach to edge detection has a sound foundation in both mathematics and physiology. However there has to our knowledge been no study of its relation to photometry: what are the Fourier phase characteristics of edges arising in natural images from the optical projection of object discontinuities?

In this report we make a systematical study of this model for one-dimensional edges in multi-dimensional images. By one-dimensional edges we mean local images features where the grey-level changes significantly in one direction and is relatively constant in perpendicular directions, such as steps, lines, and roofs having a relatively constant orientation; however the orientation of the edge is not fixed, we deal also with the selection of edge orientation. We exclude from this study the analysis of multi-directional visual features, where there are significant variations of grey-levels in two or more directions, such as corners, end-stopped edges, or strongly curved edges; these have been sudied in [9,10,41,42], and in [17,46,50,54], using methods related to the phase congruence approach.

We do not restrict the number of dimensions of the image to two, because the mathematical model can be applied to three-dimensional volumetric images; here edges and contours form surfaces across which the grey-level changes significantly. On can also use this model for the detection of spatiotemporal edges in image sequences. For example in [1,64], moving images constitute a signal on a three-dimensional space-time, and spatiotemporal edges are surfaces in space-time; in such an edge surface, each constant-time slice corresponds to the edge in the spatial image taken at that time; the orientation of that spatio-temporal surface indicates both the orientation and velocity of the spatial edge in constant-time slices. In [1] these edge surfaces are detected through an energy function obtained as a quadratic combination of convolutions of the signal with spatiotemporal filters in phase quadrature; in [64] one uses instead the sum of two filters which are in phase quadrature both in space and in time.

This report is organized as follows. In Section 2 we discuss briefly the various types of edges that can be found in natural images, from the three points of view of optics, psychophysics, and computation; in particular we recall some advantages of quadratic edge detectors, and especially of the phase congruence model.

Section 3 is devoted to the mathematical foundations of the model: we present our notation, recall briefly the properties of L^p spaces, convolution, and the Fourier transform, and then study local phase, constant phase signals, phase quadrature and the directional Hilbert transform, the analytical signal and its energy, etc. This makes our report self-contained from a mathematical point of view.

The phase congruence model of edge detection in non-periodic multi-dimensional images is described in Section 4. We specify the spatial and Fourier requirements on the two filters, and we give the mathematical properties of the energy function as well as of other quadratic operators obtained from the convolution of the image with these two filters; then we interpret traditional single-filter approaches to edge detection (such as Canny's operator [7]) in terms of energy and phase congruence. Next we give a mathematical justification to the traditional method of selecting the edge orientation by taking filters at various orientations and choosing at each point the orientation giving the greatest energy function.

Finally Section 5 discusses various questions related to this model: the digitization of the filters, the importance of uniform continuity, applications to other vision tasks, etc. That section ends with the conclusion.

2. Types of edges and visual features in natural images

As we will see later in Section 4, the phase congruence model has several interesting mathematical properties, some of which are not shared by other models; this gives thus some theoretical justification for it. However, unless one takes a purely idealistic stand, it should also be argued that it is useful for detecting "real" edges in natural images, where the "reality" of edges is considered either from a physical point of view (optical projection of scene discontinuities), or from a perceptual point of view (what humans see as edges). Hence we will discuss here photometry and psychophysics rather than mathematics, and this makes this section somewhat complementary to the remainder of the report. We will first describe several types of luminance profiles that we call edges, and which types of scene events give rise to them. Next we will examine how they are perceived by the human visual system. Finally we will briefly discuss the relevance of the phase congruence model to computer vision.

At the start, it is necessary to clarify what is generally meant by an edge in terms of the possible profiles made by the luminance function in the direction normal to the edge (the luminance is supposed to be locally quasi-constant in the directions tangent to the edge). For example, some authors (following Hubel and Wiesel [20]) restrict the word "edge" to what we call a "step edge", while they name "bar" or "line" what we call a "bar edge" or "line edge" respectively. So we will first describe several types of luminance profiles encountered in practice. Only afterwards will we explain which types of scene events give rise to them.

We show in Figure 1 seven types of one-dimensional luminance profiles, and the names we have chosen to distinguish them. These profiles can be considered as representatives of *primary* types of edges. It is possible to add to the list the grey-level inversion of a line, bar, or roof, which are not shown.

We discriminate clearly between what we call a *line* edge and a *bar* edge; the latter has a plateau between the sides. The line corresponds to what Horn [19] calls a *peak*, and it is considered by him as a fundamental type of edge. In some other studies (for example [29,40]), authors call a line edge what we classify as a bar. The confusion between the two comes from the fact that early line detectors were designed for the recognition one-pixel wide lines, and used convolutions with 3×3 masks for this purpose; now in a digital framework, there is no distinction between one pixel thick bars and lines. Note that the phase congruence model recognizes a symmetric triangular line as a feature at all scales, while it classifies a bar as a feature at coarser scales only; at finer scales, a bar is recognized as a pair of two steps.

There can be progressive or sharp steps, in the sense that the transition between the low to the high grey-level can be continuous or abrupt. The difference between the two is physically meaningful, as recognized in [19] (we will see this again later). The *round* edge was recognized by Ling [29] (who named it "convex edge") as distinct from the step; in other studies the round edge has been ignored (for example by [19]) or identified with the step (as in [40]).

Roofs are commonly acknowledged in computer vision. This is not the case for *Mach bands*, which have generally been discussed only in relation to human visual perception (see in particular [5,37,39,55]). However they occur naturally at the extremity of *extended edges*, where the grey-level changes gradually over a relatively long distance (extended edges arise for example at the border of cast shadows). Marr [34] acknowledged Mach bands and extended edges, distinguishing the latter from the featureless gradual luminance profile due to the shading of a curved surface.

As shown in Figure 2, the above-mentioned primary edge profiles can be combined in several ways in order to produce more complex edge profiles, that we call *secondary*. First, the grey-levels of several profiles can be arithmetically added; we label this combination by giving the names of the primary profiles separated by + signs. We illustrate this operation in Figure 2 (a) with the step, line, and roof from Figure 1; the two additions shown there were recognized by Horn [19] as physically significant. Second, one can put the primary edge profiles in succession; this combination is labelled by giving the names of the primary profiles separated by commas. This is shown in Figure 2 (b) with a sharp step followed by a round edge, and with two opposite round edges; these two similar new profiles emulate an inverted line edge, and they were recognized by Ling [29], who called them *valley* edges.

In their 1970 study on edge detection [18], Herskovitz and Binford noticed that steps, lines, and roofs were the most frequent edge profiles in natural images [19]. Ling [29] examined edge profiles in images of surface mounted devices, and found these three profiles, but also round edges and the compound edges of Figure 2 (b) that she called valley edges. In order to justify these experimental findings, let us now explain briefly how events in a scene geometry lead to the various types of luminance edges described above. For a similar discussion, the reader should refer to [19]; see also [18,29] for a phenomenological study of edge profiles.

Horn [19] showed that it is possible to describe the form of luminance profiles of edges in a grey-level image, according to the corresponding types surface discontinuities in the scene giving rise to that image, provided one makes some simplifying assumptions, such as:

— The surface of the objects in the scene is piecewise smooth.
— The reflectance of objects is *Lambertian*, that is: every surface patch looks equally bright from any viewing direction, and its brightness is proportional to the amount of light it receives.
— The primary illumination is *coherent*, in other words there is a single light source spanning a small solid angle (e.g., the sun, or a light bulb).

In general, things are more complicated. First, the surface is not always smooth, it can be grainy. Second, the reflectance of a patch on the surface of an object has two components [24]: a matte *body reflectance*, which is not necessarily Lambertian, and a glossy *surface reflectance*, which is not necessarily specular (mirror-like); moreover these two reflectance components may have distinct chrominances (in a coloured object, the gloss is generally whiter). Third, mutual reflections between the objects must be taken into account.

However, a simplified model of scene illumination, geometry, and reflectance can lead to qualitative distinctions among the various profiles given in the literature as models of ideal edges. As a consequence, it is possible to infer three-dimensional scene events from luminance profiles in a single two-dimensional image.

We make the simplifying assumptions that there is a single source of light, the scene geometry consists of smooth surfaces having the same reflectance, and that reflectance approximates the Lambertian rule as follows: the apparent brightness of a surface patch increases as the direction of the surface normal approaches that of incoming light.

We show in Figure 3 the situation where two faces of an object meet at a convex angle. Since that angle is not really sharp, the corresponding luminance profile will be a gradual step or bright line (as in Figure 1), or an addition of the two (as in Figure 2 (a)). As can be seen in Figure 4, when these faces meet at a concave angle, the same happens, except that we must take into account the mutual reflections between the two faces; this lead to the addition of a positive roof in the luminance profile of the edge; we can thus get a compound step plus line plus roof edge (see Figure 2 (a)).

Whe show in Figure 5 a surface occluding part of another. The border between the two in the image is very sharp (since the effect of light diffraction is negligible); hence the luminance profile of the edge will generally involve a sharp step. When the occluding surface makes a convex angle at that border, this sharp step can be flanked by a gradual line or step (the luminance profile corresponding to the convex angle); on the other hand when that occluding surface is curved along that border, the sharp step is combined with a round profile. When the occluded surface is curved, a round profile appears at the other side of the step; if this occluded surface is flat, the sharp step is accompanied on that side by a flat luminance profile. Thus quite complex luminance profiles can arise in such situations, but one observes frequently [19,29] a step or round edge (see Figure 1), or one of Ling's valley edges shown in Figure 2 (b).

When two objects are juxtaposed, their surfaces meet in a groove, and the corresponding luminance profile is generally a negative (dark) line, sometimes combined with a positive (bright) one. Finally cast shadows have generally elongated edges: the boundary between the shadowed part of the surface and the illuminated one is fuzzy, because the light source is not punctual. At the extremities of such extended edges, the luminance profiles are similar to those of Mach bands (the junction between a plateau and a ramp, see Figure 1).

We have briefly explained how various surface discontinuities lead to different types of edges in the luminance profile. Note that changes in the orientation of a surface (w.r.t. illumination or viewpoint) have in general a weak influence on the chrominance of the resulting image: the surface will appear lighter or darker, but the hue and saturation of its reflected colour will not change, except at specular highlights [24]. Thus a change in image chrominance is often an indication of a change in the chromaticity of the surface reflectance, or a transition between two surfaces having different colours [24]; such chromatic edges are generally simple steps. This justifies the restriction to grey-level images for the analysis of complex edges.

Let us now relate edges to human vision. The human visual system perceives upward and downward steps, as well as light and dark lines, as distinct events [37,39,55]. Other types of edges are generally perceived as a step, a line, or a mixture of both; for examples roofs are perceived as lines [39]. We give an illustration of this fact in Figure 6. It represents a vertical grating which is horizontally periodic; the grey-level profile is a triangular wave at the top row, a square wave at the bottom row, and at intermediate rows it is a convex linear combination of the two, evolving gradually from triangular to square wave. One perceives light and dark lines at the top, but upward and downward steps at the bottom. At the middle the grey-level profile is as in Figure 7, and there is a mixed perception of a step flanked by a line on its left; this line is similar to a Mach band. Globally, the compound feature is seen slightly to the left of its true position, and this shift in location is consistent with the phase congruence model (as is explained in detail on p. 42 of [52]). Other examples of mixed visual features can be found in [37].

The name of *Mach bands* refers to the discovery by the physicist Ernst Mach that the junction between a ramp and a plateau in the luminance leads to the perception of a narrow band (or line) at the end of the plateau; that band is light when the plateau is at the top of the ramp, and dark when the plateau is at the bottom of the ramp. As remarked in [39], such lines are also perceived at positive and negative roofs, for example in a triangular wave. We can thus consider roofs and Mach bands as similar types of features, which are equivalent up to the addition of a linear ramp signal; positive roofs or Mach bands lead to the perception of a light line, while negative ones lead to the perception of a dark line. There is not a complete agreement as to the exact position of the perceived line [5,55]: exactly at the junction, or on the side of the plateau, or on the side of the ramp? This indicates that the edge detectors of the human visual system may have a non-zero response to the underlying linear ramp (this problem will be discussed more precisely in Section 4).

A common interpretation of Mach bands is that they are visual illusions produced by the mechanism of *lateral inhibition*. As we must recognize the albedo of objects under various illumination intensities, the visual system does not measure the absolute luminance of each point, but luminance contrasts between neighbouring areas. Thus the grey-level of each point is compared to those in its neighbourhood. Then at the location of a positive roof or Mach band, the grey-level is higher than the average in its neighbourhood, leading to the perception of a light line, and conversely for the dark line seen at a negative one. This has led many authors to consider Mach bands as a "visual illusion", and not as true edges.

There are several arguments against this interpretation. First, experiments by Morrone et al. [39] on periodic vertical gratings (images whose grey-level is constant in the vertical direction and periodic in the horizontal one) have shown that the sharpness of the perceived Mach band is not related to the sharpness of the angle between the ramp and the plateau in the grey-level profile; it depends rather on Fourier phase characteristics of the image. Second, lateral inhibition should also apply to colours, since the spectrum of sunlight varies a great deal between morning and evening, and we can still perceive the intrinsic colour of objects and distinguish hues which differ by much less than the daily variation of the sunlight spectrum [3]. However in *isoluminant chromatic* images (made with colours having all the same lightness), Mach bands are not seen. This was found by coworkers of Koffka (see [25], pp. 170-171), and has been verified in more recent experiments (D. Burr, private communication). Now it is well-known that the perception of the structure of an image (figure and ground, perspective, etc.) depends on luminance changes, and vanishes for isoluminant chromatic images (see [25], pp. 126-128, where it is called the "Liebmann effect"). This has been verified in neurophysiological studies of Livingstone and Hubel (see [30,31,32]), and it is justified by the above-mentioned fact that changes in surface orientation lead to variations in the image luminance, but without changes in its chrominance. We can thus agree with Koffka that Mach bands are not sideeffects of visual mechanisms such as lateral inhibition, but rather a form of perceptual organization in the image: Koffka classifies correctly a discontinuity in the second derivative of luminance as an edge!

We have thus shown that from the point of views of both photomotry and human visual perception, it is necessary to take into account several types of edge profiles, both simple ones (as in Figure 1) and more complex ones (as in Figure 2).

Therefore considerations in the Fourier domain and related experimental results on human vision are not the only rationales behind using a pair of filters (as in the phase congruence model), instead of a single one (as in older methods). Indeed, one can assume that each one of the two filters can respond to a particular family of features; for example an odd-symmetric filter would respond maximally to steps, and an even-symmetric one would respond maximally to lines and roofs [47,51]. Indeed, it is known [45] that traditional step detectors based on odd-symmetric filters detect two neighbouring steps in a line, and lead thus to edge duplication. Even then, the justification for combining the two filters quadratically is not evident: one could for example apply separately a step detector and a line/roof detector, and combine together the two edge maps obtained separately. We will show in Section 4 that this leads again to edge duplication: for a combined step + line edge (see Figure 2 (a)), the two detectors will localize the step and the line at opposite sides of the true edge location; on the other hand, a quadratic combination the two filters is less prone to this defect, although it can lead to a small error in the localization of the edge (see for example Figure 6, where the compound edge is seen slightly to the left of its true position).

This argument does not exclude the use of quadratic combinations of three or more filters, as suggested in [47]; for example if we can describe edges as linear combinations of three types of simple profiles, it would be logical to use three filters, one adapted to each simple profile. Our justification for choosing only two filters is both intuitive and tentative:

— In human visual perception, it seems that every edge is seen as a mixture of a line and a step; this suggests a combination of two filters, an even-symmetric one leading to the perception of a line (a "line detector"), and an odd-symmetric one leading to the perception of a step (a "step detector"). This correlates with classical neurophysiological findings concerning the behaviour of simple cells in the primary visual cortex of monkeys [20,33,48].

— We can classify simple edge profiles in two groups: the first one contains odd-symmetric signals superposed on a dc level (in their Fourier decomposition, all nonzero frequencies have phase $\pm\pi/2$); the second one contains even-symmetric signals (in their Fourier decomposition, all frequencies have phase 0 or π). Thus the first group contains step edges which are odd-symmetric up to a constant, while the second one contains even-symmetric line and roof edges. Every edge would thus be decomposed into the sum of two edges from the two groups. For example Mach bands would then be classified as the superposition of a roof and a featureless linear ramp signal.

— We have not yet seen any theoretical or practical argument showing the advantage of taking more than two filters for unidirectional features. On the other hand, with two constant-phase filters in Fourier quadrature (that is, their respective constant Fourier phases differ by $\pi/2$, and their Fourier amplitudes are equal), we have an elegant mathematical theory of phase congruence developed in Sections 3 and 4.

We do not exclude the possibility that future studies may suggest the need for more than two filters;

for example round edges or valley edges (see Figure 2 (b)) might necessitate special filters.

Note that some authors [1,13,17,26,27,45,46,47,54] have applied the quadratic approach (taking the sum of squares of convolutions of the image with two filters), in the case where the two filters have indeed constant Fourier phase (respectively 0 and $\pi/2$), but have different Fourier amplitudes, and so do not satisfy the requirement of Fourier quadrature of the phase congruence model. Serious reasons should be given for omitting this constraint on the filters, because we will see in Section 4 that it leads to many interesting results. One possible justification would be that line and step edges in natural images have different Fourier amplitude spectra, so that the two filters in the edge detector should be adapted to this fact. There can also be mathematical considerations; see for example [27], where the pair of filters consisting of a n-th derivative of a Gaussian and of its derivative is shown to satisfy the causality property for the scale-space edge map, while the filters in Fourier phase quadrature used in the phase congruence model do not satisfy this property (we will discuss this fact in more detail later on).

What are the appropriate choices for the two filters C and S used to build the "energy function" $(I * C)^2 + (I * S)^2$? The phase congruence model gives several requirements for C and S in Fourier domain, but they are largely unsufficient to specify these functions. Well-known models of the receptive field profiles of simple cells in the monkey visual cortex (the "edge and bar detectors" of Hubel and Wiesel [20]), indicate that (in the two-dimensional case of static images) these profiles are similar to those of Gabor cosine and sine functions [8,33]; in other words the even-symmetric and odd-symmetric functions C and S superficially look like

$$\exp\left[-\frac{x_t^2}{w_t^2} - \frac{x_n^2}{w_n^2}\right] \cdot \cos[2\pi\nu x_n] \qquad \text{and} \qquad \exp\left[-\frac{x_t^2}{w_t^2} - \frac{x_n^2}{w_n^2}\right] \cdot \sin[2\pi\nu x_n] \tag{2.1}$$

respectively, where x_t and x_n stand for the coordinates in the edge tangential and normal directions respectively, ν is a fixed frequency, while w_n and w_t are constants measuring the width of these functions in the normal and tangential directions; normally w_t is significantly larger than w_n [46]. Note that the two functions given here are not in Fourier phase quadrature, in particular the integral over $\mathbb{R}^2$ of the even-symmetric function is not zero, and we have already pointed [51] some disadvantages of this fact for edge detection. There have also been criticisms of this model on experimental grounds [23]. Alternate analytical formulations for C and/or S have been given [7,23,36,37], but they always give profiles qualitatively similar to (2.1) in the normal direction x_n: for C a central positive lobe flanked by two negative lobes, plus sometimes ripples beyond them; for S, a positive and a negative lobe on each side the origin, plus sometimes ripples beyond them. See Figure 8. In the tangential direction, one can take for both C and S a smoothing function looking like a Gaussian.

The following practical question arises immediately: does the phase congruence model accurately detect and localize edges in natural images? We can give here a partial theoretical answer. The ideal steps, lines, and roofs shown in Figure 1 have constant phase at the edge location (the phase is zero for lines and roofs, but $-\pi/2$ for positive-going steps); as we will see in Section 4, the energy function obtained as the sum of squares of convolutions of the signal with two constant phase filters in phase quadrature, has an absolute maximum at this edge position. Ramp edges and Mach bands are a linear combination of a symmetric roof and a linear ramp signal; by choosing the two filters in such a way that they have a zero response on linear signals, the energy function will be the same as for a symmetric roof, and it will have an absolute maximum at the edge position. The compound step + line or step + line + roof edges shown in Figure 2 (a) are the sum of a signal with constant zero phase and another one with constant $-\pi/2$ phase; hence they have all Fourier phases comprised between 0 and $-\pi/2$ at the edge location; it is thus likely that these phases will be maximally congruent at a point near the edge location, where the energy function will reach a local maximum. For example in the grating of Figure 6, whose grey-level profile in the middle rows is shown in Figure 7, maximum phase congruence is achieved slightly to the left of the true edge location, and this is consistent with visual perception.

On the other hand, from a practical point of view, experiments made among others by Morrone, Owens and, Venkatesh [37,38,45,56,63] indicate that this model gives satisfactory results in natural images. Further experiments with various filter designs will be necessary in order to compare this model to others.

A related question is whether the phase congruence model can give false edges, in other words if one can get local maxima of the energy function which do not correspond to true edges. In [45], various one-dimensional edge models were tested against an "ideal" binary edge element modeled as a Dirac delta impulse; in some cases the resulting energy function, besides its marked absolute maximum at the origin, makes ripples, leading to spurious local maxima, in other words "echoes" of the feature located at the origin. It could be possible to do the same testing with idealized edge grey-level profiles similar to those of Figure 1. For example, an ideal line would be a Dirac delta (or a symmetric triangle, as in Figure 1), an ideal step a Heaviside step function, an ideal roof or Mach band would have its ramps on the two sides extending to infinity. Such ideal edges do not occur in natural images, but they can be used as a mathematical check for an edge detector: verify that when applying the detector to the ideal edge, that edge is properly detected and localized, and no additional feature is detected. We feel that the phase congruence model is incomplete, in the sense that filter specification in the Fourier domain is unsufficient, and that some requirements in the spatial domain must be made for filters; for example we gave very general constraints on pp. 35 and 36 of [52] in order to avoid the detection of false edges in ideal steps, lines and roofs. More research should be conducted on non-Fourier requirements for quadratic edge detectors. The problem of choosing a "good" pair of filters is not yet solved.

Another possible answer to this question is to restrict edges to *regional* maxima of the energy function, that is points where that function has a maximum within a certain radius r corresponding to the scale of analysis (in fact, to the spatial extent of the filters), so that purely local maxima are eliminated. Indeed, for ideal edges, the energy function reaches an *absolute* maximum at the edge location, and we might plausibly deduce that for several standard edges distant from each other by at least $2r$, each one may give a regional maximum of the energy function within radius r. We discuss this approach in Section 4. Further research should elucidate criteria for selecting "good" maxima of the energy function, or equivalently, "good" maxima of phase congruence: should a point where Fourier phases are incoherent, but nevertheless less so than in its neighbourhood, be considered as a feature?

One property of human vision which must be taken into account for artificial vision, is that each visual attribute (for example a feature) corresponds to a size scale. For example a black cat seen from afar looks like a black blob; at moderate distance, limbs are distinguished; closer yet, the fur appears as a texture; within hand reach, individual hair are perceived. This was seriously recognized by Marr [35], who linked this fact to the existence of banks of visual filters tuned to different size scales. Since then, it is customary to analyse images with filters at several size scales (usually 3 or 4), with a scale factor of 2. Alternately, theoretical models of "scale-space" have been studied, where the scale varies continuously, and is combined with the underlying visual space as an additional dimension; see for instance [16] for a recent study. One should note that the nature of an edge can be scale-dependent. We illustrate this in Figure 9, where an edge profile is shown with various spatial magnification factors; whenever it is enlarged, what we see correponds to what will be detected in the original profile at a smaller scale. Here it appears that a bar edge at coarse scale becomes a pair of step edges at a finer scale, and then a set of four Mach bands at a still finer scale. Thus, whenever one speaks of an edge, one must specify its scale; in practice, it corresponds to the size scale of the filters used to detect it: filters with a wide grey-level profile detect features at a coarse scale, and those with a narrow profile detect features at a fine scale.

Marr and Hildreth [36] required from significant features that they appear at all scales used, preferably at the same position. In practice the position of feature changes as scale varies. One can however associate to each edge detection method a model of "ideal" edges which should be detected and localized at the same position for all possible scales, arbitrarily small or large. As we will also see in Section 4 when discussing models based on a single filter, a usual edge detector localizing edges at maxima of the absolute value of the filtered image has in general an underlying ideal edge, for which it gives an optimal response at all scales. We can mention in particular that for step detectors based on Gaussian-smoothed gradients, such as Canny's [7], a signal with constant Fourier phase $\pi/2$ is a perfect downward step (the phase is $-\pi/2$ for a perfect upward step); on the other hand for the phase congruence model, a perfect edge is any signal having constant Fourier phase, and this is an example of the fact that the phase congruence model generalizes some previous methods by allowing a wider family of edges to be detected.

Since it must be expected that the position of real edges changes with scale, it has become customary to analyse the evolution of edges in continuous scale-space [16]. One requirement that has been put forward is that of *causality*: as scale increases from fine to coarse, edges can move, vanish, or split, but cannot be created *ex nihilo*; this is related to the maximum principle for parabolic partial differential equations, for example the heat diffusion equation verified by Gaussian smoothing and derived models (which are causal), such as the Marr-Hildreth Laplacian of Gaussian zero-crossing model. Kube and Perona [27] have shown that for many choices of the pair of filters (in particular, Hilbert transform pairs) involved in an energy function, causality is not verified: edges can appear at a coarse scale, which do not correspond to edges at finer scales; they illustrated this fact with the pair consisting of the second derivative of a Gaussian and its Hilbert transform. On the other hand the pair of filters consisting of the n-th derivative of a Gaussian ($n \geq 1$) and of its derivative is shown to satisfy the causality requirement. Thus the phase congruence model has one drawback compared to some other models. This defect should be weighted against its advantages. However there are some reasons to suspect that causality failures (new edges arising from nothing as scale increases) might correspond to spurious local maxima of the energy function. A typical pattern of such causality failure is shown in Figure 10 (see also Figures 2 and 3 of [27]). We see here that a purely local maximum arises in the energy function, which could be eliminated by taking as edge points only regional maxima of the energy function, as explained above. It would be interesting to know if the two approaches suggested above for authenticating edges could allow the elimination of all causality failures.

This section has shown the various problems associated with edge detection, and the need to take into account various types of luminance profiles and their evolution across scale-space. This justifies the exploration of new edge detectors involving several filters. We will now start studying mathematical properties of the phase congruence model, based on convolution with two constant phase filters in Fourier quadrature.

3. Mathematical foundations of the phase congruence model

In order to make this report self-contained, we recall the mathematics underlying the phase congruence model; this will essentially be Fourier analysis in $L^1 + L^2$ and for tempered functions, and some elementary facts derived from it. We introduce first our notation:

Let $\mathcal{E} = \mathbb{R}^d$ for some integer $d \geq 1$; we will consider all signals in the spatial or Fourier domain as functions $\mathcal{E} \to \mathbb{C}$ (or $\mathcal{E} \to \mathbb{R}$). In practical situations we will often have $d = 2$ for static images, or $d = 3$ for volumetric images or image sequences (cfr. the spatiotemporal edge model of [1,64]). We write: x, y, z, etc. for real or complex variables; $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, etc. for vectors in $\mathcal{E}$; f, g, etc. for functions $\mathbb{R} \to \mathbb{C}$; F, G, etc. for functions $\mathcal{E} \to \mathbb{C}$. We write $\mathbf{x} \cdot \mathbf{y}$ for the scalar product of $\mathbf{x}$ and $\mathbf{y} \in \mathcal{E}$, and $|\mathbf{x}|$ for the Euclidean norm of $\mathbf{x}$, that is $|\mathbf{x}| = (\mathbf{x} \cdot \mathbf{x})^{1/2}$. For $x \in \mathbb{C}$, let $\overline{x}$ be its complex conjugate and $|x|$ its absolute value. For a function $F : \mathcal{E} \to \mathbb{C}$, we define $\overline{F}$ and $|F|$ by $\overline{F}(\mathbf{x}) = \overline{F(\mathbf{x})}$ and $|F|(\mathbf{x}) = |F(\mathbf{x})|$.

The *reflection* F_ρ of a function F is given by $F_\rho(\mathbf{x}) = F(-\mathbf{x})$. We say that F is *even-symmetric* if $F_\rho = F$, *odd-symmetric* if $F_\rho = -F$, and *conjugate-symmetric* if $F_\rho = \overline{F}$. We say that the function F *vanishes at infinity* if $\lim_{|\mathbf{x}| \to \infty} F(\mathbf{x}) = 0$. We define the *signum* function sgn on $\mathbb{C}$ by

$$\operatorname{sgn}(x) = \begin{cases} x/|x| & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

In particular for $x \in \mathbb{R}$, we have $\operatorname{sgn}(x) = 1$ if $x > 0$ and $\operatorname{sgn}(x) = -1$ if $x < 0$.

We use on the space $\mathcal{E}$ the Lebesgue measure and integral (see [11]). All subsets of $\mathcal{E}$ and all functions on $\mathcal{E}$ are implicitly assumed to be Lebesgue measurable. We write $d\mathbf{x}$ for $dx_1 \cdots dx_d$, the d-dimensional Lebesgue measure element in an integral (cfr. [60]). Thus we write $\int_{\mathcal{S}} F(\mathbf{x})\, d\mathbf{x}$, $\int_{\mathcal{S}} d\mathbf{x}\, F(\mathbf{x})$, or simply $\int_{\mathcal{S}} F$ for the integral of F over a (measurable) set $\mathcal{S} \subseteq \mathcal{E}$. Note that "integrable" means "integrable in absolute value": a (measurable) function F is integrable over $\mathcal{S}$ if and only if $|F|$ is integrable over $\mathcal{S}$, in other words $\int_{\mathcal{S}} |F(\mathbf{x})\, d\mathbf{x}| < +\infty$.

We use the customary qualification "almost" in order to mean "except in a set of measure zero", and the abbreviations "a.a." and "a.e." for "almost all" and "almost everywhere". For example, the function F is a.e. even-symmetric if $F(\mathbf{x}) = F(-\mathbf{x})$ almost everywhere. We define the equivalence relation $\equiv$ on functions by setting $F \equiv G$ for two functions F, G if $F(\mathbf{x}) = G(\mathbf{x})$ a.e.; in particular $F(\mathbf{x}) = G(\mathbf{x})$ for all points $\mathbf{x}$ at which both F and G are continuous. Clearly the equivalence $\equiv$ is compatible with algebraic operations, and for integrable functions, $F \equiv G$ implies that $\int F = \int G$.

For any $\mathbf{u} \in \mathcal{E}$, let $\tau_{\mathbf{u}}$ be the translation by $\mathbf{u}$, which moves horizontally by $\mathbf{u}$ the graph of a function F, that is $\tau_{\mathbf{u}}(F)$ is defined by $\tau_{\mathbf{u}}(F)(\mathbf{x}) = F(\mathbf{x} - \mathbf{u})$. We define $\operatorname{cis}_{\mathbf{u}}$, the "cisoid" function of frequency $\mathbf{u}$, by setting for $\mathbf{x} \in \mathcal{E}$:

$$\operatorname{cis}_{\mathbf{u}}(\mathbf{x}) = \exp(2\pi i\, \mathbf{u} \cdot \mathbf{x}) = \cos(2\pi\, \mathbf{u} \cdot \mathbf{x}) + i \sin(2\pi\, \mathbf{u} \cdot \mathbf{x}). \tag{3.1}$$

Note that we consider frequency in cycles per unit, and not angular frequency in radians per unit; in this we follow [11,60].

Let $\mathbf{n}$ be a unit vector in $\mathcal{E}$ ($\mathbf{n} \cdot \mathbf{n} = 1$); in the next section, it will be interpreted as the unit vector normal to the edge. It partitions the space $\mathcal{E}$ into the three sets

$$\begin{aligned} \mathcal{P}_{\mathbf{n}} &= \{\mathbf{x} \in \mathcal{E} \mid \mathbf{n} \cdot \mathbf{x} > 0\}, \\ \mathcal{N}_{\mathbf{n}} &= \{\mathbf{x} \in \mathcal{E} \mid \mathbf{n} \cdot \mathbf{x} < 0\}, \\ \mathcal{E}_{\mathbf{n}} &= \{\mathbf{x} \in \mathcal{E} \mid \mathbf{n} \cdot \mathbf{x} = 0\}. \end{aligned} \tag{3.2}$$

We define also the three functions $\mathrm{pos}_{\mathbf{n}}$, $\mathrm{neg}_{\mathbf{n}}$, and $\mathrm{sgn}_{\mathbf{n}}$ on $\mathcal{E}$ by setting for $\mathbf{x} \in \mathcal{E}$:

$$\begin{aligned} \mathrm{pos}_{\mathbf{n}}(\mathbf{x}) &= \begin{cases} 1 & \text{if } \mathbf{n} \cdot \mathbf{x} > 0, \\ 0 & \text{if } \mathbf{n} \cdot \mathbf{x} \leq 0; \end{cases} \\ \mathrm{neg}_{\mathbf{n}}(\mathbf{x}) &= \begin{cases} 1 & \text{if } \mathbf{n} \cdot \mathbf{x} < 0, \\ 0 & \text{if } \mathbf{n} \cdot \mathbf{x} \geq 0; \end{cases} \\ \mathrm{sgn}_{\mathbf{n}}(\mathbf{x}) &= \begin{cases} +1 & \text{if } \mathbf{n} \cdot \mathbf{x} > 0, \\ 0 & \text{if } \mathbf{n} \cdot \mathbf{x} = 0, \\ -1 & \text{if } \mathbf{n} \cdot \mathbf{x} < 0. \end{cases} \end{aligned} \tag{3.3}$$

(Cfr. the definition of the signum function sgn above). Note that $\mathrm{sgn}_{\mathbf{n}} = \mathrm{pos}_{\mathbf{n}} - \mathrm{neg}_{\mathbf{n}}$.

Let $\mathcal{L}_{\mathbf{n}}$ be the vector space generated by $\mathbf{n}$. We can consider $\mathcal{E}$ as the product of the two orthogonal subspaces $\mathcal{E}_{\mathbf{n}}$ and $\mathcal{L}_{\mathbf{n}}$, the latter being identified with $\mathbb{R}$ (every $t \in \mathbb{R}$ corresponding to $t\mathbf{n} \in \mathcal{L}_{\mathbf{n}}$). Thus every $\mathbf{x} \in \mathcal{E}$ can be written in a unique way as the pair $(\mathbf{y}, t) \in \mathcal{E}_{\mathbf{n}} \times \mathbb{R}$, where $t = \mathbf{x} \cdot \mathbf{n}$ and $\mathbf{y} = \mathbf{x} - t\mathbf{n}$, in other words $\mathbf{x} = \mathbf{y} + t\mathbf{n}$. We illustrate this in Figure 11 for $d = 2$. Given a function $F : \mathcal{E} \to \mathbb{C}$, every $\mathbf{y} \in \mathcal{E}_{\mathbf{n}}$ gives the $\mathcal{E}_{\mathbf{n}}$-section $F_{\mathbf{y}} : \mathbb{R} \to \mathbb{C} : t \mapsto F(\mathbf{y}, t)$, while every $t \in \mathbb{R}$ gives the $\mathcal{L}_{\mathbf{n}}$-section $F^t : \mathcal{E}_{\mathbf{n}} \to \mathbb{C} : \mathbf{y} \mapsto F(\mathbf{y}, t)$ (see [11], p. 63). For an integrable function $F : \mathcal{E} \to \mathbb{C}$, we define the function $F_{/\mathbf{n}} : \mathbb{R} \to \mathbb{R}$ by

$$F_{/\mathbf{n}}(t) = \int_{\mathcal{E}_{\mathbf{n}}} F(\mathbf{y}, t)\, d\mathbf{y} = \int_{\mathcal{E}_{\mathbf{n}}} F(\mathbf{y} + t\mathbf{n})\, d\mathbf{y}; \tag{3.4}$$

by Fubini's theorem, $F_{/\mathbf{n}}$ is integrable.

3.1. L^p spaces and convolution

Let p be such that $1 \leq p \leq \infty$. For $p < \infty$, the L^p *norm* $\|\ \|_p$ of is defined by $\|F\|_p = \left(\int_{\mathcal{E}} |F(\mathbf{x})|^p\, d\mathbf{x}\right)^{1/p}$ for every function $F : \mathcal{E} \to \mathbb{C}$. For $p = \infty$, the L^∞ norm $\|F\|_\infty$ of F is the *essential supremum* of all $|F(\mathbf{x})|$, in other words the least $m \in [0, \infty]$ such that $|F(\mathbf{x})| \leq m$ almost everywhere.

Let L^p be the set of functions F such that $\|F\|_p < \infty$; for example L^1 is the space of integrable functions; functions in L^2 are called *square-integrable*; functions in L^∞ are called *essentially bounded*. L^p is a vector space in which $\|\ \|_p$ is a norm [11], and for any $F, G \in L^p$, $F \equiv G$ if and only if $\|F - G\|_p = 0$. Write $L^p + L^q$ for the space of functions of the form $F_p + F_q$, with $F_p \in L^p$ and $F_q \in L^q$.

Let $L^p/{\equiv}$ be the set of equivalence classes of $\equiv$ over L^p. Then the L^p norm makes $L^p/{\equiv}$ into a metric space, where the distance between two functions F and G is $\|F - G\|_p$. A well-known theorem states that the metric space $L^p/{\equiv}$ is *complete* (i.e., every Cauchy sequence converges).

For $p < \infty$, functions in L^p satisfy the property of *L^p-continuity*, namely that for $F \in L^p$,

$$\lim_{\mathbf{u} \to \mathbf{0}} \|F - \tau_{\mathbf{u}}(F)\|_p = 0. \tag{3.5}$$

A proof can be found in Proposition 8.5 of [11]. Note that (3.5) does not necessarily hold for $p = \infty$.

For any two functions F and G, their *convolution* $F * G$ is given by

$$[F * G](\mathbf{x}) = \int_{\mathcal{E}} F(\mathbf{x} - \mathbf{t}) G(\mathbf{t}) \, d\mathbf{t} \tag{3.6}$$

(whenever the integral exists). This operation is bilinear and commutative. Moreover, it commutes with translation, that is $F * \tau_{\mathbf{u}}(G) = \tau_{\mathbf{u}}(F * G) = \tau_{\mathbf{u}}(F) * G$ for all $\mathbf{u} \in \mathcal{E}$. The convolution by a function in L^1 is a stable operation w.r.t. the L^p norm (see Theorem 1.3 of [60] and Theorem 8.7 of [11]):

— **Young's inequality:** *Let $F \in L^1$ and $G \in L^p$ ($1 \leq p \leq \infty$). Then $(F * G)(x)$ is defined almost everywhere, $F * G$ belongs to L^p, and $\|F * G\|_p \leq \|F\|_1 \|G\|_p$.*

In particular the convolution operation is a bilinear product operation in $L^1/\equiv$, which is associative in the sense that for $F, G, H \in L^1$, $F * (G * H) \equiv (F * G) * H$.

A slightly stronger result holds for the convolution of a function in L^p with one in $L^{p/p-1}$ (see Theorem 8.8 of [11]):

— **p-p' convolution property:** *Let $F \in L^p$ and $G \in L^{p'}$, where $1 \leq p, p' \leq \infty$ and $(1/p) + (1/p') = 1$. Then $(F * G)(x)$ is defined for all $\mathbf{x} \in \mathcal{E}$, $F * G$ is uniformly continuous and bounded: for all $x \in \mathcal{E}$, $\big|[F * G](x)\big| \leq \|F\|_p \|G\|_{p'}$. Furthermore, $F * G$ vanishes at infinity, provided that one of the following is satisfied:*

(a) *$p, p' < \infty$;*

(b) *$p = \infty$ and F vanishes at infinity;*

(c) *$p' = \infty$ and G vanishes at infinity.*

A particular case is $p = 1$ and $p' = \infty$: the convolution of a function in L^1 and one in L^∞ is uniformly continuous and bounded. Another one is given by $p = p' = 2$.

The above two properties are interesting enough to justify the choice of functions in L^1 for the filters applied to the image. Besides, grey-levels of pictures usually belong to a bounded range; so in view of the p-p' convolution property, if we want to have the picture convolved with a mask to get its grey-levels in the same range, we must require the convolution function to be in L^1. Furthermore, the uniform continuity of the result of the convolution is interesting in view of digitization, as we will explain in Section 5.

In the sequel, we will generally restrict ourselves to the spaces L^1, L^2, and L^∞ and their sums. For example in Section 4, all visual signals will belong to $L^1 + L^2 + L^\infty$.

3.2. The Fourier transform and bitempered functions

As said above, we follow [60] in considering that frequency is in cycles per unit, and not angular frequency in radians per unit. Thus the factor 2π precedes the frequency in the argument of an imaginary exponential (cfr. (3.1)). As in [60], we write $\widehat{F}$ for the Fourier transform of a function F; the Fourier transform of an expression $(\ldots)$ will be written $(\ldots)^\wedge$.

The *Fourier transform* $\widehat{F}$ of an integrable function F is defined pointwise by the *Fourier integral*:

$$\widehat{F}(\mathbf{u}) = \int_{\mathcal{E}} F(\mathbf{x}) \exp[-2\pi i \, \mathbf{u} \cdot \mathbf{x}] \, d\mathbf{x}.$$

In particular $\int_{\mathcal{E}} F = \widehat{F}(0)$; also, given the decomposition of $\mathcal{E}$ as the product of $\mathcal{E}_{\mathbf{n}} \times \mathcal{L}_{\mathbf{n}}$ (or $\mathcal{E}_{\mathbf{n}} \times \mathrm{I\!R}$), the function $F_{/\mathbf{n}}$ defined in (3.4) satisfies:

$$(F_{/\mathbf{n}})^{\wedge}(\nu) = \widehat{F}(\mathbf{0}, \nu) = \widehat{F}(\nu \mathbf{n}), \tag{3.7}$$

as we can easily check with Fubini's theorem. The Fourier integral has the following distinctive property:

— **Riemann-Lebesgue theorem:** *For $F \in L^1$, $\widehat{F}$ is uniformly continuous and bounded on $\mathcal{E}$: $|\widehat{F}(\mathbf{u})| \leq \|F\|_1$ for all $\mathbf{u} \in \mathcal{E}$; moreover $\widehat{F}$ vanishes at infinity.*

Note also that *integrable functions are the only functions for which the Fourier transform is defined pointwise*. Indeed, the Fourier integral preserves the L^2 norm of functions in $L^1 \cap L^2$, and the Fourier transform is thus defined in $L^2/\!\equiv$ as the unique isometry extending the Fourier integral on the dense set $(L^1 \cap L^2)/\!\equiv$. Thus the Fourier transform of a square-integrable function is a square-integrable function, unique up to equivalence by $\equiv$, in other words it is not defined pointwise, but up to a set of measure zero. The distinctive property of square-integrable functions w.r.t. the Fourier transform is well-known:

— **Plancherel theorem:** *For $F \in L^2$, $\widehat{F} \in L^2$, $\|\widehat{F}\|_2 = \|F\|_2$, and $F_\rho \equiv \widehat{\widehat{F}}$.*

For a function F which is both integrable and square-integrable, the two definitions of $\widehat{F}$ in L^1 and L^2 coincide up to equivalence by $\equiv$. This allows us to define the Fourier transform on $L^1 + L^2$ as follows: for $F = F_1 + F_2$ with $F_1 \in L^1$ and $F_2 \in L^2$, we define $\widehat{F}$ as $\widehat{F_1} + \widehat{F_2}$; the latter is defined up to equivalence by $\equiv$, in other words up to a set of measure zero, and in this sense it does not depend on the choice of the functions F_1 and F_2 in the decomposition of F.

The following three fundamental formulas are more readily expressed in the framework of $L^1 + L^2$, although they can be considered in a more general setting:

— **Convolution formula:** *Let $F \in L^1$ and $G \in L^1 + L^2$. Then $[F * G]^{\wedge} \equiv \widehat{F}\widehat{G}$ (and the equality holds pointwise for G integrable).*

— **Dual convolution formula:** *Let $F, G \in L^2$. Then $[FG]^{\wedge} = \widehat{F} * \widehat{G}$.*

— **Multiplication formula:** *Let either $F, G \in L^1$, or $F, G \in L^2$, or $F \in L^1 + L^2$ and $G \in L^1 \cap L^2$. Then $F\widehat{G}$ and $\widehat{F}G$ are integrable and $\int_{\mathcal{E}} F\widehat{G} = \int_{\mathcal{E}} \widehat{F}G$.*

For functions outside $L^1 + L^2$, the definition of the Fourier transform relies on tempered distributions. We recall briefly the relevant theory; the reader is referred to Section 1.3 of [60], or to Sections 8.1 and 8.5 of [11], for further details. A *Schwartz function* is a C^∞ function such that itself and all its derivatives, multiplied by any polynomial, remain bounded. A *tempered distribution* is a continuous linear functional on the space of Schwartz functions; the tempered distribution ψ associates to a Schwartz function S the value $\langle \psi, S \rangle$. A *tempered function* is a function F such that there is some $N \geq 0$ for which the function $\mathbf{x} \mapsto (1 + |\mathbf{x}|)^{-N} F(\mathbf{x})$ is integrable; for examples functions in L^p $(1 \leq p \leq \infty)$ are tempered. A tempered function F induces a tempered distribution F^{dist} given by $\langle F^{\mathrm{dist}}, S \rangle = \int_{\mathcal{E}} FS$; we generally identify F with F^{dist}, and write thus $\langle F, S \rangle = \int_{\mathcal{E}} FS$. Note that two tempered functions F and G give rise to the same tempered distribution (i.e., $F^{\mathrm{dist}} = G^{\mathrm{dist}}$) if and only if $F \equiv G$.

The Fourier transform $\widehat{\psi}$ of a tempered distribution ψ is the unique tempered distribution given by $\langle \widehat{\psi}, S \rangle = \langle \psi, \widehat{S} \rangle$ for all Schwartz functions S. For a function $F \in L^1 + L^2$, the definition of $\widehat{F}$

in the sense of tempered distributions coincides with that given above, thanks to the multiplication formula $\int_{\mathcal{E}} \widehat{F} S = \int_{\mathcal{E}} F \widehat{S}$ which holds for any Schwartz function S.

As we will consider phase and amplitude in the Fourier domain, we will be interested by functions whose Fourier transform is a function. Let us thus introduce a personal terminology. We define a *bitempered function* as a tempered function whose Fourier transform (in the tempered distribution sense) is given by a function. More precisely, it is a tempered function F for which there is a tempered function G such that we have $\int_{\mathcal{E}} GS = \int_{\mathcal{E}} F\widehat{S}$ for every Schwartz function S; here G is the Fourier transform $\widehat{F}$ of F, and it is unique in the sense of tempered distributions, in other words as a function it is unique up to equivalence by $\equiv$, and it is defined up to a set of measure zero.

From now on, every function will implicitly be assumed to be bitempered, and general properties of the Fourier transform will be considered in the framework of bitempered functions. First, the Fourier transform commutes with reflexion:

$$\left(F_\rho\right)^\wedge = \left(\widehat{F}\right)_\rho; \tag{3.8}$$

we will write $F^\vee$ for $\left(F_\rho\right)^\wedge = \left(\widehat{F}\right)_\rho$. When F is integrable we have

$$F^\vee(\mathbf{u}) = \int_{\mathcal{E}} F(\mathbf{x}) \exp[2\pi i\, \mathbf{u}\cdot\mathbf{x}]\, d\mathbf{x}.$$

We recall the well-known:

— **Fourier inversion formula:** *For every function F, $F_\rho \equiv \widehat{\widehat{F}}$, in other words $F \equiv \widehat{F}^\vee$.*

We deduce the *uniqueness property*: given two functions F, G, $\widehat{F} \equiv \widehat{G}$ if and only if $F \equiv G$; furthermore, for $F, G \in L^1$, $\widehat{F} \equiv \widehat{G}$ implies the pointwise equality $\widehat{F} = \widehat{G}$. Next:

$$\widehat{\overline{F}}(\mathbf{u}) = \overline{\widehat{F}(-\mathbf{u})}; \tag{3.9}$$

this can be rewritten as:

$$\widehat{\overline{F}} = \overline{F^\vee} \qquad \text{and} \qquad \overline{F}^\vee = \overline{\widehat{F}}. \tag{3.10}$$

The Fourier duality between translation and modulation is expressed as follows in our notation; for $\mathbf{h} \in \mathcal{E}$ we have:

$$[\tau_{\mathbf{h}}(F)]^\wedge = \mathrm{cis}_{-\mathbf{h}} \cdot \widehat{F}; \tag{3.11}$$

$$[\mathrm{cis}_{\mathbf{h}} \cdot F]^\wedge = \tau_{\mathbf{h}}(\widehat{F}). \tag{3.12}$$

Given a bijective linear transform α of $\mathcal{E}$, the function G defined by $G(\mathbf{x}) = F(\alpha(\mathbf{x}))$ has its Fourier transform given by $\widehat{G}(\mathbf{u}) = |\det(\alpha)|^{-1}\widehat{F}(\alpha^{-T}(\mathbf{u}))$, where $\det(\alpha)$ is the determinant of α and α^{-T} is the inverse transpose of α. In particular if α is an isometry (α is its own inverse transpose), then $\widehat{G}(\mathbf{u}) = \widehat{F}(\alpha(\mathbf{u}))$, in other words the Fourier transform commutes with α. For example reflection commutes with the Fourier transform (cfr. (3.8)); the Fourier transform commutes also with rotations of $\mathcal{E}$.

Note that by (3.8) and the uniqueness property, F is a.e. even-symmetric (resp., a.e. odd-symmetric) if and only if $\widehat{F}$ is a.e. even-symmetric (resp., a.e. odd-symmetric). By (3.9) and the uniqueness property, F is a.e. real-valued if and only if $\widehat{F}$ is a.e. conjugate-symmetric; in particular for F real-valued, F is a.e. even-symmetric (resp., a.e. odd-symmetric), if and only if $\widehat{F}$ is a.e. real (resp., a.e. imaginary).

Further properties of the Fourier transform will be considered in Subsection 3.4, where we will study a multidimensional generalization of Fourier quadrature and the Hilbert transform. We give now the first consequence of the Riemann-Lebesgue theorem and the Fourier inversion formula:

Lemma 3.1. *Let F be a function such that $\widehat{F}$ is integrable. Define F_u by*

$$F_u(\mathbf{x}) = \int_{\mathcal{E}} \widehat{F}(\mathbf{u}) \exp[2\pi i\, \mathbf{u}\cdot\mathbf{x}]\, d\mathbf{u} \tag{3.13}$$

for $\mathbf{x} \in \mathcal{E}$. Then $F \equiv F_u$, F_u is uniformly continuous and bounded on $\mathcal{E}$, and it vanishes at infinity.

In the remainder of this report, we will use the notation F_u introduced here.

3.3. Local Fourier phase

We will define here local Fourier phase, and show how to obtain from it results concerning pointwise values of a function.

Given a function F, write $F^{\mathcal{A}}$ for the *Fourier amplitude* of F, that is $F^{\mathcal{A}}(\mathbf{u}) = \left|\widehat{F}(\mathbf{u})\right|$, and F^{Φ} for the *Fourier phase* of F, that is $\widehat{F}(\mathbf{u}) = F^{\mathcal{A}}(\mathbf{u}) \exp\left[iF^{\Phi}(\mathbf{u})\right]$. When F is real-valued, by (3.9) $\widehat{F}$ is a.e. conjugate-symmetric, so that $F^{\mathcal{A}}$ is a.e. even-symmetric and F^{Φ} is a.e. odd-symmetric.

Given a function F and a point $\mathbf{x} \in \mathcal{E}$, we call the *Fourier transform of F at $\mathbf{x}$* the Fourier transform of the function resulting from F when the origin of $\mathcal{E}$ is shifted to $\mathbf{x}$; in other words it is

$$[\tau_{-\mathbf{x}}(F)]^{\wedge} = \mathrm{cis}_{\mathbf{x}} \cdot \widehat{F}.$$

Here, the Fourier amplitudes are those of F, but the Fourier phases are advanced proportionally to $2\pi\,\mathbf{x}$:

$$\left[\tau_{-\mathbf{x}}(F)\right]^{\Phi}(\mathbf{u}) = F^{\Phi}(\mathbf{u}) + 2\pi\,\mathbf{x}\cdot\mathbf{u}.$$

Let us write $F^{\Phi}(\mathbf{u},\mathbf{x})$ for the *Fourier phase of F at $\mathbf{x}$ for the frequency* $\mathbf{u}$, in other words

$$F^{\Phi}(\mathbf{u},\mathbf{x}) = F^{\Phi}(\mathbf{u}) + 2\pi\,\mathbf{x}\cdot\mathbf{u}. \tag{3.14}$$

The function F^{Φ} with two arguments $\mathbf{u}$ and $\mathbf{x}$ in the Fourier and spatial domain respectively, is called the *local Fourier phase of F*. Now for a function F such that $\widehat{F}$ is integrable, (3.13) can be rewritten in terms of Fourier amplitude and local Fourier phase:

Lemma 3.2. *Let F be a function such that $\widehat{F}$ is integrable. Then*

$$F_u(\mathbf{x}) = \int_{\mathcal{E}} F^{\mathcal{A}}(\mathbf{u}) \exp\left[i\, F^{\Phi}(\mathbf{u},\mathbf{x})\right] d\mathbf{u} \tag{3.15}$$

and

$$F_u(\mathbf{x})\overline{F_u(\mathbf{x})} = \int_{\mathcal{E}}\int_{\mathcal{E}} F^{\mathcal{A}}(\mathbf{u})F^{\mathcal{A}}(\mathbf{v}) \cos\left[F^{\Phi}(\mathbf{u},\mathbf{x}) - F^{\Phi}(\mathbf{v},\mathbf{x})\right] d\mathbf{u}d\mathbf{v} \tag{3.16}$$

for $\mathbf{x} \in \mathcal{E}$. Furthermore, if F is real-valued, then

$$F_u(\mathbf{x}) = \int_{\mathcal{E}} F^{\mathcal{A}}(\mathbf{u}) \cos\left[F^{\Phi}(\mathbf{u},\mathbf{x})\right] d\mathbf{u} = 2\int_{\mathcal{P}_{\mathbf{n}}} F^{\mathcal{A}}(\mathbf{u}) \cos\left[F^{\Phi}(\mathbf{u},\mathbf{x})\right] d\mathbf{u} \tag{3.17}$$

for $\mathbf{x} \in \mathcal{E}$ and for any half-space $\mathcal{P}_{\mathbf{n}}$.

Proof. For every $\mathbf{x} \in \mathcal{E}$ we have by definition of phase and local phase (see (3.14)):

$$\begin{aligned}&\widehat{F}(\mathbf{u}) \exp[2\pi i\,\mathbf{u}\cdot\mathbf{x}] = F^{\mathcal{A}}(\mathbf{u}) \exp\left[iF^{\Phi}(\mathbf{u})\right] \exp[2\pi i\,\mathbf{u}\cdot\mathbf{x}] \\ &= F^{\mathcal{A}}(\mathbf{u}) \exp\left[iF^{\Phi}(\mathbf{u}) + 2\pi i\,\mathbf{u}\cdot\mathbf{x}\right] = F^{\mathcal{A}}(\mathbf{u}) \exp\left[i\,F^{\Phi}(\mathbf{u},\mathbf{x})\right].\end{aligned}$$

Hence (3.15) follows from (3.13). We obtain then:

$$\begin{aligned} F_u(\mathbf{x})\overline{F_u(\mathbf{x})} &= \int_{\mathcal{E}} F^{\mathcal{A}}(\mathbf{u}) \exp\bigl[i\, F^{\Phi}(\mathbf{u},\mathbf{x})\bigr]\, d\mathbf{u} \int_{\mathcal{E}} \overline{F^{\mathcal{A}}(\mathbf{v}) \exp\bigl[i\, F^{\Phi}(\mathbf{v},\mathbf{x})\bigr]}\, d\mathbf{v} \\ &= \int_{\mathcal{E}}\int_{\mathcal{E}} F^{\mathcal{A}}(\mathbf{u})F^{\mathcal{A}}(\mathbf{v}) \exp\bigl[i\bigl(F^{\Phi}(\mathbf{u},\mathbf{x}) - F^{\Phi}(\mathbf{v},\mathbf{x})\bigr)\bigr]\, d\mathbf{u}d\mathbf{v}. \end{aligned}$$

Since the left member of this equality is real, so is the right one, hence this double integral is equal to its real part, in other words we get (3.16). When F is real-valued, (3.15) reduces to its real part, that is

$$F_u(\mathbf{x}) = \int_{\mathcal{E}} F^{\mathcal{A}}(\mathbf{u}) \cos\bigl[F^{\Phi}(\mathbf{u},\mathbf{x})\bigr]\, d\mathbf{u},$$

which is the left-hand equality in (3.17). Now $F^{\Phi}(\mathbf{u})$ is a.e. odd-symmetric (since F is real), and so $F^{\Phi}(\mathbf{u},\mathbf{x})$ is a.e. odd-symmetric in $\mathbf{u}$ by (3.14); also $F^{\mathcal{A}}$ is a.e. even-symmetric, and we deduce that $F^{\mathcal{A}}(\mathbf{u}) \cos\bigl[F^{\Phi}(\mathbf{u},\mathbf{x})\bigr]$ is a.e. even-symmetric in $\mathbf{u}$. Hence for any unit vector $\mathbf{n}$ we have

$$\int_{\mathcal{P}_{\mathbf{n}}} F^{\mathcal{A}}(\mathbf{u}) \cos\bigl[F^{\Phi}(\mathbf{u},\mathbf{x})\bigr]\, d\mathbf{u} = \int_{\mathcal{N}_{\mathbf{n}}} F^{\mathcal{A}}(\mathbf{u}) \cos\bigl[F^{\Phi}(\mathbf{u},\mathbf{x})\bigr]\, d\mathbf{u},$$

and so the right-hand equality in (3.17) holds. ∎

Lemmas 3.1 and 3.2 give us an example where pointwise values of the function $F_u \equiv F$ can be given in terms of $\widehat{F}$ if the latter is integrable. We will continue in this way in order to show that a continuous function with constant zero Fourier phase at a point has an integrable Fourier transform and an absolute maximum at that point. This will be a consequence of the following two results:

Lemma 3.3. *Let $G \not\equiv 0$ be integrable and having real non-negative values. Then:*

- (a) *For $\mathbf{t} \neq \mathbf{0}$, $|\widehat{G}(\mathbf{t})| < \widehat{G}(\mathbf{0})$.*
- (b) *Given H such that $|H(\mathbf{x})| \leq G(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{E}$, then for $\mathbf{t} \neq \mathbf{0}$, either $|\widehat{H}(\mathbf{t})| < \widehat{G}(\mathbf{0})$, or there is some φ such that $H \equiv e^{i\varphi}\mathrm{cis}_{\mathbf{t}} \cdot G$, that is $\widehat{H} = e^{i\varphi}\tau_{\mathbf{t}}(\widehat{G})$.*

Proof. For $\mathbf{t} \neq \mathbf{0}$ we have

$$|\widehat{H}(\mathbf{t})| = \Bigl|\int_{\mathcal{E}} \mathrm{cis}_{-\mathbf{t}} \cdot H\Bigr| \leq \int_{\mathcal{E}} |\mathrm{cis}_{-\mathbf{t}} \cdot H| = \int_{\mathcal{E}} |H| \leq \int_{E} G = \widehat{G}(\mathbf{0}).$$

Thus $|\widehat{H}(\mathbf{t})| \leq \widehat{G}(\mathbf{0})$, and the equality holds if and only if $|H| \equiv G$ and $\mathrm{cis}_{-\mathbf{t}} \cdot H$ has constant complex argument a.e. on $\mathcal{E}$, in other words $\mathrm{cis}_{-\mathbf{t}} \cdot H \equiv e^{i\varphi}G$, that is $H \equiv e^{i\varphi}\mathrm{cis}_{\mathbf{t}} \cdot G$. By the uniqueness property in L^1 and (3.12), the latter is equivalent to $\widehat{H} = [e^{i\varphi}\mathrm{cis}_{\mathbf{t}} \cdot G]^{\wedge} = e^{i\varphi}\tau_{\mathbf{t}}(\widehat{G})$. Thus ($b$) holds.

Taking $H = G$, as $e^{i\varphi}\mathrm{cis}_{\mathbf{t}}(\mathbf{x}) \neq 1$ almost everywhere, the equality $G \equiv e^{i\varphi}\mathrm{cis}_{\mathbf{t}} \cdot G$ implies that $G \equiv 0$. Thus $G \not\equiv 0$ gives $|\widehat{G}(\mathbf{t})| < \widehat{G}(\mathbf{0})$, and ($a$) holds. ∎

Proposition 3.4. *Let F be a function such that F is a.e. bounded in a neighbourhood of the origin and $\widehat{F}$ has real non-negative values. Then $\widehat{F}$ is integrable.*

Proof. For $r > 0$, let V_r be the set of all $\mathbf{x} \in \mathcal{E}$ having $|\mathbf{x}| < r$. There are $m, r > 0$ such that for almost all $\mathbf{x} \in V_r$ we have $|F(\mathbf{x})| < m$. Let the function G be given by $G(\mathbf{x}) = \exp[-\pi\,|\mathbf{x}|^2]$; for every integer n, define the two functions H_n and K_n as follows:

$$\begin{aligned} H_n(\mathbf{x}) &= G(n^{-1}\,\mathbf{x}) = \exp\bigl[-\pi\,|\mathbf{x}|^2/n^2\bigr], \\ K_n(\mathbf{x}) &= n^d G(n\,\mathbf{x}) = n^d \exp\bigl[-\pi n^2\,|\mathbf{x}|^2\bigr]. \end{aligned}$$

It is well-known (see [60]) that $G = \widehat{G}$, and so that $K_n = \widehat{H}_n$ and $H_n = \widehat{K}_n$. As H_n, K_n are Schwartz functions, the multiplication formula implies that FK_n and $\widehat{F}H_n$ are integrable, and

$$\int_{\mathcal{E}} FK_n = \int_{\mathcal{E}} F\widehat{H}_n = \int_{\mathcal{E}} \widehat{F}H_n. \tag{3.18}$$

As $|F(\mathbf{x})| < m$ a.e. in V_r, we get

$$\Big|\int_{V_r} FK_n\Big| \leq \int_{V_r} |FK_n| \leq m \cdot \int_{V_r} |K_n| \leq m \cdot \int_{\mathcal{E}} |K_n| = m \cdot \int_{\mathcal{E}} K_n = m \cdot \widehat{K}_n(\mathbf{0}) = m \cdot H_n(\mathbf{0}) = m.$$

Take an integer n_0 such that $2\pi r^2 n_0^2 > d$; then the calculation of $\partial K_n/\partial n$ shows that for $n \geq n_0$ and $|\mathbf{x}| \geq r$ we have $0 \leq K_n(\mathbf{x}) \leq K_{n_0}(\mathbf{x})$. Hence:

$$\Big|\int_{\mathcal{E}\setminus V_r} FK_n\Big| \leq \int_{\mathcal{E}\setminus V_r} |FK_n| \leq \int_{\mathcal{E}\setminus V_r} |F|K_{n_0} \leq \int_{\mathcal{E}} |F|K_{n_0}.$$

Combining both inequations above, for $n \geq n_0$ we have

$$\Big|\int_{\mathcal{E}} FK_n\Big| \leq m + \int_{\mathcal{E}} |F|K_{n_0}. \tag{3.19}$$

Hence $\int_{\mathcal{E}} FK_n$ remains bounded for $n \to \infty$. The functions H_n are positive and increase with n. As $\widehat{F}$ has non-negative real values, the functions $\widehat{F}H_n$ are non-negative and increase with n; moreover for $n \to \infty$, $H_n(\mathbf{x}) \to 1$, and so $\widehat{F}H_n \to \widehat{F}$. Hence by the Lebesgue monotone convergence theorem, (3.18), and (3.19), we obtain

$$\int_{\mathcal{E}} \widehat{F} = \int_{\mathcal{E}} \lim_{n\to\infty} \widehat{F}H_n = \lim_{n\to\infty} \int_{\mathcal{E}} \widehat{F}H_n = \lim_{n\to\infty} \int_{\mathcal{E}} FK_n < \infty,$$

that is $\widehat{F}$ is integrable. ∎

Note that the requirement that F is a.e. bounded in a neighbourhood of the origin is necessary (in fact F_u is bounded everywhere by the Riemann-Lebesgue theorem). We can now state the main result of this subsection:

PROPOSITION 3.5. *Let F be a function and let $\mathbf{x} \in \mathcal{E}$ such that F is a.e. bounded in a neighbourhood of $\mathbf{x}$ and for every $\mathbf{u} \in \mathcal{E}$ we have $F^{\Phi}(\mathbf{u}, \mathbf{x}) = 0$. Then $\widehat{F}$ is integrable and:*

- (a) *For $\mathbf{y} \neq \mathbf{x}$, $|F_u(\mathbf{y})| < F_u(\mathbf{x})$.*
- (b) *Given a function K such that $|\widehat{K}| \leq |\widehat{F}|$, then for $\mathbf{y} \neq \mathbf{x}$, either $|K_u(\mathbf{y})| < F_u(\mathbf{x})$, or there is some φ such that $K_u = e^{i\varphi}\tau_{\mathbf{y}-\mathbf{x}}(F_u)$.*

PROOF. By definition of local Fourier phase, for every $\mathbf{u} \in \mathcal{E}$ we have $\big[\tau_{-\mathbf{x}}(F)\big]^{\Phi}(\mathbf{u}) = 0$, in other words $\big[\tau_{-\mathbf{x}}(F)\big]^{\wedge}$ has real non-negative values. Moreover $\tau_{-\mathbf{x}}(F)$ is a.e. bounded in a neighbourhood of the origin, and we deduce from Proposition 3.4 that $\big[\tau_{-\mathbf{x}}(F)\big]^{\wedge}$ is integrable; as $\Big|\big[\tau_{-\mathbf{x}}(F)\big]^{\wedge}\Big| = |\widehat{F}|$ by (3.11), $\widehat{F}$ is integrable. Let $G = \big[\tau_{-\mathbf{x}}(F)\big]^{\wedge}$. By Lemma 3.1, for every $\mathbf{y} \in \mathcal{E}$,

$$F_u(\mathbf{y}) = \tau_{-\mathbf{x}}(F_u)(\mathbf{y} - \mathbf{x}) = [\tau_{-\mathbf{x}}(F)]_u(\mathbf{y} - \mathbf{x}) = G^{\vee}(\mathbf{y} - \mathbf{x}) = \widehat{G}(\mathbf{x} - \mathbf{y}),$$

and in particular $F_u(\mathbf{x}) = \widehat{G}(\mathbf{0})$; hence by Lemma 3.3 (a), for $\mathbf{y} \neq \mathbf{x}$ we have

$$|F_u(\mathbf{y})| = |\widehat{G}(\mathbf{x} - \mathbf{y})| < \widehat{G}(\mathbf{0}) = F_u(\mathbf{x});$$

therefore (a) holds. Let K be such that $|\widehat{K}| \leq |\widehat{F}|$; set $H = \left[\tau_{-\mathbf{x}}(K)\right]^{\wedge}$, and we have then $|H| \leq G$. As above, Lemma 3.1 gives for every $\mathbf{y} \in \mathcal{E}$: $K_u(\mathbf{y}) = \widehat{H}(\mathbf{x}-\mathbf{y})$. Then by Lemma 3.3 (b) for $\mathbf{y} \neq \mathbf{x}$ we have either

$$K_u(\mathbf{y}) = \widehat{H}(\mathbf{x}-\mathbf{y}) < \widehat{G}(\mathbf{0}) = F_u(\mathbf{x}),$$

or $\widehat{H} = e^{i\varphi}\tau_{\mathbf{x}-\mathbf{y}}(\widehat{G})$, in other words every $\mathbf{z} \in \mathcal{E}$ gives

$$K_u(\mathbf{z}) = \widehat{H}(\mathbf{x}-\mathbf{z}) = e^{i\varphi}\tau_{\mathbf{x}-\mathbf{y}}(\widehat{G})(\mathbf{x}-\mathbf{z}) = e^{i\varphi}\widehat{G}(\mathbf{y}-\mathbf{z}) = e^{i\varphi}F_u(\mathbf{x}-\mathbf{y}+\mathbf{z}),$$

that is $K_u = e^{i\varphi}\tau_{\mathbf{y}-\mathbf{x}}(F_u)$, and (b) holds. ∎

As a continuous function F is bounded in a neighbourhood of any point and satisfies $F = F_u$, we deduce:

Corollary 3.6. *Let F be a continuous function and let $\mathbf{x} \in \mathcal{E}$ such that for every $\mathbf{u} \in \mathcal{E}$ we have $F^{\Phi}(\mathbf{u}, \mathbf{x}) = 0$. Then for $\mathbf{y} \neq \mathbf{x}$, $|F(\mathbf{y})| < F(\mathbf{x})$.*

3.4. *The Hilbert transform and phase quadrature*

The Hilbert transform is defined for functions $\mathbb{R} \to \mathbb{C}$, and it plays an important role in the one-dimensional phase congruence model. In the earliest version of the model proposed by Morrone and Owens [38], the energy function was defined as the sum of squares of the one-dimensional signal and its Hilbert transform. In the second version proposed afterwards by Morrone and Burr [37], this energy function was defined as the sum of squares of the convolutions of the one-dimensional signal with two filters forming a Hilbert transform pair. In [51] we have highlighted the hidden mathematical asumptions underlying both versions of the phase congruence model. As we will deal with oriented edges in two-dimensional images, a muldimensional model of oriented energy functions must be built. Hence we will extend the Hilbert transform in an anisotropic way to functions $\mathbb{R}^d \to \mathbb{C}$; this will be the *directional Hilbert transform*, which will be defined w.r.t. a unit vector $\mathbf{n}$.

Two functions $f, g : \mathbb{R} \to \mathbb{C}$ are said to be *in quadrature*, or to form a *quadrature pair* if we have $\widehat{g}(\nu) = -i\,\mathrm{sgn}(\nu)\widehat{f}(\nu)$ for almost all $\nu \in \mathbb{R}$; when f and g are real-valued, this means that they have the same Fourier amplitude, but that for positive frequencies the phase of g is shifted by $-\pi/2$ (equivalently, for negative frequencies the phase of g is shifted by $\pi/2$). Note that the order of f and g in this relation does not really matter, since the ordered quadrature pair (f, g) implies the ordered quadrature pair $(g, -f)$, and that f and g are generally squared in an energy function.

Let $f : \mathbb{R} \to \mathbb{C}$ be in L^p, where $1 \leq p < \infty$; its *Hilbert transform* $\mathcal{H}[f]$ can be defined in two ways:

$$\begin{aligned} \mathcal{H}[f](x) &= \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|t| \geq \varepsilon} \frac{f(x-t)}{t}\,dt, \\ &= \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{\mathbb{R}} f(x-t)\frac{t}{t^2+\varepsilon^2}\,dt, \end{aligned} \qquad (\varepsilon > 0). \tag{3.20}$$

In fact (see [60], p. 218, or [57], p. 255), the two expressions coincide on the so-called *Lebesgue set* of f; thus they are equal almost everywhere, and in particular on all points at which f is continuous. The Hilbert transform is linear, translation-invariant ($g(x) = f(x-h)$ implies $\mathcal{H}[g](x) = \mathcal{H}[f](x-h)$), scale-invariant (for $a > 0$, $g(x) = f(x/a)$ implies $\mathcal{H}[g](x) = \mathcal{H}[f](x/a)$), and antisymmetric ($\mathcal{H}[f_\rho] = -\mathcal{H}[f]_\rho$).

A result due to M. Riesz (see [57], p. 287, or [60], p. 188) states that for $1 < p < \infty$, there is some constant A_p such that for every f in L^p, $\mathcal{H}[f]$ is also in L^p, with $\|\mathcal{H}[f]\|_p \leq A_p\,\|f\|_p$. For $p = 2$ we have the following well-known property (see [57], p. 257):

— **Quadrature formula in $\mathbf{L^2}$:** *For any f in L^2, $\mathcal{H}[f]^\wedge(\nu) = -i\,\mathrm{sgn}(\nu)\widehat{f}(\nu)$ for almost all $\nu \in \mathbb{R}$.*

Thus Hilbert transform pairs of functions in L^2 are in quadrature; we have then $\|\mathcal{H}[f]\|_2 = \|f\|_2$ by the Plancherel theorem. We get also $\mathcal{H}\big[\mathcal{H}[f]\big] \equiv -f$, since both have the same Fourier transform; this is called the *skewed symmetry* of the Hilbert transform.

Let us now consider functions defined on the space $\mathcal{E} = \mathbb{R}^d$, where $d > 1$. We will show how to extend the notions of quadrature and Hilbert transform defined for $d = 1$ to the multidimensional case. This generalization will be directional, in the sense that the multidimensional quadrature and Hilbert transform will be defined w.r.t. a particular direction.

Let $\mathbf{n}$ be any unit vector. Two functions $F, G : \mathcal{E} \to \mathbb{C}$ are said to be *in $\mathbf{n}$-quadrature*, or to form an *$\mathbf{n}$-quadrature pair* if we have $\widehat{G}(\mathbf{u}) = -i\,\mathrm{sgn}_\mathbf{n}(\mathbf{u})\widehat{F}(\mathbf{u})$ for almost all $\mathbf{u} \in \mathcal{E}$; when F and G are real-valued, this means that they have the same Fourier amplitude, but that the phase of G is shifted by $-\pi/2$ for frequencies in $\mathcal{P}_\mathbf{n}$ (or equivalently by $\pi/2$ for frequencies in $\mathcal{N}_\mathbf{n}$). Note that the following four equalities are equivalent:

$$\begin{aligned}
\widehat{G}(\mathbf{u}) &= -i\,\mathrm{sgn}_\mathbf{n}(\mathbf{u})\widehat{F}(\mathbf{u});\\
\widehat{F}(\mathbf{u}) &= i\,\mathrm{sgn}_\mathbf{n}(\mathbf{u})\widehat{G}(\mathbf{u});\\
\widehat{G}(\mathbf{u}) &= i\,\mathrm{sgn}_{-\mathbf{n}}(\mathbf{u})\widehat{F}(\mathbf{u});\\
\widehat{F}(\mathbf{u}) &= -i\,\mathrm{sgn}_{-\mathbf{n}}(\mathbf{u})\widehat{G}(\mathbf{u}).
\end{aligned}$$

Hence the order of F and G in such a relation does not really matter; anyway F and G will be squared in the energy function. Recall that $\mathcal{E} = \mathcal{E}_\mathbf{n} \times \mathcal{L}_\mathbf{n}$, with $\mathcal{L}_\mathbf{n}$ being identified with $\mathbb{R}$, so that every $\mathbf{x} \in \mathcal{E}$ can be written in a unique way as the pair $(\mathbf{y}, t) \in \mathcal{E}_\mathbf{n} \times \mathbb{R}$; then the $\mathbf{n}$-quadrature formula becomes: $\widehat{G}(\mathbf{y}, t) = -i\,\mathrm{sgn}(t)\widehat{F}(\mathbf{y}, t)$ for almost all $(\mathbf{y}, t) \in \mathcal{E}_\mathbf{n} \times \mathbb{R}$.

In order to give the properties of $\mathbf{n}$-quadrature, we must adapt to the present situation a general result concerning multidimensional Fourier analysis. Recall from the beginning of this section that given a function $F : \mathcal{E} \to \mathbb{C}$, every $\mathbf{y} \in \mathcal{E}_\mathbf{n}$ gives the $\mathcal{E}_\mathbf{n}$-section $F_\mathbf{y} : \mathbb{R} \to \mathbb{C} : t \mapsto F(\mathbf{y}, t)$, while every $t \in \mathbb{R}$ gives the $\mathcal{L}_\mathbf{n}$-section $F^t : \mathcal{E}_\mathbf{n} \to \mathbb{C} : \mathbf{y} \mapsto F(\mathbf{y}, t)$. We define the *Fourier transform inside $\mathcal{E}_\mathbf{n}$* as the transformation $\mathcal{F}^\mathbf{n}$ of functions $F : \mathcal{E} \to \mathbb{C}$ which applies to every $\mathcal{L}_\mathbf{n}$-section F^t the Fourier transform for functions $\mathcal{E}_\mathbf{n} \to \mathbb{C}$. More precisely, if $F \in L^1 + L^2$, then $F^t \in L^1 + L^2$ for almost every $t \in \mathbb{R}$, and we set $\big[\mathcal{F}^\mathbf{n}(F)\big]^t = \big[F^t\big]^\wedge$, in other words, the function $\mathcal{E}_\mathbf{n} \to \mathbb{C} : \mathbf{y} \mapsto \mathcal{F}^\mathbf{n}(F)(\mathbf{y}, t)$ is the Fourier transform of the function $\mathcal{E}_\mathbf{n} \to \mathbb{C} : \mathbf{y} \mapsto F(\mathbf{y}, t)$. For example if F is integrable, for every $\mathbf{v} \in \mathcal{E}_\mathbf{n}$ and $t \in \mathbb{R}$ we have

$$\mathcal{F}^\mathbf{n}(F)(\mathbf{v}, t) = \int_{\mathcal{E}_\mathbf{n}} F(\mathbf{y}, t)\, \exp[-2\pi i\, \mathbf{v} \cdot \mathbf{y}]\, d\mathbf{y}.$$

Note in particular that the function $F_{/\mathbf{n}}$ defined in (3.4) satisfies the identity $F_{/\mathbf{n}}(t) = \mathcal{F}^\mathbf{n}(F)(0, t)$.

We define similarly $\mathcal{F}_\mathbf{n}$, the *Fourier transform inside $\mathcal{L}_\mathbf{n}$*, as follows: for $F \in L^1 + L^2$, we have $\big[\mathcal{F}_\mathbf{n}(F)\big]_\mathbf{y} = \big[F_\mathbf{y}\big]^\wedge$ for almost all $\mathbf{y} \in \mathcal{E}_\mathbf{n}$; in particular when F is integrable, for every $\mathbf{y} \in \mathcal{E}_\mathbf{n}$ and $\nu \in \mathbb{R}$ we have

$$\mathcal{F}_\mathbf{n}(F)(\mathbf{y}, \nu) = \int_{\mathcal{E}_\mathbf{n}} F(\mathbf{y}, t)\, \exp[-2\pi i\, \nu t]\, dt.$$

It is easily verified that for every Schwartz function S the multiplication formula $\int_\mathcal{E} \mathcal{F}^\mathbf{n}(F)S = \int_\mathcal{E} F\mathcal{F}^\mathbf{n}(S)$ is verified. Thus we can extend the definition of $\mathcal{F}^\mathbf{n}$ to tempered distribution by setting

$\langle \mathcal{F}^{\mathbf{n}}(\psi), S\rangle = \langle \psi, \mathcal{F}^{\mathbf{n}}(S)\rangle$ for every tempered distribution ψ and Schwartz functions S. Note that the two definitions of $\mathcal{F}^{\mathbf{n}}$ coincide a.e.: given a tempered function F such that F^t is bitempered for almost every $t \in \mathbb{R}$, we have $\big[\mathcal{F}^{\mathbf{n}}(F)\big]^t \equiv \big[F^t\big]^\wedge$ for almost every $t \in \mathbb{R}$. Similarly we set $\langle \mathcal{F}_{\mathbf{n}}(\psi), S\rangle = \langle \psi, \mathcal{F}_{\mathbf{n}}(S)\rangle$ for every Schwartz functions S and tempered distribution ψ.

A fundamental property of multidimensional Fourier analysis, the *decomposability* of the Fourier transform, states that a multidimensional Fourier transform on a product space can be decomposed into a sequence of Fourier transforms on each of the subspaces in the product. More specifically, given a unit vector $\mathbf{n}$, every function F satisfies

$$\widehat{F} \equiv \mathcal{F}^{\mathbf{n}}(\mathcal{F}_{\mathbf{n}}(F)) \equiv \mathcal{F}_{\mathbf{n}}(\mathcal{F}^{\mathbf{n}}(F)); \tag{3.21}$$

in particular, if $F(\mathbf{y}, t) = F_1(\mathbf{y})F_2(t)$ for two functions $F_1 : \mathcal{E}_{\mathbf{n}} \to \mathbb{C}$ and $F_2 : \mathbb{R} \to \mathbb{C}$, then $\widehat{F}(\mathbf{v}, \nu) = \widehat{F}_1(\mathbf{v})\widehat{F}_2(\nu)$ a.e. For F integrable, (3.21) follows from a straightforward application of Fubini's theorem; for F square-integrable, it follows from the fact $\mathcal{F}^{\mathbf{n}} \circ \mathcal{F}_{\mathbf{n}}$, $\mathcal{F}_{\mathbf{n}} \circ \mathcal{F}^{\mathbf{n}}$, and the Fourier transform are isometries of $L^2/\!\equiv$ coinciding on the dense set $(L^1 \cap L^2)/\!\equiv$; for other functions (or tempered distributions), (3.21) is obtained through Schwartz functions.

Let us now return to directional quadrature. We assume temporarily that all functions are in $L^1 + L^2$. The following result links $\mathbf{n}$-quadrature to quadrature for functions on $\mathbb{R}$:

Lemma 3.7. *Given $F, G : \mathcal{E} \to \mathbb{C}$, F and G are in $\mathbf{n}$-quadrature if and only if for almost all $\mathbf{y} \in \mathcal{E}_{\mathbf{n}}$, $F_{\mathbf{y}}$ and $G_{\mathbf{y}}$ are in quadrature. In particular for every function $H : \mathcal{E}_{\mathbf{n}} \to \mathbb{C}$:*

- (*i*) *For $f, g : \mathbb{R} \to \mathbb{C}$ in quadrature, the functions $Hf : (\mathbf{y}, t) \mapsto H(\mathbf{y})f(t)$ and $Hg : (\mathbf{y}, t) \mapsto H(\mathbf{y})g(t)$ are in $\mathbf{n}$-quadrature.*
- (*ii*) *For $F, G : \mathcal{E} \to \mathbb{C}$ in $\mathbf{n}$-quadrature, the functions $HF : (\mathbf{y}, t) \mapsto H(\mathbf{y})F(\mathbf{y}, t)$ and $HG : (\mathbf{y}, t) \mapsto H(\mathbf{y})G(\mathbf{y}, t)$ are in $\mathbf{n}$-quadrature.*

Proof. Let $\widetilde{F} = \mathcal{F}_{\mathbf{n}}(F)$ and $\widetilde{G} = \mathcal{F}_{\mathbf{n}}(G)$, the Fourier transforms inside $\mathcal{L}_{\mathbf{n}}$ of F and G respectively, in other words $\widetilde{F}_{\mathbf{y}} = \big[F_{\mathbf{y}}\big]^\wedge$ and $\widetilde{G}_{\mathbf{y}} = \big[G_{\mathbf{y}}\big]^\wedge$ for almost all $\mathbf{y} \in \mathcal{E}_{\mathbf{n}}$. By the decomposability property of the Fourier transform, $\widehat{F} \equiv \mathcal{F}^{\mathbf{n}}(\widetilde{F})$ and $\widehat{G} \equiv \mathcal{F}^{\mathbf{n}}(\widetilde{G})$, that is $\widehat{F}^t \equiv \big[\widetilde{F}^t\big]^\wedge$ and $\widehat{G}^t \equiv \big[\widetilde{G}^t\big]^\wedge$ for almost every $t \in \mathbb{R}$. Hence the fact that (F, G) is an $\mathbf{n}$-quadrature pair:

- for almost all $(\mathbf{y}, t) \in \mathcal{E}_{\mathbf{n}} \times \mathbb{R}$, $\widehat{G}(\mathbf{y}, t) = -i\,\mathrm{sgn}(t)\widehat{F}(\mathbf{y}, t)$,

is equivalent to:

- for almost all $t \in \mathbb{R}$, $\big[\widetilde{G}^t\big]^\wedge \equiv \widehat{G}^t \equiv -i\,\mathrm{sgn}(t)\widehat{F}^t \equiv -i\,\mathrm{sgn}(t)\big[\widetilde{F}^t\big]^\wedge$.

Now since $\big[\widetilde{F}^t\big]^\wedge$ is the Fourier transform of $\widetilde{F}^t$ for variables in $\mathcal{E}_{\mathbf{n}}$, w.r.t. which $\mathrm{sgn}(t)$ is a constant, we have $-i\,\mathrm{sgn}(t)\big[\widetilde{F}^t\big]^\wedge = \big[-i\,\mathrm{sgn}(t)\widetilde{F}^t\big]^\wedge$; thus the above statement is equivalent to:

- for almost all $t \in \mathbb{R}$, $\big[\widetilde{G}^t\big]^\wedge \equiv \big[-i\,\mathrm{sgn}(t)\widetilde{F}^t\big]^\wedge$.

By the uniqueness property of the Fourier transform, this is equivalent to:

- for almost all $t \in \mathbb{R}$, $\widetilde{G}^t \equiv -i\,\mathrm{sgn}(t)\widetilde{F}^t$,

that is,

- for almost all $(\mathbf{y}, t) \in \mathcal{E}_{\mathbf{n}} \times \mathbb{R}$, $\widetilde{G}(\mathbf{y}, t) = -i\,\mathrm{sgn}(t)\widetilde{F}(\mathbf{y}, t)$,

in other words to:

- for almost all $\mathbf{y} \in \mathcal{E}_{\mathbf{n}}$, we have $\widetilde{G}_{\mathbf{y}}(t) = -i\,\mathrm{sgn}(t)\widetilde{F}_{\mathbf{y}}(t)$ for almost all $t \in \mathbb{R}$,

and by definition of $\widetilde{F}$ and $\widetilde{G}$, this is:

- for almost all $\mathbf{y} \in \mathcal{E}_{\mathbf{n}}$, we have $\big[G_{\mathbf{y}}\big]^\wedge(t) = -i\,\mathrm{sgn}(t)\big[F_{\mathbf{y}}\big]^\wedge(t)$ for almost all $t \in \mathbb{R}$, which means that $(F_{\mathbf{y}}, G_{\mathbf{y}})$ is a quadrature pair. Hence F and G are in $\mathbf{n}$-quadrature if and only if for almost all $\mathbf{y} \in \mathcal{E}_{\mathbf{n}}$, $F_{\mathbf{y}}$ and $G_{\mathbf{y}}$ are in quadrature.

Take $H : \mathcal{E}_{\mathbf{n}} \to \mathbb{C}$. Given $f, g : \mathbb{R} \to \mathbb{C}$ in quadrature, for every $\mathbf{y} \in \mathcal{E}_{\mathbf{n}}$ we have $[Hf]_{\mathbf{y}} = H(\mathbf{y})f$ and $[Hg]_{\mathbf{y}} = H(\mathbf{y})g$; as $H(\mathbf{y})$ is a constant w.r.t. the variable t of f and g, the fact that f and g are in quadrature implies that $[Hf]_{\mathbf{y}}$ and $[Hg]_{\mathbf{y}}$ are in quadrature for all $\mathbf{y} \in \mathcal{E}_{\mathbf{n}}$, so that Hf and Hg will be in $\mathbf{n}$-quadrature. Given $F, G : \mathcal{E} \to \mathbb{C}$ in $\mathbf{n}$-quadrature, for almost all $\mathbf{y} \in \mathcal{E}_{\mathbf{n}}$, $F_{\mathbf{y}}$ and $G_{\mathbf{y}}$ are in quadrature; now $[HF]_{\mathbf{y}} = H(\mathbf{y})F_{\mathbf{y}}$ and $[Hg]_{\mathbf{y}} = H(\mathbf{y})G_{\mathbf{y}}$; as $H(\mathbf{y})$ is a constant w.r.t. the variable t of $F_{\mathbf{y}}$ and $G_{\mathbf{y}}$, $[HF]_{\mathbf{y}}$ and $[Hg]_{\mathbf{y}}$ will be in quadrature for almost all $\mathbf{y} \in \mathcal{E}_{\mathbf{n}}$, so that Hf and Hg will be in $\mathbf{n}$-quadrature. ∎

In Proposition 3 of [51] we showed that for two integrable functions $f, g : \mathbb{R} \to \mathbb{C}$ in quadrature, $\int_{\mathbb{R}} f = \int_{\mathbb{R}} g = 0$. The following is a multidimensional generalization of this result:

Proposition 3.8. *Let F, G be integrable functions forming an $\mathbf{n}$-quadrature pair. Then for almost all $\mathbf{y} \in \mathcal{E}_{\mathbf{n}}$ we have $\int_{\mathbb{R}} F_{\mathbf{y}} = \int_{\mathbb{R}} G_{\mathbf{y}} = 0$; in particular $\int_{\mathcal{E}} F = \int_{\mathcal{E}} G = 0$.*

Proof. Since F and G are integrable and in $\mathbf{n}$-quadrature, for almost all $\mathbf{y} \in \mathcal{E}_{\mathbf{n}}$, $F_{\mathbf{y}}$ and $G_{\mathbf{y}}$ are integrable (by Fubini's theorem) and in quadrature (by Lemma 3.7). For such a $\mathbf{y}$ we have $\left[G_{\mathbf{y}}\right]^{\wedge}(\nu) = -i \left[F_{\mathbf{y}}\right]^{\wedge}(\nu)$ for almost all $\nu > 0$ and $\left[G_{\mathbf{y}}\right]^{\wedge}(\nu) = i \left[F_{\mathbf{y}}\right]^{\wedge}(\nu)$ for almost all $\nu < 0$. By the Riemann-Lebesgue theorem, $\left[F_{\mathbf{y}}\right]^{\wedge}$ and $\left[G_{\mathbf{y}}\right]^{\wedge}$ are continuous; the continuity at 0 gives then both $\left[G_{\mathbf{y}}\right]^{\wedge}(0) = i \left[F_{\mathbf{y}}\right]^{\wedge}(0)$ and $\left[G_{\mathbf{y}}\right]^{\wedge}(0) = -i \left[F_{\mathbf{y}}\right]^{\wedge}(0)$, in other words $\int_{\mathbb{R}} G_{\mathbf{y}} = \left[G_{\mathbf{y}}\right]^{\wedge}(0) = 0 = \left[F_{\mathbf{y}}\right]^{\wedge}(0) = \int_{\mathbb{R}} F_{\mathbf{y}}$. By Fubini's theorem we have $\int_{\mathcal{E}} F = \int_{\mathcal{E}_{\mathbf{n}}} d\mathbf{y} \int_{\mathbb{R}} F_{\mathbf{y}} = \int_{\mathcal{E}_{\mathbf{n}}} d\mathbf{y}\, 0 = 0$ and similarly $\int_{\mathcal{E}} G = 0$. ∎

In electrical engineering parlance, "F and G have zero dc level", in other words the convolution of F or G with a constant function will be identically zero.

We introduce now some further notation. Let ξ be the identity function $x \mapsto x$ on $\mathbb{R}$; for a function $f : \mathbb{R} \to \mathbb{C}$, we will consider the function $\xi f : x \mapsto xf(x)$. Write x_m for the coordinate of the vector $\mathbf{x}$ in the direction of a unit vector $\mathbf{m}$ (thus $x_m = \mathbf{x} \cdot \mathbf{m}$). Let $\xi_{\mathbf{m}}$ be the function $\mathcal{E} \to \mathbb{R} : \mathbf{x} \mapsto x_m = \mathbf{x} \cdot \mathbf{m}$; for a function $F : \mathcal{E} \to \mathbb{C}$, we will consider the function $\xi_{\mathbf{m}} F : \mathbf{x} \mapsto x_m F(\mathbf{x})$. Write $D_{\mathbf{m}}$ for the operation of taking the derivative in the direction of $\mathbf{m}$, in other words $D_{\mathbf{m}}(F) = \partial F / \partial x_m$. The following property of the Fourier transform will be used:

— **$\mathbf{L^1}$ Fourier derivative formula:** *If F and $\xi_{\mathbf{m}} F$ are both integrable, then $\widehat{F}$ is derivable in x_m, and $D_{\mathbf{m}}(\widehat{F}) = -2\pi i \, [\xi_{\mathbf{m}} F]^{\wedge}$.*

In Proposition 4 of [51] we showed that given two integrable functions $f, g : \mathbb{R} \to \mathbb{C}$ in quadrature, and such that the two functions ξf and ξg are integrable, then the latter are in quadrature. The following is a multidimensional generalization of this result:

Proposition 3.9. *Given two (not necessarily distinct) unit vectors $\mathbf{m}$ and $\mathbf{n}$, let F, G be integrable functions forming an $\mathbf{n}$-quadrature pair, and such that $\xi_{\mathbf{m}} F$ and $\xi_{\mathbf{m}} G$ are integrable. Then $\xi_{\mathbf{m}} F$ and $\xi_{\mathbf{m}} G$ are in $\mathbf{n}$-quadrature, and $\int_{\mathcal{E}} \xi_{\mathbf{m}} F = \int_{\mathcal{E}} \xi_{\mathbf{m}} G = 0$.*

Proof. By the L^1 Fourier derivative formula, we have $[\xi_{\mathbf{m}} F]^{\wedge} = (-2\pi i)^{-1} D_{\mathbf{m}}(\widehat{F})$ and $[\xi_{\mathbf{m}} G]^{\wedge} = (-2\pi i)^{-1} D_{\mathbf{m}}(\widehat{G})$. Since F and G are in $\mathbf{n}$-quadrature and $\widehat{F}$ and $\widehat{G}$ are continuous (by the Riemann-Lebesgue theorem), we have $\widehat{G}(\mathbf{u}) = -i \widehat{F}(\mathbf{u})$ for all $\mathbf{u} \in \mathcal{P}_{\mathbf{n}}$ and $\widehat{G}(\mathbf{u}) = i \widehat{F}(\mathbf{u})$ for all $\mathbf{u} \in \mathcal{N}_{\mathbf{n}}$; it follows then that $D_{\mathbf{m}}(\widehat{G})(\mathbf{u}) = -i \, D_{\mathbf{m}}(\widehat{F})(\mathbf{u})$ for all $\mathbf{u} \in \mathcal{P}_{\mathbf{n}}$ and $D_{\mathbf{m}}(\widehat{G})(\mathbf{u}) = i \, D_{\mathbf{m}}(\widehat{F})(\mathbf{u})$ for all

$\mathbf{u} \in \mathcal{N}_{\mathbf{n}}$. Hence for all $\mathbf{u} \notin \mathcal{E}_{\mathbf{n}}$ we have

$$[\xi_{\mathbf{m}} G]^{\wedge}(\mathbf{u}) = (-2\pi i)^{-1} D_{\mathbf{m}}(\widehat{G})(\mathbf{u}) = -i \operatorname{sgn}_{\mathbf{n}}(\mathbf{u})(-2\pi i)^{-1} D_{\mathbf{m}}(\widehat{F})(\mathbf{u}) = -i \operatorname{sgn}_{\mathbf{n}}(\mathbf{u})[\xi_{\mathbf{m}} F]^{\wedge}(\mathbf{u}),$$

and so $\xi_{\mathbf{m}} F$ and $\xi_{\mathbf{m}} G$ are in $\mathbf{n}$-quadrature. It follows then from Proposition 3.8 that $\int_{\mathcal{E}} \xi_{\mathbf{m}} F = \int_{\mathcal{E}} \xi_{\mathbf{m}} G = 0$. ∎

Note that Propositions 3.8 and 3.9 are not necessarily true if we assume F and G square-integrable instead of integrable; this question is discussed in more detail in Subsection 3.4 of [52].

Let us now define a multidimensional generalization of the Hilbert transform which will produce pairs of functions in $\mathbf{n}$-quadrature. For $F : \mathcal{E} \to \mathbb{C}$, its *Hilbert transform in the direction* $\mathbf{n}$, or $\mathbf{n}$-*directional Hilbert transform*, is the function $\mathcal{H}_{\mathbf{n}}[F]$ defined by

$$\big(\mathcal{H}_{\mathbf{n}}[F]\big)_{\mathbf{y}} = \mathcal{H}\big[F_{\mathbf{y}}\big] \qquad \text{for all} \quad \mathbf{y} \in \mathcal{E}_{\mathbf{n}}. \tag{3.22}$$

When $F \in L^p$ (for $1 \leq p < \infty$), then (by Fubini's theorem) $F_{\mathbf{y}} \in L^p$ for almost all $\mathbf{y} \in \mathcal{E}_{\mathbf{n}}$, and for any such $\mathbf{y}$, $\mathcal{H}\big[F_{\mathbf{y}}\big]$ is defined; hence $\mathcal{H}_{\mathbf{n}}[F]$ is defined almost everywhere. By (3.20) and the definition of the $\mathcal{E}_{\mathbf{n}}$-sections, we have:

$$\begin{aligned}\mathcal{H}_{\mathbf{n}}[F](\mathbf{y}, x) &= \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|t| \geq \varepsilon} \frac{F(\mathbf{y}, x-t)}{t}\, dt, \\ &= \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{\mathbb{R}} F(\mathbf{y}, x-t) \frac{t}{t^2 + \varepsilon^2}\, dt,\end{aligned} \qquad (\varepsilon > 0), \tag{3.23}$$

or with $\mathbf{x} = (\mathbf{y}, x) = \mathbf{y} + x\mathbf{n}$ and $(\mathbf{y}, x-t) = \mathbf{x} - t\mathbf{n}$:

$$\begin{aligned}\mathcal{H}_{\mathbf{n}}[F](\mathbf{x}) &= \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|t| \geq \varepsilon} \frac{F(\mathbf{x} - t\mathbf{n})}{t}\, dt, \\ &= \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{\mathbb{R}} F(\mathbf{x} - t\mathbf{n}) \frac{t}{t^2 + \varepsilon^2}\, dt,\end{aligned} \qquad (\varepsilon > 0). \tag{3.24}$$

The two formulas given here coincide almost everywhere. This definition (3.24) can be found in [59], p. 49. As the one-dimensional Hilbert transform, this multidimentional directional Hilbert transform is linear, translation-invariant ($G(\mathbf{x}) = F(\mathbf{x} - \mathbf{h})$ implies $\mathcal{H}_{\mathbf{n}}[G](\mathbf{x}) = \mathcal{H}_{\mathbf{n}}[F](\mathbf{x} - \mathbf{h})$), scale-invariant (for $a > 0$, $G(\mathbf{x}) = F(a^{-1}\mathbf{x})$ implies $\mathcal{H}_{\mathbf{n}}[G](\mathbf{x}) = \mathcal{H}_{\mathbf{n}}[F](a^{-1}\mathbf{x})$), and antisymmetric ($\mathcal{H}_{\mathbf{n}}[F_\rho] = -\mathcal{H}_{\mathbf{n}}[F]_\rho$). The Hilbert transform quadrature formula and skewed symmetry in L^2 extend naturally to $\mathcal{H}_{\mathbf{n}}$:

LEMMA 3.10. *Given a square-integrable function $F : \mathcal{E} \to \mathbb{C}$, then $\mathcal{H}_{\mathbf{n}}[F]^{\wedge}(\mathbf{u}) = -i \operatorname{sgn}_{\mathbf{n}}(\mathbf{u}) \widehat{F}(\mathbf{u})$ for almost all $\mathbf{u} \in \mathbb{R}$, in other words F and $\mathcal{H}_{\mathbf{n}}[F]$ are in $\mathbf{n}$-quadrature. Moreover, $\|\mathcal{H}_{\mathbf{n}}[F]\|_2 = \|F\|_2$ and $\mathcal{H}_{\mathbf{n}}\big[\mathcal{H}_{\mathbf{n}}[F]\big] \equiv -F$.*

PROOF. By Fubini's theorem, $F_{\mathbf{y}} \in L^2$ for almost all $\mathbf{y} \in \mathcal{E}_{\mathbf{n}}$. For such a $\mathbf{y}$ we have by the Hilbert transform quadrature formula in L^2: $\big[\big(\mathcal{H}_{\mathbf{n}}[F]\big)_{\mathbf{y}}\big]^{\wedge}(t) = \mathcal{H}\big[F_{\mathbf{y}}\big]^{\wedge}(t) = -i \operatorname{sgn}(t) \cdot \big[F_{\mathbf{y}}\big]^{\wedge}(t)$ for almost all $t \in \mathbb{R}$, in other words $\big(\mathcal{H}_{\mathbf{n}}[F]\big)_{\mathbf{y}}$ and $F_{\mathbf{y}}$ are in quadrature. By Lemma 3.7, F and $\mathcal{H}_{\mathbf{n}}[F]$ are in $\mathbf{n}$-quadrature. It follows that $|\mathcal{H}_{\mathbf{n}}[F]^{\wedge}(\mathbf{x})| = |\widehat{F}(\mathbf{x})|$ almost everywhere, and so the Plancherel theorem gives $\|\mathcal{H}_{\mathbf{n}}[F]\|_2 = \|\mathcal{H}_{\mathbf{n}}[F]^{\wedge}\|_2 = \|\widehat{F}\|_2 = \|F\|_2$. Since $\mathcal{H}_{\mathbf{n}}[F] \in L^2$, $\mathcal{H}_{\mathbf{n}}[F]$ and $\mathcal{H}_{\mathbf{n}}\big[\mathcal{H}_{\mathbf{n}}[F]\big]$ are in $\mathbf{n}$-quadrature. Thus $\mathcal{H}_{\mathbf{n}}\big[\mathcal{H}_{\mathbf{n}}[F]\big]^{\wedge}(\mathbf{u}) = -i \operatorname{sgn}_{\mathbf{n}}(\mathbf{u}) \mathcal{H}_{\mathbf{n}}[F]^{\wedge}(\mathbf{u}) = (-i \operatorname{sgn}_{\mathbf{n}}(\mathbf{u}))^2 \widehat{F}(\mathbf{u})$ a.e., and so $\mathcal{H}_{\mathbf{n}}\big[\mathcal{H}_{\mathbf{n}}[F]\big] \equiv -F$. ∎

There is an analogue of items (i) and (i) of Lemma 3.7 for the $\mathbf{n}$-directional Hilbert transform. For every function $G : \mathcal{E}^0_{\mathbf{n}} \to \mathbb{C}$ we have:

(a) For $f : \mathbb{R} \to \mathbb{C}$ in L^p $(1 \leq p < \infty)$, the functions $Gf : (\mathbf{y}, t) \mapsto G(\mathbf{y})f(t)$ and $G\mathcal{H}[f] : (\mathbf{y}, t) \mapsto G(\mathbf{y})\mathcal{H}[f](t)$ verify $G\mathcal{H}[f] = \mathcal{H}_{\mathbf{n}}[Gf]$.

(b) For $F : \mathcal{E} \to \mathbb{C}$ in L^p $(1 \leq p < \infty)$, the functions $GF : (\mathbf{y}, t) \mapsto G(\mathbf{y})F(\mathbf{y}, t)$ and $G\mathcal{H}_{\mathbf{n}}[F] : (\mathbf{y}, t) \mapsto G(\mathbf{y})\mathcal{H}_{\mathbf{n}}[F](\mathbf{y}, t)$ verify $G\mathcal{H}_{\mathbf{n}}[F] = \mathcal{H}_{\mathbf{n}}[GF]$.

These facts are a straightforward consequence of the definition (3.22) of $\mathcal{H}_{\mathbf{n}}$. They can be useful for building $\mathbf{n}$-directional Hilbert transform pairs of functions $\mathcal{E} \to \mathbb{C}$ from Hilbert transform pairs of functions $\mathbb{R} \to \mathbb{C}$.

Let us give some examples of square-integrable Hilbert transform pairs; we do not justify here their properties, see [51,52] for more details. In [51] we considered the Gaussian

$$G_\sigma(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[\frac{-x^2}{2\sigma^2}\right], \tag{3.25}$$

whose Hilbert transform is the function

$$K_\sigma(x) = \frac{2}{\sqrt{\pi}}\, G_\sigma(x) \int_0^{x/\sigma\sqrt{2}} \exp(s^2)\, ds,$$

which is asymptotically in $1/\pi x$. The Hilbert transform of the n-th derivative $G_\sigma^{(n)}$ of G_σ is the n-th derivative $K_\sigma^{(n)}$ of K_σ, which is asymptotically proportional to $1/\xi^{n+1}$. The Hilbert transform a difference of Gaussians $G_{\sigma_1} - G_{\sigma_2}$ (where $\sigma_1 \neq \sigma_2$) is the difference $K_{\sigma_1} - K_{\sigma_2}$, which is asymptotically proportional to $1/\xi^3$. Thus functions involving the Hilbert transform K_σ of the Gaussian G_σ have a relatively slow decay, in addition to their computational complexity.

In [37], data obtained in psychophysical measurements of human visual response led to considering the function $f : \mathbb{R} \to \mathbb{R}$ given by its Fourier transform:

$$\widehat{f}(\nu) = \exp\Big(-\frac{[\ln(|\nu|/P)]^2}{2[q \ln 2]^2}\Big).$$

Here P is the peak frequency of f, and q its half bandwidth in octaves at height $\exp(-1/2)$. It can be shown that both $\widehat{f}$ and $\nu \mapsto \mathrm{sgn}(\nu)\widehat{f}(\nu)$ are Schwartz functions, so that h and $\mathcal{H}[f]$ are Schwartz functions.

The two functions

$$\begin{aligned} f_n(x) &= \Re\big\{(1 - ix)^{-n-1}\big\} = \frac{\Re\big\{(1 + ix)^{n+1}\big\}}{(1 + x^2)^{n+1}} \\ \text{and} \qquad g_n(x) &= \Im\big\{(1 - ix)^{-n-1}\big\} = \frac{\Im\big\{(1 + ix)^{n+1}\big\}}{(1 + x^2)^{n+1}}, \end{aligned} \tag{3.26}$$

where $n \geq 0$, were proposed in [23], where they were called *Cauchy functions*; we have

$$\widehat{f_n}(\nu) = (n!)^{-1}(2\pi)^{n+1}|\nu|^n \exp[-2\pi|\nu|].$$

The functions f_n and g_n are square-integrable and in quadrature, so that $g_n = \mathcal{H}[f_n]$; they are also integrable for $n \geq 1$. The constant n determines the number of ripples in the profiles of f_n and g_n. In [23] the values $n = 3$ and $n = 5$ were taken in order to define filters modeling some properties of the human visual response. The main interest of Cauchy functions resides in their computational simplicity.

In Subsection 3.4 of [52] we studied further properties of the one-dimensional Hilbert transform $\mathcal{H}$ and of the multidimensional directional Hilbert transform $\mathcal{H}_{\mathbf{n}}$; in particular we showed that for $1 < p < \infty$ the skewed symmetry is also valid for functions in L^p, while the quadrature formula holds for functions in $L^1 \cap L^p$. We recalled also the properties of these transforms for functions in L^1, and showed how to obtain Hilbert transform pairs in L^1.

Note that there is another multidimensional generalization of the Hilbert transform to d dimensions, namely the *M. Riesz transform* (see [60], p. 224); its definition and property in the Fourier domain for functions in L^2 were briefly recalled on pp. 25,26 of [52]. We do not use this transform, because it does not lead to a Fourier phase quadrature, since it modifies also the Fourier amplitude of a square-integrable function F.

3.5. The complex-valued function associated to a quadrature pair, and its energy

Given a square-integrable function $f : \mathbb{R} \to \mathbb{R}$, the complex-valued function $f + i\,\mathcal{H}[f]$ has some interesting mathematical properties, in particular the fact that it can be extended to a function of a complex variable which is analytic in the upper half of the complex plane. The use of this function in signal processing has been proposed by Gabor [14], who called it the *analytic signal* associated to f, and who introduced the term *energy* in order to designate the square of its absolue value, namely $f^2 + \mathcal{H}[f]^2 = \left|f + i\,\mathcal{H}[f]\right|^2$. Owens [44] used extensively this function $f + i\,\mathcal{H}[f]$ and its analytic extension in the upper half of the complex plane in order to describe the behaviour of the energy function $f^2 + \mathcal{H}[f]^2$ by means of complex analysis. We will consider here the complex-valued function $F + i\,G$ for an $\mathbf{n}$-quadrature pair of functions on $\mathcal{E}$.

Given two functions $F, G : \mathcal{E} \to \mathbb{C}$, the fact that F and G are in $\mathbf{n}$-quadrature is equivalent to each of the following identities:

$$\begin{aligned} [F + i\,G]^\wedge &\equiv 2\,\mathrm{pos}_{\mathbf{n}} \cdot \widehat{F}; \\ [F - i\,G]^\wedge &\equiv 2\,\mathrm{neg}_{\mathbf{n}} \cdot \widehat{F}. \end{aligned} \tag{3.27}$$

More precisely:

Lemma 3.11. *Consider the following four conditions:*

- (a) *For almost all* $\mathbf{u} \in \mathcal{P}_{\mathbf{n}}$, $[F + i\,G]^\wedge(\mathbf{u}) = 2\,\widehat{F}(\mathbf{u})$.
- (b) *For almost all* $\mathbf{u} \in \mathcal{P}_{\mathbf{n}}$, $[F - i\,G]^\wedge(\mathbf{u}) = 0$.
- (c) *For almost all* $\mathbf{u} \in \mathcal{N}_{\mathbf{n}}$, $[F + i\,G]^\wedge(\mathbf{u}) = 0$.
- (b) *For almost all* $\mathbf{u} \in \mathcal{N}_{\mathbf{n}}$, $[F - i\,G]^\wedge(\mathbf{u}) = 2\,\widehat{F}(\mathbf{u})$.

Then (a) *is equivalent to* (b), (c) *is equivalent to* (d), *and* (F, G) *is an* $\mathbf{n}$*-quadrature pair if and only if one of* (a) *or* (b), *and one of* (c) *or* (d), *hold. Furthermore, if* F *and* G *are real-valued, then each one of* (a, b, c, d) *is equivalent to the fact that* F *and* G *are in* $\mathbf{n}$*-quadrature.*

For example, (a) and (c) together mean the first identity in (3.27), while (b) and (d) together mean the second one. The first conclusion in this Lemma is straightforward; the second one follows from the fact that for F and G real-valued, (3.9) gives $\overline{[F \pm i\,G]^\wedge(-\mathbf{u})} = [F \mp i\,G]^\wedge(\mathbf{u})$, so that (b) is equivalent to (c).

We consider the space $\mathcal{F}^+_{\mathbf{n}}$ of all functions $H : \mathcal{E} \to \mathbb{C}$ such that $\widehat{H}(\mathbf{u}) = 0$ a.e. on $\mathcal{N}_{\mathbf{n}}$; we have similarly the space $\mathcal{F}^-_{\mathbf{n}} = \mathcal{F}^+_{-\mathbf{n}}$ of all functions $H : \mathcal{E} \to \mathbb{C}$ such that $\widehat{H}(\mathbf{u}) = 0$ a.e. on $\mathcal{P}_{\mathbf{n}}$. Note that by Lemma 3.11, $H \in \mathcal{F}^+_{\mathbf{n}}$ if and only if $\Re H$ and $\Im H$ are in **n**-quadrature, and $H \in \mathcal{F}^-_{\mathbf{n}}$ if and only if $\Re H$ and $-\Im H$ are in **n**-quadrature. Given $F, G : \mathcal{E} \to \mathbb{C}$, by Lemma 3.11 (see items (*b*) and (*c*)), F and G are in **n**-quadrature if and only if both $F + i\,G \in \mathcal{F}^+_{\mathbf{n}}$ and $F - i\,G \in \mathcal{F}^-_{\mathbf{n}}$; for F and G real-valued, each one of the two conditions $F + i\,G \in \mathcal{F}^+_{\mathbf{n}}$ and $F - i\,G \in \mathcal{F}^-_{\mathbf{n}}$ is equivalent to F and G being in **n**-quadrature. For an **n**-quadrature pair (F, G), we have $F + i\,G = (\Re F - \Im G) + i\,(\Re G + \Im F)$, so that the functions $F_1 = \Re(F + i\,G) = \Re F - \Im G$ and $G_1 = \Im(F + i\,G) = \Re G + \Im F$ are in **n**-quadrature. Note also that by (3.9) $\overline{F}$ and $\overline{G}$ are in **n**-quadrature, so that both $(\Re F, \Re G)$ and $(\Im F, \Im G)$ are **n**-quadrature pairs (from which we deduce that $\Re F - \Im G$ and $\Re G + \Im F$ are in **n**-quadrature). Thus **n**-quadrature pairs can be reduced to the case of real-valued functions, and we can write elements of $\mathcal{F}^+_{\mathbf{n}}$ under the form $F + i\,\widetilde{F}$ with $F, \widetilde{F}$ real-valued and in **n**-quadrature.

Clearly $\mathcal{F}^+_{\mathbf{n}}$ is a vector space on $\mathbb{C}$; in particular for any angle θ and $F, \widetilde{F}$ real-valued and in **n**-quadrature,

$$e^{i\,\theta}(F + i\,\widetilde{F}) = \big(\cos\theta\, F - \sin\theta\, \widetilde{F}\big) + i\,\big(\sin\theta\, F + \cos\theta\, \widetilde{F}\big)$$

belongs to $\mathcal{F}^+_{\mathbf{n}}$, so that $\cos\theta\, F - \sin\theta\, \widetilde{F}$ and $\sin\theta\, F + \cos\theta\, \widetilde{F}$ are in **n**-quadrature; moreover we have a.e.:

$$\big(\cos\theta\, F - \sin\theta\, \widetilde{F}\big)^{\wedge}(\mathbf{u}) = \exp[i\,\theta\, \mathrm{sgn}_{\mathbf{n}}(\mathbf{u})]\, \widehat{F}(\mathbf{u}).$$

Given two **n**-quadrature pairs $(F, \widetilde{F})$ and $(G, \widetilde{G})$ of square-integrable functions, $F + i\,\widetilde{F}$ and $G + i\,\widetilde{G}$ are square-integrable and in $\mathcal{F}^+_{\mathbf{n}}$; by the dual convolution formula and the fact that the convolution of two functions vanishing on $\mathcal{N}_{\mathbf{n}}$ vanishes on $\mathcal{N}_{\mathbf{n}}$, their product

$$(F + i\,\widetilde{F})(G + i\,\widetilde{G}) = (FG - \widetilde{F}\widetilde{G}) + i\,(F\widetilde{G} + \widetilde{F}G)$$

belongs to $\mathcal{F}^+_{\mathbf{n}}$; similarly

$$(F - i\,\widetilde{F})(G - i\,\widetilde{G}) = (FG - \widetilde{F}\widetilde{G}) - i\,(F\widetilde{G} + \widetilde{F}G)$$

belongs to $\mathcal{F}^-_{\mathbf{n}}$, so that $FG - \widetilde{F}\widetilde{G}$ and $F\widetilde{G} + \widetilde{F}G$ are in **n**-quadrature. Here $\widetilde{F} = \mathcal{H}_{\mathbf{n}}[F]$ and $\widetilde{G} = \mathcal{H}_{\mathbf{n}}[G]$. In the one-dimensional case where $\mathcal{E} = \mathbb{R}$ and the **n**-directional Hilbert transform reduces to the ordinary one, such a formula $FG - \widetilde{F}\widetilde{G}$ was used in [44] as a new type of product of F and G, and it was written $F \odot G$. The advantage of such a product $F \odot G$ is that the energy associated to it, namely $(F \odot G)^2 + (F\widetilde{G} + \widetilde{F}G)^2$, is the ordinary product of the energies $F^2 + \widetilde{F}^2$ and $G^2 + \widetilde{G}^2$ associated to F and G; for example if G has constant energy, the energy of $F \odot G$ is proportional to that of F.

Let us recall briefly how in the one-dimensional case such a complex-valued fonction extends to a function of a complex variable which is analytic in the upper half of the complex plane. Given a square-integrable function $f : \mathbb{R} \to \mathbb{R}$, let $g = f + i\,\mathcal{H}[f]$ and $h = \widehat{g}$, that is $h(\nu) = 2\,\widehat{f}(\nu)$ for a.a. $\nu > 0$ and $h(\nu) = 0$ for a.a. $\nu < 0$. We extend g to a function G of a complex variable by extending the inverse Fourier transform of h to a complex variable z:

$$G(z) = \int_0^\infty h(\nu)\, \exp[2\pi i\,\nu z]\, d\nu \qquad (z \in \mathbb{C}), \tag{3.28}$$

in other words for $z = x + i\,y$ we have:

$$G(x + i\,y) = \int_0^\infty h(\nu)\, \exp[-2\pi\nu y]\, \exp[2\pi i\,\nu x]\, d\nu.$$

It can be shown that $G(z)$ is well defined and derivable for $\Im z > 0$ (by deriving w.r.t. z under the integral in (3.28)), in other words G is analytic on the upper half-plane $\Im z > 0$. Now g can be obtained from G as its "non-tangential limit"; this means that for every $a > 0$, for x real and $\Im z > 0$, $z \to x$ subject to $|\Re(z-x)| < a|\Im(z-x)|$ gives $G(z) \to g(x)$. See [15,53] for more details; a short exposition of this theory is given in [44].

Let us illustrate this theory with a concrete example. We define h by

$$h(\nu) = \begin{cases} (n!)^{-1}(2\pi)^{n+1}\nu^n \exp[-2\pi\nu] & \text{for } \nu \geq 0, \\ 0 & \text{for } \nu < 0, \end{cases}$$

where $n \geq 0$. Then one can show by induction on n that the complex function $G(z)$ defined by (3.28) is given by $G(z) = (1 - i\,z)^{-n-1}$; it is indeed analytic for $\Im z > -1$ and it has a pole at $z = -i$. In particular for x real, $G(x)$ is its own "non-tangential limit", that is $G(x) = g(x) = (1 - i\,x)^{-n-1}$ (where $h = \widehat{g}$), and we get thus the quadrature pair $(\Re g, \Im g)$, namely the Cauchy functions given in (3.26).

Let us now describe the Fourier transform of the energy function $F^2 + G^2$ for F and G in $\mathbf{n}$-quadrature:

Lemma 3.12. *Let the functions F and G be square-integrable and in $\mathbf{n}$-quadrature. Then*

$$\left(F^2 + G^2\right)^\wedge = [F + i\,G]^\wedge * [F - i\,G]^\wedge = 4\,(\mathrm{pos}_\mathbf{n} \cdot \widehat{F}) * (\mathrm{neg}_\mathbf{n} \cdot \widehat{F}). \tag{3.29}$$

Furthermore if F is real-valued and has constant phase on $\mathcal{P}_\mathbf{n}$, then

$$\left(F^2 + G^2\right)^\wedge = 4\,(\mathrm{pos}_\mathbf{n} \cdot F^\mathcal{A}) * (\mathrm{neg}_\mathbf{n} \cdot F^\mathcal{A}), \tag{3.30}$$

a non-negative real-valued function.

Proof. (3.29) is obtained by combining the dual convolution formula with (3.27). If F has constant phase on $\mathcal{P}_\mathbf{n}$, there is a constant angle φ such that $F^\Phi(\mathbf{u}) = \varphi$ for $\mathbf{u} \in \mathcal{P}_\mathbf{n}$; if F is also real-valued, then $F^\Phi(\mathbf{u}) = -\varphi$ for $\mathbf{u} \in \mathcal{N}_\mathbf{n}$, and we get thus

$$\mathrm{pos}_\mathbf{n} \cdot \widehat{F} = e^{i\,\varphi}\,\mathrm{pos}_\mathbf{n} \cdot F^\mathcal{A} \qquad \text{and} \qquad \mathrm{neg}_\mathbf{n} \cdot \widehat{F} = e^{-i\,\varphi}\,\mathrm{neg}_\mathbf{n} \cdot F^\mathcal{A},$$

so that (3.29) gives

$$\left(F^2 + G^2\right)^\wedge = 4\,e^{i\,\varphi}e^{-i\,\varphi}\,(\mathrm{pos}_\mathbf{n} \cdot F^\mathcal{A}) * (\mathrm{neg}_\mathbf{n} \cdot F^\mathcal{A}),$$

in other words (3.30). ∎

The argument used here, combined with the results of Subsection 3.3, will be central to our analysis of the phase congruence model.

4. Edge detection in the phase congruence model

We will now describe the phase congruence model for edge detection and its mathematical properties, using the results of Section 3. We deal successively with the choice of the filters and their basic properties, the relation of the proposed edge detector with phase congruence, its behaviour on the standard edge profiles shown in Section 2, other quadratic combinations of the filters, in particular single-filter approaches to edge detection (such as Canny's operator [7]), the problem of orientation selection (that is, how to determine the orientation of the edges detected at a given position), and finally the authentification of edges, especially in scale-space.

4.1. *Axioms and basic properties*

When the dimension d of our space $\mathcal{E}$ is at least two, we will allow our filters to rotate in order to adapt to the normal orientation of the edge; so the orientation of the filters will be given by a unit vector $\mathbf{n}$, which will be considered as normal to the corresponding edge orientation; in order to specify the filters, we will choose a standard orientation given by a canonical unit vector $\mathbf{c}$. When $d = 1$, the filters will be fixed, so that $\mathbf{n}$ and $\mathbf{c}$ will simply be the unit 1 in $\mathbb{R}$.

We take as filters two (bitempered) functions $\mathcal{E} \to \mathbb{R}$, an even-symmetric function C (for "cosine"), and an odd-symmetric function S (for "sine"). When $d \geq 2$, the two functions C and S have their orientation corresponding to the canonical unit vector $\mathbf{c}$.

Write I for the image in which edges must be detected; I will be a function $\mathcal{E} \to \mathbb{R}$, and it will be convolved with the filters.

As edge detection involves very particular filters being applied to rather arbitrary images, we choose to impose sharp constraints on C and S, but loose ones on I. The first requirement is a two-dimensional generalization of the Morrone-Burr condition [37]:

Axiom 1. *$C \not\equiv 0$, C is even-symmetric, S is odd-symmetric, $\widehat{C}$ has real non-negative values, and (C, S) is a $\mathbf{c}$-quadrature pair.*

Let us write A for the Fourier amplitude $C^{\mathcal{A}}$ of C; then the following is a reformulation of the requirement concerning the Fourier transforms of the filters:

$$\text{For all} \quad \mathbf{u} \in \mathcal{E}, \qquad \begin{aligned} \widehat{C}(\mathbf{u}) &= A(\mathbf{u}), \\ \widehat{S}(\mathbf{u}) &= -i \operatorname{sgn}_{\mathbf{c}}(\mathbf{u}) A(\mathbf{u}), \end{aligned} \qquad \text{where} \quad A(\mathbf{u}) \geq 0. \tag{4.1}$$

Note that several previous studies (in particular [37,51]) have considered one-dimensional functions satisfying $\widehat{S}(\nu) = i \operatorname{sgn}(\nu)\widehat{C}(\nu)$, in another words such that $(C, -S)$ (rather than (C, S)) is a quadrature pair, but again this does not matter, we have then only to take $-S$ instead of S.

If C is bounded in a neighbourhood of the origin, then by Proposition 3.4, $\widehat{C}$ is integrable, and as $|\widehat{S}| = |\widehat{C}|$, it follows that $\widehat{S}$ is also integrable. By Lemma 3.1, the two functions $C_u = \widehat{C}^{\vee}$ and $S_u = \widehat{S}^{\vee}$ defined as in (3.13) are bounded and uniformly continuous, they vanish at infinity, and we have $C_u \equiv C$ and $S_u \equiv S$. It follows then that $I * C = I * C_u$ and $I * S = I * S_u$, in other words C and S considered as filters have the same behaviour as C_u and S_u. We can thus take C_u and S_u in place of C and S. Hence the following requirement is natural:

Axiom 2. *C and S are continuous.*

As explained just before, negating this axiom amounts to requiring that C is unbounded around the origin. C and S being bounded, if they are also integrable, then by Proposition 6.10 of [11], C and S will belong to L^p for every $p \in [1, \infty]$, and in particular they will be square-integrable. Thus requiring square-integrability on C and S is less restrictive than requiring integrability. As we choose to take the strongest restrictions on the filters and the weakest ones on the images, we assume the following:

Axiom 3. *C and S are integrable.*

Another reason for postulating integrability instead of square-integrability, is that a function which is square-integrable but not integrable has a slow decay: asymptotically it cannot decrease faster than $1/|\mathbf{x}|^d$.

Summarizing what we said above, Axioms 2 and 3 imply the following:

Proposition 4.1. *$\widehat{C}$ and $\widehat{S}$ are integrable, bounded and uniformly continuous, they vanish at infinity, the Fourier inversion formulas $C = \widehat{C}^\vee$ and $S = \widehat{S}^\vee$ hold pointwise (as a Fourier integral), C and S vanish at infinity, and $C, S \in L^p$ for every $p \geq 1$.*

Axioms 1, 2, and 3 are the basis of our theory; they are the only ones that we use in the case of one-dimensional signals, since all further axioms explicitly require $d \geq 2$. These three axioms lead to the following fundamental result:

Proposition 4.2. *For $F = C$, S, $C + i\,S$, and $C - i\,S$, we have:*

- (*i*) *$\int_{\mathcal{E}} F = 0$, and $H * F = 0$ for every constant $H : \mathcal{E} \to \mathbb{R}$.*
- (*ii*) *For $I \in L^1 + L^2$, $I * F$ is square-integrable, bounded, uniformly continuous, and it vanishes at infinity. Furthermore $(I * F)^\wedge = \widehat{I}\widehat{F}$ is integrable and square-integrable, and the equality $I * F = [\widehat{I}\widehat{F}]^\vee$ holds pointwise (as a Fourier integral).*

Proof. By Proposition 3.8, $\int_{\mathcal{E}} C = \int_{\mathcal{E}} S = 0$, so that the same holds for $F = C + i\,S$ or $C - i\,S$. If $H(\mathbf{x}) = c$ for all $\mathbf{x} \in \mathcal{E}$, then $(H * F)(\mathbf{x}) = c \cdot \int_{\mathcal{E}} F = 0$ for all $\mathbf{x} \in \mathcal{E}$; thus (*i*) holds.

We know that C is integrable and bounded; hence it is also square-integrable. Take $I = I_1 + I_2$, where $I_1 \in L^1$ and $I_2 \in L^2$. Since $I_1 \in L^1$ and $C \in L^2$, and since $I_2 \in L^2$ and $C \in L^1$, by Young's inequality we have $I_1 * C, I_2 * C \in L^2$. Since I_1 is integrable while C is bounded and vanishes at infinity, the p-p' convolution property for $p = 1$ implies that $I_1 * C$ will be bounded, uniformly continuous, and will vanish at infinity; since $I_2, C \in L^2$, the p-p' convolution property for $p = 2$ implies that $I_2 * C$ will also be bounded, uniformly continuous, and vanishing at infinity. Thus $I * C = I_1 * C + I_2 * C$ will share the common properties of $I_1 * C$ and $I_2 * C$, namely being square-integrable, bounded, uniformly continuous, and vanishing at infinity.

Since I_1 and C are integrable, $\widehat{I_1}$ and $\widehat{C}$ are bounded by the Riemann-Lebesgue theorem; hence $\widehat{I_1}\widehat{C}$ is bounded. We know also that $\widehat{C}$ is integrable by Lemma 4.1; as $\widehat{I_1}$ is bounded, it follows that $\widehat{I_1}\widehat{C}$ is integrable. Being bounded and integrable, $\widehat{I_1}\widehat{C}$ will also be square-integrable by Proposition 6.10 of [11]. As I_2 and C are square-integrable, $\widehat{I_2}$ and $\widehat{C}$ are square-integrable by the Plancherel Theorem, and it follows from Hölder's inequality that $\widehat{I_2}\widehat{C}$ is integrable. As $\widehat{C}$ is also bounded, $\widehat{I_2}\widehat{C}$ will be square-integrable. Thus $\widehat{I}\widehat{C} = \widehat{I_1}\widehat{C} + \widehat{I_2}\widehat{C}$ will share the common properties of $\widehat{I_1}\widehat{C}$ and $\widehat{I_2}\widehat{C}$, namely being integrable and square-integrable. The convolution formula gives $(I * C)^\wedge = \widehat{I}\widehat{C}$.

Since $I * C$ is continuous and its Fourier transform $\widehat{I}\widehat{C}$ is integrable, by Lemma 3.1 the Fourier inversion formula $I * C = [\widehat{I}\widehat{C}]^\vee$ will hold pointwise as a Fourier integral.

The same argument works with S, $C + i\,S$, and $C - i\,S$, instead of C, and so (*ii*) holds. ∎

This theoretical result is also interesting from a practical point of view. The fact that $I * C$ and $I * S$ are bounded implies that we can control the dynamic range of these convolutions, and in particular avoid overflows in computer implementations; the fact that they vanish at infinity means that significant edges of the image I will all be localized in a bounded domain; the fact that they are uniformly continuous is interesting in view of digitization, as we will see in Section 5.

The above three axioms, and their consequences (Propositions 4.1 and 4.2), are the basis for the phase congruence model for edge detection in the case of one-dimensional signals, or more generally for the detection of edges having a fixed orientation; this will be the subject of the next subsection. We will now give four more axioms in order to deal with edges having various orientations when $d \geq 2$.

The filters C and S are oriented along the direction given by the canonical unit vector $\mathbf{c}$, and can thus be used to detect edges whose normal orientation is parallel to $\mathbf{c}$. Given an edge whose normal orientation is parallel to a unit vector $\mathbf{n}$, we will use two filters $C_\mathbf{n}$ and $S_\mathbf{n}$ oriented along the direction of $\mathbf{n}$. The natural way to do this is to take a rotation R of $\mathcal{E}$ such that $\mathbf{n} = R(\mathbf{c})$, and to apply that rotation R to the filters: we set $C_\mathbf{n} = R(C)$ and $S_\mathbf{n} = R(S)$, where $R(F)(\mathbf{x}) = F(R^{-1}(\mathbf{x}))$ for any function $F : \mathcal{E} \to \mathbb{C}$, in particular for $F = C, S$. The first thing to require is that all rotations mapping $\mathbf{c}$ to $\mathbf{n}$ give the same result for $C_\mathbf{n}$ and $S_\mathbf{n}$, or equivalently that C and S are invariant under every rotation fixing $\mathbf{c}$. When $d = 2$, there is a unique rotation mapping $\mathbf{c}$ to $\mathbf{n}$, and so the choice of $C_\mathbf{n}$ and $S_\mathbf{n}$ is unambiguous; however for $d \geq 3$ we need an explicit requirement:

Axiom 4. *For $d \geq 3$, C and S are invariant under every rotation of $\mathcal{E}_\mathbf{c}$.*

This invariance can also be expressed as follows: for $d \geq 3$, given $\mathbf{y}, \mathbf{z} \in \mathcal{E}_\mathbf{c}$ such that $|\mathbf{y}| = |\mathbf{z}|$ and $t \in \mathbb{R}$, we have $C(\mathbf{y} + t\mathbf{c}) = C(\mathbf{z} + t\mathbf{c})$, and similarly for S. As the Fourier transform commutes with rotations, $\widehat{C}$ and $\widehat{S}$ will also be invariant under every rotation of $\mathcal{E}_\mathbf{c}$.

For every unit vector $\mathbf{n}$, we will have corresponding filters $C_\mathbf{n}$ and $S_\mathbf{n}$ obtained by $C_\mathbf{n} = R(C)$ and $S_\mathbf{n} = R(S)$ for every rotation R satisfying $R(\mathbf{c}) = \mathbf{n}$. The implicit assumption underlying this construction is that the space $\mathcal{E}$ is isotropic, and that this isotropy extends to edge detection, in other words that edge detection commutes with rotation (it is already translation-invariant, thanks to the use of convolutions). This assumption is natural in the case of planar or volumetric static images; however it is somewhat questionable in the case of image sequences, where the three-dimensional space $\mathcal{E}$ is in fact a space-time, whose group of symmetries is not the one of rotations with translations. Therefore we do not exclude that for image sequences other choices of filters $C_\mathbf{n}$ and $S_\mathbf{n}$, and in particular other axioms, could be necessary.

We recall from (3.4) that for every integrable function F on $\mathcal{E}$, and every unit vector $\mathbf{n}$, we define the function $F_{/\mathbf{n}}$ on $\mathbb{R}$ by

$$F_{/\mathbf{n}}(t) = \int_{\mathcal{E}_\mathbf{n}} F(\mathbf{y} + t\mathbf{n})\, d\mathbf{y},$$

that $F_{/\mathbf{n}}$ is integrable, and that its Fourier transform satisfies (see (3.7))

$$(F_{/\mathbf{n}})^\wedge(\nu) = \widehat{F}(\nu\mathbf{n}).$$

Thus (4.1) gives:

$$\text{For all} \quad \nu \in \mathbb{R}, \qquad \begin{array}{l} (C_{/\mathbf{n}})^\wedge(\nu) = A(\nu\mathbf{n}), \\ (S_{/\mathbf{n}})^\wedge(\nu) = -i\,\mathrm{sgn}_\mathbf{c}(\mathbf{n})\mathrm{sgn}\nu\, A(\nu\mathbf{n}), \end{array} \qquad (4.2)$$

Our next two requirements deal with $C_{/\mathbf{n}}$ and $S_{/\mathbf{n}}$:

Axiom 5. *Let $d \geq 2$; for every unit vector $\mathbf{n}$, $C_{/\mathbf{n}}$ and $S_{/\mathbf{n}}$ are continuous.*

Given a unit vector $\mathbf{n}$, the following three statements are equivalent:

(*i*) $C_{/\mathbf{n}}(t) = 0$ for all $t \in \mathbb{R}$;

(*ii*) $C_{/\mathbf{n}}(t) = 0$ for almost all $t \in \mathbb{R}$;

(*iii*) $\widehat{C}(\nu \cdot \mathbf{n}) = 0$ for all $\nu \in \mathbb{R}$.

The equivalence between (*i*) and (*ii*) follows from Axiom 5, and the one between (*ii*) and (*iii*) from the fact that the Fourier transform of the integrable function $C_{/\mathbf{n}}$ is the continuous function $\nu \mapsto \widehat{C}(\nu\mathbf{n})$. We have similar equivalences for S. As C and S are integrable and in $\mathbf{c}$-quadrature, we have $\widehat{C}(\mathbf{u}) = \widehat{S}(\mathbf{u}) = 0$ for every $\mathbf{u} \in \mathcal{E}_\mathbf{c}$ (see the proof of Proposition 3.8). We derive thus the following:

Lemma 4.3. *Let $d \geq 2$; for every unit vector $\mathbf{d} \in \mathcal{E}_\mathbf{c}$, the functions $C_{/\mathbf{d}}$ and $S_{/\mathbf{d}}$ are identically* 0.

When the unit vector $\mathbf{n}$ does not belong to $\mathcal{E}_\mathbf{c}$, we exclude the result of Lemma 4.3:

Axiom 6. *Let $d \geq 2$; for every unit vector $\mathbf{n} \notin \mathcal{E}_\mathbf{c}$, $C_{/\mathbf{n}}$ is not identically zero, in other words $\widehat{C}(\nu \cdot \mathbf{n}) \neq 0$ for some $\nu \in \mathbb{R}$.*

From (4.2) and Axioms 5 and 6 we derive the following consequence:

Proposition 4.4. *Let $d \geq 2$; for every unit vector $\mathbf{n} \in \mathcal{P}_\mathbf{c}$, $C_{/\mathbf{n}}$ and $S_{/\mathbf{n}}$ satisfy Axioms 1, 2, and 3 with $d = 1$.*

We can thus apply Propositions 4.1 and 4.2 with $C_{/\mathbf{n}}$ and $S_{/\mathbf{n}}$ instead of C and S. The following result shows that using $C_{/\mathbf{n}}$ and $S_{/\mathbf{n}}$ with one-dimensional signals is equivalent to using C and S with images whose grey-level is constant along $\mathcal{E}_\mathbf{n}$ and varies along the direction $\mathbf{n}$ only:

Proposition 4.5. *Let $d \geq 2$, let $\mathbf{n}$ be a unit vector, and let P be a function $\mathbb{R} \to \mathbb{R}$. Let $I : \mathcal{E} \to \mathbb{R}$ be given by*

$$\forall \mathbf{y} \in \mathcal{E}_\mathbf{n}, \forall x \in \mathbb{R}, \qquad I(\mathbf{y} + x\mathbf{n}) = P(x). \tag{4.3}$$

Then every integrable function $\mathcal{E} \to \mathbb{R}$ gives

$$\forall \mathbf{y} \in \mathcal{E}_\mathbf{n}, \forall x \in \mathbb{R}, \qquad (I * F)(\mathbf{y} + x\mathbf{n}) = (P * F_{/\mathbf{n}})(x). \tag{4.4}$$

This holds in particular for $F = C$, S, $C + i\,S$, or $C - i\,S$.

This follows by applying Fubini's theorem. The one-dimensional function P is called the *profile* of I. We give now our last axiom; it means that when a vector in $\mathcal{P}_\mathbf{c}$ is rotated away from $\mathcal{L}_\mathbf{c}$ towards $\mathcal{E}_\mathbf{c}$, its value by $\widehat{C}$ decreases; the two possible ways of decreasing (strictly everywhere, or not) give two versions (strong and weak) of that axiom:

Axiom 7. *Let $d \geq 2$; for every unit vector $\mathbf{d} \in \mathcal{E}_\mathbf{c}$ and for every $\nu > 0$, the function f of variable $\theta \in [0, \pi/2]$ defined by*

$$f(\theta) = A\big(\nu(\cos\theta\mathbf{c} + \sin\theta\mathbf{d})\big),$$

- (a) **(Strong version)** *is strictly decreasing on $[0, \pi/2]$, that is for $0 \leq \theta < \theta' \leq \pi/2$ we have $f(\theta) > f(\theta')$.*
- (b) **(Weak version)** *is decreasing on $[0, \pi/2]$, with a unique maximum at 0, that is for $0 < \theta < \theta' \leq \pi/2$ we have $f(0) > f(\theta) \geq f(\theta')$.*

To see how restrictive is the strong version of that axiom, let us note that it implies a stronger version of Axiom 6: for every unit vector $\mathbf{n} \notin \mathcal{E}_\mathbf{c}$, there is some $\mathbf{d} \in \mathcal{E}_\mathbf{c}$ and some $\theta \in [0, \pi/2[$ such that $\mathbf{n} = \cos\theta\mathbf{c} + \sin\theta\mathbf{d}$. Then for *every* $\nu \neq 0$ we have

$$\widehat{C}(\nu \cdot \mathbf{n}) = \widehat{C}\big(\nu(\cos\theta\mathbf{c} + \sin\theta\mathbf{d})\big) > \widehat{C}\big(\nu(\cos\frac{\pi}{2}\mathbf{c} + \sin\frac{\pi}{2}\mathbf{d})\big) = \widehat{C}(\nu\mathbf{d}) = 0.$$

Now let us explain how C and S are used to detect edges in the phase congruence model. This model has traditionally been described in the case of one-dimensional signals [37,38,51]: assuming $d = 1$, we convolve the image I with the two filters C and S, and we define the *energy function* E by $E = (I * C)^2 + (I * S)^2$. Note that several authors, in particular [37,44], define $\sqrt{E}$ as the energy function, but this does not really matter. The edges in I are made of the points of $\mathcal{E}$ at which the energy E reaches a maximum. Generally it is assumed that one means here a purely local maximum, but in [51] we suggested that one can take regional maxima instead. We will discuss this point in more detail in Subsection 4.5.

As described here, the model can also be applied to the case where $d > 1$, but all edges have their normal orientation parallel to $\mathbf{c}$. Here edge points are maxima of E in the direction of $\mathbf{c}$: $E(\mathbf{p}) > E(\mathbf{p} + \varepsilon\mathbf{c})$ locally.

Now let us describe the model in the case where $n > 1$ and the orientation of edges is allowed to vary. For every unit vector $\mathbf{n}$, we have the oriented energy function $E_{\mathbf{n}} = (I * C_{\mathbf{n}})^2 + (I * S_{\mathbf{n}})^2$. For every point $\mathbf{p} \in \mathcal{E}$, we define the edge energy $E(\mathbf{p})$ and edge normal orientation $\mathbf{N}(\mathbf{p})$ at $\mathbf{p}$ as follows: $E(\mathbf{p})$ is the maximum of all $E_{\mathbf{n}}(\mathbf{p})$ for varying unit vectors $\mathbf{n}$ in $\mathcal{E}$, and $\mathbf{N}(\mathbf{p})$ is the unit vector $\mathbf{n}$ for which this maximum of $E_{\mathbf{n}}(\mathbf{p})$ is reached. Then the point $\mathbf{p}$ belongs to the edge if and only if E has at $\mathbf{p}$ a maximum in the direction of $\mathbf{N}(\mathbf{p})$: $E(\mathbf{p}) > E(\mathbf{p} + \varepsilon\mathbf{N}(\mathbf{p}))$ locally.

The relation between the energy function and phase congruence will be analysed in Subsection 4.2; this will be explicited in particular for the edge profiles described in Figure 1. In Subsection 4.3 we will relate other quadratic operators, in particular traditional ones based on a single filter, to Fourier phase. The selection of edge orientation will be dealt with in Subsection 4.4. Finally Subsection 4.5 will discuss the problem of selecting local or regional maxima as edge localization.

4.2. Phase congruence and standard edge profiles

We take a real-valued image I in $L^1 + L^2$. We suppose momentarily that edges are all oriented along a single direction; we can thus assume that the edge normal orientation is along the canonical unit vector $\mathbf{c}$, and so we consider the filters in the orientation of $\mathbf{c}$ only: $C = C_{\mathbf{c}}$ and $S = S_{\mathbf{c}}$.

In view of Subsection 3.5, we define the complex-valued filter $D = C + i\,S$, and we set

$$\begin{aligned} \Gamma &= I * C, \\ \Sigma &= I * S, \\ \text{and} \qquad \Delta &= I * D = \Gamma + i\,\Sigma. \end{aligned} \tag{4.5}$$

We define the energy E by

$$E = |\Delta|^2 = \Gamma^2 + \Sigma^2 = (I * C)^2 + (I * S)^2. \tag{4.6}$$

From (4.1), we write $\widehat{C} = A$, and then we have $\widehat{S} = -i\,\mathrm{sgn}_{\mathbf{c}}\,A$ and $\widehat{D} = 2\mathrm{pos}_{\mathbf{c}}\,A$.

We will see now that at every point $\mathbf{p}$ the energy function $E(\mathbf{p})$ measures the degree to which all phases $I^{\Phi}(\mathbf{u}, \mathbf{p})$ for frequencies $\mathbf{u} \in \mathcal{P}_{\mathbf{c}}$ are clustered around a unique value.

Proposition 4.6. *Assume that $I \in L^1 + L^2$. Then $\widehat{E} = 4\,(\mathrm{pos}_{\mathbf{c}}\,\widehat{I}A) * (\mathrm{neg}_{\mathbf{c}}\,\widehat{I}A)$, an integrable function. Furthermore:*

(*i*) *If I^{Φ} is constant on $\mathcal{P}_{\mathbf{c}}$, then $\widehat{E} = 4\,(\mathrm{pos}_{\mathbf{c}}\,I^{\mathcal{A}}A) * (\mathrm{neg}_{\mathbf{c}}\,I^{\mathcal{A}}A)$, a real-valued non-negative function, and E^{Φ} is constant zero.*

(*ii*) *If for a given $\mathbf{p} \in \mathcal{E}$ we have $I^{\Phi}(\mathbf{u}, \mathbf{p})$ constant for all $\mathbf{u} \in \mathcal{P}_{\mathbf{c}}$, then $E^{\Phi}(\mathbf{u}, \mathbf{p}) = 0$ for all $\mathbf{u} \in \mathcal{E}$, and $E(\mathbf{p}) > E(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{E}$ such that $\mathbf{x} \neq \mathbf{p}$.*

(*iii*) *For every $\mathbf{p} \in \mathcal{E}$ we have*

$$E(\mathbf{p}) = 4 \iint_{\mathcal{P}_{\mathbf{c}} \times \mathcal{P}_{\mathbf{c}}} I^{\mathcal{A}}(\mathbf{u})A(\mathbf{u})I^{\mathcal{A}}(\mathbf{v})A(\mathbf{v}) \cos\bigl[I^{\Phi}(\mathbf{u}, \mathbf{p}) - I^{\Phi}(\mathbf{v}, \mathbf{p})\bigr]\, d\mathbf{u}d\mathbf{v}. \tag{4.7}$$

PROOF. Γ and Σ are in **c**-quadrature. By Proposition 4.2, Γ, Σ, Δ, and $\overline{\Delta}$ are square-integrable and we have $\widehat{\Delta} = \widehat{I}\widehat{D} = 2\widehat{I}\text{pos}_{\mathbf{c}}\, A$ and $\widehat{\overline{\Delta}} = 2\widehat{I}\text{neg}_{\mathbf{c}}\, A$; hence $\widehat{E} = 4\,(\text{pos}_{\mathbf{c}}\,\widehat{I}A) * (\text{neg}_{\mathbf{c}}\,\widehat{I}A)$ by Lemma 3.12 (see (3.29)). By Proposition 4.2 again, $\widehat{\Delta}$ and $\widehat{\overline{\Delta}}$ are integrable, so that $\widehat{E}$, being the convolution of two integrable functions, is integrable.

Now $\widehat{\Delta}(\mathbf{u}) = 0$ for $\mathbf{u} \notin \mathcal{P}_{\mathbf{c}}$, while for $\mathbf{u} \in \mathcal{P}_{\mathbf{c}}$ we have $\Delta^{\mathcal{A}}(\mathbf{u}) = 2\,I^{\mathcal{A}}(\mathbf{u})A(\mathbf{u})$ and $\Delta^{\Phi}(\mathbf{u}) = I^{\Phi}(\mathbf{u})$. By Lemma 3.12 (see (3.30)), if I^{Φ} is constant on $\mathcal{P}_{\mathbf{c}}$, we get $\widehat{E} = 4\,(\text{pos}_{\mathbf{c}}\,I^{\mathcal{A}}A) * (\text{neg}_{\mathbf{c}}\,I^{\mathcal{A}}A)$, and since $\widehat{E}$ is non-negative real, E^{Φ} is constant zero. Therefore (*i*) follows.

By definition, the phase of I at point $\mathbf{p}$ is the Fourier phase of $\tau_{-\mathbf{p}}(I)$. Now $\tau_{-\mathbf{p}}(I)$ leads to the energy function $\tau_{-\mathbf{p}}(E)$. Suppose that $\tau_{-\mathbf{p}}(I)$ has constant Fourier phase. Then by (*i*), $\tau_{-\mathbf{p}}(E)$ has zero Fourier phase. Since $\tau_{-\mathbf{p}}(E)$ is continuous and the Fourier transform of $\tau_{-\mathbf{p}}(E)$ is integrable, Corollary 3.6 implies that $\tau_{-\mathbf{p}}(E)(\mathbf{0}) > \tau_{-\mathbf{p}}(E)(\mathbf{z})$ for all $\mathbf{z} \neq \mathbf{0}$. Therefore (*ii*) holds.

Finally (*iii*) follows from (3.16) in Lemma 3.2, with Δ instead of F. ∎

This result, especially (*iii*), is at the basis of the model. Indeed, at a given point $\mathbf{p}$, the more all Fourier phases $I^{\Phi}(\mathbf{u},\mathbf{p}) = I^{\Phi}(\mathbf{u}) + 2\pi\,\mathbf{p}\cdot\mathbf{u}$ for $\mathbf{u} \in \mathcal{P}_{\mathbf{c}}$ are clustered together, the more all $\cos\big[I^{\Phi}(\mathbf{u},\mathbf{p}) - I^{\Phi}(\mathbf{v},\mathbf{p})\big]$ appearing in (4.7) are close to 1, and the higher is $E(\mathbf{p})$. Therefore maxima of E correspond to *points of maximal phase congruence* for frequencies in the half-plane $\mathcal{P}_{\mathbf{c}}$. In particular by (*ii*), a point where all phases become strictly equal gives an absolute maximum of E. Thus edges whose normal direction is parallel to **c** will be localized at points where E has a maximum in the normal direction. Whether such maxima are purely local or should be over a certain range will be discussed later in Subsection 4.5.

Note that when the image signal has only one frequency in $\mathcal{P}_{\mathbf{c}}$, there is no phase congruence, and in fact the energy function is constant. Indeed for a cosine function $H : \mathcal{E} \to \mathbb{R}$ given by $H(\mathbf{x}) = \cos(2\pi\,\mathbf{u}\cdot\mathbf{x} + \varphi)$, where $\mathbf{u} \in \mathcal{E}$ and φ is an angle, we have

$$(H * D)(\mathbf{x}) = A(\mathbf{u})\exp\big[i\,\text{sgn}_{\mathbf{c}}(\mathbf{u})(2\pi\,\mathbf{u}\cdot\mathbf{x} + \varphi)\big],$$

so that $E(\mathbf{x}) = A(\mathbf{u})^2$ for every $\mathbf{x} \in \mathcal{E}$.

The next result shows how the value around which all phases at $\mathbf{p}$ are clustered is obtained from the argument of the complex number $\Delta(\mathbf{p})$. This generalizes a similar finding in [63], given there in the case of one-dimensional periodic signals. For any $\mathbf{x} \in \mathcal{E}$, define the angle $\varphi(\mathbf{x})$ (uniquely modulo 2π) by

$$\Delta(\mathbf{x}) = |\Delta(\mathbf{x})| \cdot e^{i\,\varphi(\mathbf{x})}, \tag{4.8}$$

in other words

$$\Gamma(\mathbf{x}) = |\Delta(\mathbf{x})| \cdot \cos\varphi(\mathbf{x}) \qquad \text{and} \qquad \Sigma(\mathbf{x}) = |\Delta(\mathbf{x})| \cdot \sin\varphi(\mathbf{x}). \tag{4.9}$$

This will be the angle around which all phases cluster:

PROPOSITION 4.7. *For every* $\mathbf{p} \in \mathcal{E}$,

$$\Delta(\mathbf{p}) = |\Delta(\mathbf{p})| \cdot e^{i\,\varphi(\mathbf{p})} = 2\int_{\mathcal{P}_{\mathbf{c}}} I^{\mathcal{A}}(\mathbf{u})A(\mathbf{u})\exp\big[i\,I^{\Phi}(\mathbf{u},\mathbf{p})\big]\,d\mathbf{u}, \tag{4.10}$$

$$\int_{\mathcal{P}_{\mathbf{c}}} I^{\mathcal{A}}(\mathbf{u})A(\mathbf{u})\sin\big[I^{\Phi}(\mathbf{u},\mathbf{p}) - \varphi(\mathbf{p})\big]\,d\mathbf{u} = 0, \tag{4.11}$$

and

$$2\int_{\mathcal{P}_{\mathbf{c}}} I^{\mathcal{A}}(\mathbf{u})A(\mathbf{u})\cos\big[I^{\Phi}(\mathbf{u},\mathbf{p}) - \theta\big]\,d\mathbf{u}\begin{cases} = |\Delta(\mathbf{p})| & \text{for } \theta = \varphi(\mathbf{p}); \\ < |\Delta(\mathbf{p})| & \text{for } \theta \neq \varphi(\mathbf{p}). \end{cases} \tag{4.12}$$

Furthermore if for all $\mathbf{u} \in \mathcal{P}_{\mathbf{c}}$, $I^{\Phi}(\mathbf{u},\mathbf{p}) = \theta$, *where* θ *is constant, then* $\theta = \varphi(\mathbf{p})$.

PROOF. (4.10) follows from (3.15), combined with (4.1). Now (4.10) gives

$$|\Delta(\mathbf{p})| = 2\exp\bigl[-i\,\varphi(\mathbf{p})\bigr]\int_{\mathcal{P}_\mathbf{c}} I^{\mathcal{A}}(\mathbf{u})A(\mathbf{u})\exp\bigl[i\,I^{\Phi}(\mathbf{u},\mathbf{p})\bigr]\,d\mathbf{u}$$
$$= 2\int_{\mathcal{P}_\mathbf{c}} I^{\mathcal{A}}(\mathbf{u})A(\mathbf{u})\exp\bigl[i\,\bigl(I^{\Phi}(\mathbf{u},\mathbf{p})-\varphi(\mathbf{p})\bigr)\bigr]\,d\mathbf{u}.$$

The imaginary part of this equality gives (4.11), and the real part of it gives the equality in (4.12) for $\theta = \varphi(\mathbf{p})$. Now for $\theta \neq \varphi(\mathbf{p})$, we have

$$|\Delta(\mathbf{p})| > \Re\bigl(e^{i\,(\varphi(\mathbf{p})-\theta)}|\Delta(\mathbf{p})|\bigr) = \Re\bigl(e^{-i\,\theta}\Delta(\mathbf{p})\bigr) = \Re\Bigl(2e^{-i\,\theta}\int_{\mathcal{P}_\mathbf{c}} I^{\mathcal{A}}(\mathbf{u})A(\mathbf{u})\exp\bigl[i\,I^{\Phi}(\mathbf{u},\mathbf{p})\bigr]\,d\mathbf{u}\Bigr)$$
$$=2\Re\Bigl(\int_{\mathcal{P}_\mathbf{c}} I^{\mathcal{A}}(\mathbf{u})A(\mathbf{u})\exp\bigl[i\,\bigl(I^{\Phi}(\mathbf{u},\mathbf{p})-\theta\bigr)\bigr]\,d\mathbf{u}\Bigr) = 2\int_{\mathcal{P}_\mathbf{c}} I^{\mathcal{A}}(\mathbf{u})A(\mathbf{u})\cos\bigl[I^{\Phi}(\mathbf{u},\mathbf{p})-\theta\bigr]\,d\mathbf{u},$$

giving the inequality in (4.12). Finally if $I^{\Phi}(\mathbf{u},\mathbf{p}) = \theta$ for all $\mathbf{u} \in \mathcal{P}_\mathbf{c}$, then (4.10) gives

$$|\Delta(\mathbf{p})|\cdot e^{i\,\varphi(\mathbf{p})} = 2e^{i\,\theta}\int_{\mathcal{P}_\mathbf{c}} I^{\mathcal{A}}(\mathbf{u})A(\mathbf{u})\,d\mathbf{u},$$

from which we deduce that $\theta = \varphi(\mathbf{p})$. ∎

Let us comment this result, and the equations within it. Equation (4.10) shows that $e^{i\,\varphi(\mathbf{p})}$ is a weighted linear combination (with non-negative weights) of all $\exp\bigl[i\,I^{\Phi}(\mathbf{u},\mathbf{p})\bigr]$ for $\mathbf{u} \in \mathcal{P}_\mathbf{c}$; in particular when all $I^{\Phi}(\mathbf{u},\mathbf{p})$ are equal, they must coincide with $\varphi(\mathbf{p})$. Equation (4.11) is another way of expressing that $\varphi(\mathbf{p})$ is some form of average between the $I^{\Phi}(\mathbf{u},\mathbf{p})$, $\mathbf{u} \in \mathcal{P}_\mathbf{c}$. Finally (4.12) implies that

$$|\Delta(\mathbf{p})| = 2\max_{\theta\in[0,2\pi[}\int_{\mathcal{P}_\mathbf{c}} I^{\mathcal{A}}(\mathbf{u})A(\mathbf{u})\cos\bigl[I^{\Phi}(\mathbf{u},\mathbf{p})-\theta\bigr]\,d\mathbf{u}. \tag{4.13}$$

Here, the closer are all $I^{\Phi}(\mathbf{u},\mathbf{p})$ to θ, the higher are all $\cos\bigl[I^{\Phi}(\mathbf{u},\mathbf{p})-\theta\bigr]$, and the higher is the resulting integral. This means that $\varphi(\mathbf{p})$ represents the angle which is on the average the closest to each $I^{\Phi}(\mathbf{u},\mathbf{p})$, in other words the angle of maximum congruence of the phases $I^{\Phi}(\mathbf{u},\mathbf{p})$ at $\mathbf{p}$ for $\mathbf{u} \in \mathcal{P}_\mathbf{c}$. We call thus $\varphi(\mathbf{p})$ the *average phase at* $\mathbf{p}$.

We define the *phase congruence function* of image I as the function $\bigl(\int_{\mathcal{E}} I^{\mathcal{A}}A\bigr)^{-1}\cdot|\Delta|$; indeed (4.13) and the symmetry of $I^{\mathcal{A}}A$ give:

$$\frac{|\Delta(\mathbf{p})|}{\int_{\mathcal{E}} I^{\mathcal{A}}A} = \max_{\theta\in[0,2\pi[}\frac{\int_{\mathcal{P}_\mathbf{c}} I^{\mathcal{A}}(\mathbf{u})A(\mathbf{u})\cos\bigl[I^{\Phi}(\mathbf{u},\mathbf{p})-\theta\bigr]\,d\mathbf{u}}{\int_{\mathcal{P}_\mathbf{c}} I^{\mathcal{A}}(\mathbf{u})A(\mathbf{u})\,d\mathbf{u}}. \tag{4.14}$$

This function measures the degree to which all Fourier phases at $\mathbf{p}$ are concentrated around $\varphi(\mathbf{p})$. Clearly it does not change when I is multiplied by a constant factor, and it takes values in the interval $[0,1]$. It has value 1 at points $\mathbf{p}$ where all $I^{\Phi}(\mathbf{u},\mathbf{p})$ for $\mathbf{u} \in \mathcal{P}_\mathbf{c}$ are equal, in other words I has constant Fourier phase at $\mathbf{p}$. Such a phase congruence function was considered in [63] in the case of one-dimensional periodic signals.

Note that in all the above equations, to each frequency $\mathbf{u}$ corresponds the non-negative weight $I^{\mathcal{A}}(\mathbf{u})A(\mathbf{u})$; thus the value of the average phase $\varphi(\mathbf{p})$, and the existence of a maximum of E at point $\mathbf{p}$, depend not only on the phases $I^{\Phi}(\mathbf{u},\mathbf{p})$, but also on the amplitude spectra of both I and C. In particular, they are sensitive to the choice of C and S; another pair of filters satisfying the same requirements could lead to other values for $\varphi(\mathbf{p})$, and to other positions for the maxima of the energy function E. However this change should in general be moderate, since in practice filters used in the phase congruence model have in general similar grey-level profiles (somewhat like in (2.1)), and their Fourier transforms have similar shapes (A has a single positive lobe in $\mathcal{P}_{\mathbf{c}}$).

Let us now see how behave Fourier phases of edge profiles described in Section 2, in particular whether such edges will be properly detected and localized by the phase congruence model. As in (4.3), we assume that the image I forms a one-dimensional profile P along direction $\mathbf{c}$, namely:

$$\forall \mathbf{y} \in \mathcal{E}_{\mathbf{c}}, \forall x \in \mathrm{I\!R}, \qquad I(\mathbf{y} + x\mathbf{c}) = P(x).$$

By Proposition 4.5 (see (4.4)), we have:

$$\forall \mathbf{y} \in \mathcal{E}_{\mathbf{c}}, \forall x \in \mathrm{I\!R}, \qquad (I * C)(\mathbf{y} + x\mathbf{c}) = (P * C_{/\mathbf{c}})(x) \quad \text{and} \quad (I * S)(\mathbf{y} + x\mathbf{c}) = (P * S_{/\mathbf{c}})(x).$$

By Proposition 4.4, we can apply then the phase congruence model with P and $D/\mathbf{c}$ instead of I and D, and so we have only to look at the Fourier transform of P. For the sake of simplicity, we assume that P is integrable.

We consider first an ideal line edge profile. In Proposition 4.7 of [52] we showed the following: *Let $P : \mathrm{I\!R} \to \mathrm{I\!R}$ be integrable, even-symmetric, having non-negative values, and convex on $\mathrm{I\!R}^+$, in other words such that for $0 \leq a < b$ and $0 \leq \lambda \leq 1$ we have $P\big(\lambda a + (1-\lambda)b\big) \leq \lambda P(a) + (1-\lambda)P(b)$. Then P has constant zero Fourier phase.*

If we look at the ideal line edge profile shown in Figure 1, we see that it satisfies the above hypothesis, so that such a profile has constant zero phase, and by Proposition 4.6, the energy function will have an absolute maximum at the edge position. On the other hand, the ideal bar profile shown next to it in Figure 1 will have a Fourier transform of the form $\widehat{P}(\nu) = \sin(a\nu)/(b\nu)$, where a and b are positive constants, so that its Fourier phases alternate between 0 and π. We explained in Section 2 that following Horn's model [19], we considered the line as physically more realistic than the bar. We see here that this physical choice has also a mathematical advantage.

Note that there are other line profiles which do not satisfy the above hypothesis, but which have nevertheless constant zero phase (for example a Gaussian).

Let us now consider ideal step edge profiles. In Proposition 4.8 of [52] we showed the following: *Let $P : \mathrm{I\!R} \to \mathrm{I\!R}$ be integrable, odd-symmetric, having decreasing non-negative values on $\mathrm{I\!R}^+$, in other words such that for $0 \leq a < b$ we have $P(a) \geq P(b) \geq 0$. Then P has constant $-\pi/2$ Fourier phase.*

It follows that an odd-symmetric *sharp* step, where the grey-level jumps discontinuously from negative to positive, and then decreases, will have constant $-\pi/2$ phase. This is however not the necessarily the case with the gradual step shown in Figure 1.

Note that there are other step profiles which do not satisfy the above hypothesis, but which have nevertheless constant $-\pi/2$ phase (for example minus the derivative of a Gaussian).

Let us now consider roofs and Mach bands. A symmetric roof as the one illustrated in Figure 1 has constant zero phase; the same can be said with a a roof obtained by scaling up a line edge. Non-symmetric roofs and Mach bands are a linear combination of a symmetric roof and a linear ramp $\xi_{\mathbf{m}}$ (for a unit vector $\mathbf{m}$, for example $\mathbf{m} = \mathbf{c}$); now the convolution of $\xi_{\mathbf{m}}$ with C gives $\xi_{\mathbf{m}} \int C - \int \xi_{\mathbf{m}} C$, and similarly for S. In order for the two convolutions $\xi_{\mathbf{m}} * C$ and $\xi_{\mathbf{m}} * S$ to be properly defined, we must require $\xi_{\mathbf{m}} C$ and $\xi_{\mathbf{m}} S$ to be integrable. But then $\int C = \int S = 0$ by Proposition 3.8, while

Proposition 3.9 gives $\int \xi_{\mathbf{m}} C = \int \xi_{\mathbf{m}} S = 0$, and hence we get $\xi_{\mathbf{m}} * C = \xi_{\mathbf{m}} * S = 0$. Therefore the linear ramp does not contribute to the energy function, which is the same as for the underlying symmetric roof.

Thus all ideal ramp edges and Mach bands give, up to a constant factor, the same energy function; in particular the same edges should be detected, at the same locations. We said in Section 2 that there is not a complete agreement as to the exact position of the line perceived by human observers in a Mach band or roof edge [5,55]. This indicates that the edge detectors of the human visual system may have a non-zero response to the underlying linear ramp, contradicting the above argument. We might also consider that the linear ramp does not extend to infinity, that beyond a given radius it becomes a bounded function, and that the convolution of this bounded ramp with the filters C and S will not be constant, even at the points inside the linear part of the ramp.

We finally consider compound edges consisting of the linear superposition of a line and a step (see Figure 2). We can assume that the line and step have constant phases 0 and $-\pi/2$ (resp. $\pi/2$), so that all phases of the compound edge will be in the same quadrant $[-\pi/2, 0]$ (resp. $[0, \pi/2]$). Thus the congruence of phases will be relatively high at that edge position, but we might have a higher congruence at a neighbouring location. Consider for example a middle row in Figure 6, whose grey-level profile, the addition of a square wave and a triangular wave, is shown in Figure 7. As a periodic function of x, it can be decomposed as a Fourier series of the form

$$a + \sum_{n=1}^{\infty} \big(\frac{\alpha}{n^2} \cos[2\pi n f x] - \frac{\beta}{n} \sin[2\pi n f x] \big),$$

where f is the fundamental frequency, a is a constant corresponding to the frequency 0, and the constants $\alpha, \beta > 0$ determine the degree of mixture between the two waves; thus the Fourier coefficient for frequency nf will be $\frac{\alpha}{n^2} + i\frac{\alpha}{n}$, and in particular the corresponding Fourier phase will be $\arctan[n\beta/\alpha]$. Hence the phase increases with frequency, so that for small $\varepsilon > 0$, the phases

$$P^{\Phi}(nf, -\varepsilon) = \arctan\big[\frac{n\beta}{\alpha}\big] - 2\pi n f \varepsilon$$

at $-\varepsilon$ will generally be more congruent than at the origin. This accords with our visual perception of Figure 6, where in a middle row the feature appears as a combination of an edge and of a Mach band extending slightly to the left of the feature's true position.

Therefore the edge localization given by the phase congruence model can be slightly to the left (or to the right) of the true edge position, but this is not a serious problem, since this offset is generally small. As we will see in the next subsection, a much worse problem would arise with usual methods where one applies the two filters separately, the even-symmetric one in order to detect lines, and the odd-symmetric one in order to detect steps: the two edges detected by both filters would not coincide.

4.3. Other quadratic operators, and the relation to classical edge detectors

We will study here all quadratic combinations of $I * C$ and $I * S$, and give their interpretation in terms of phases and phase congruence; in particular we will consider traditional edge detectors using a single filter, and show that they lead to an edge model which is a restricted form of the phase congruence approach.

We already have the quadratic operator associating to the image I its energy $E = |\Delta|^2 = \Gamma^2 + \Sigma^2$. We introduce two new quadratic combinations of Γ and Σ:

$$\Omega = \Gamma^2 - \Sigma^2 = \Re(\Delta^2) \qquad \text{and} \qquad \Psi = 2\Gamma\Sigma = \Im(\Delta^2), \tag{4.15}$$

in other words

$$\Omega + i\,\Psi = \Delta^2 = (\Gamma + i\,\Sigma)^2. \tag{4.16}$$

We can also write

$$\Omega = \frac{\Delta^2 + \overline{\Delta}^2}{2} \qquad \text{and} \qquad \Psi = \frac{\Delta^2 - \overline{\Delta}^2}{2i}.$$

We have then

$$\Omega^2 + \Psi^2 = |\Delta^2|^2 = E^2.$$

Note that since Γ and Σ are in $\mathbf{c}$-quadrature, $\Omega = \Gamma \odot \Gamma$ and $\Psi = \Gamma \odot \Sigma$ according to the definition of $\odot$ by [44] (cfr. Subsection 3.5). We have the following interpretation of Ω and Ψ in terms of phases:

PROPOSITION 4.8. *Assume that $I \in L^1 + L^2$. Then Ω and Ψ are in $\mathbf{c}$-quadrature, and $\widehat{\Omega}$ and $\widehat{\Psi}$ are integrable. Furthermore:*

(*i*) *If for a given $\mathbf{p} \in \mathcal{E}$ we have $I^\Phi(\mathbf{u},\mathbf{p}) = \phi$ for all $\mathbf{u} \in \mathcal{P}_\mathbf{c}$, then $\Omega^\Phi(\mathbf{u},\mathbf{p}) = 2\phi$ and $\Psi^\Phi(\mathbf{u},\mathbf{p}) = 2\phi - \pi/2$ for all $\mathbf{u} \in \mathcal{P}_\mathbf{c}$.*

(*ii*) *For every $\mathbf{p} \in \mathcal{E}$ we have*

$$\begin{aligned} \Omega(\mathbf{p}) &= 4 \iint_{\mathcal{P}_\mathbf{c}\times\mathcal{P}_\mathbf{c}} I^\mathcal{A}(\mathbf{u})A(\mathbf{u})I^\mathcal{A}(\mathbf{v})A(\mathbf{v}) \cos\bigl[I^\Phi(\mathbf{u},\mathbf{p}) + I^\Phi(\mathbf{v},\mathbf{p})\bigr]\, d\mathbf{u}d\mathbf{v}; \\ \Psi(\mathbf{p}) &= 4 \iint_{\mathcal{P}_\mathbf{c}\times\mathcal{P}_\mathbf{c}} I^\mathcal{A}(\mathbf{u})A(\mathbf{u})I^\mathcal{A}(\mathbf{v})A(\mathbf{v}) \sin\bigl[I^\Phi(\mathbf{u},\mathbf{p}) + I^\Phi(\mathbf{v},\mathbf{p})\bigr]\, d\mathbf{u}d\mathbf{v}. \end{aligned} \tag{4.17}$$

PROOF. An argument similar to that in Lemma 3.12 and in the proof of Proposition 4.6 gives $\widehat{\Delta} = 2\mathrm{pos}_\mathbf{c}\, A\widehat{I}$, $(\Delta^2)^\wedge = \widehat{\Delta} * \widehat{\Delta}$, and the latter is integrable. Similarly $\bigl(\overline{\Delta}^2\bigr)^\wedge$ is integrable, and as $\Omega = \Re(\Delta^2)$ and $\Psi = \Im(\Delta^2)$, both $\widehat{\Omega}$ and $\widehat{\Psi}$ are integrable. Since $\widehat{\Delta}$ vanishes outside $\mathcal{P}_\mathbf{c}$, so does $(\Delta^2)^\wedge = \widehat{\Delta} * \widehat{\Delta}$, and from Subsection 3.5 it follows that Ω and Ψ are in $\mathbf{c}$-quadrature.

If $I^\Phi(\mathbf{u}) = \phi$ for all $\mathbf{u} \in \mathcal{P}_\mathbf{c}$, then $\widehat{\Delta} = 2e^{i\phi}\mathrm{pos}_\mathbf{c}\, AI^\mathcal{A}$ and $\widehat{\Omega} + i\,\widehat{\Psi} = (\Delta^2)^\wedge = 4e^{2i\phi}(\mathrm{pos}_\mathbf{c}\, AI^\mathcal{A}) * (\mathrm{pos}_\mathbf{c}\, AI^\mathcal{A})$, from which we get (with (3.27)) that

$$\forall \mathbf{u} \in \mathcal{P}_\mathbf{c}, \qquad \widehat{\Omega}(\mathbf{u}) = 2e^{2i\phi}\Bigl[(\mathrm{pos}_\mathbf{c}\, AI^\mathcal{A}) * (\mathrm{pos}_\mathbf{c}\, AI^\mathcal{A})\Bigr](\mathbf{u}),$$

so that $\Omega^\Phi(\mathbf{u}) = 2\phi$. As Ω and Ψ are in $\mathbf{c}$-quadrature, we get $\Psi^\Phi(\mathbf{u}) = 2\phi - \pi/2$ for $\mathbf{u} \in \mathcal{P}_\mathbf{c}$.

If $I^\Phi(\mathbf{u},\mathbf{p}) = \phi$ for all $\mathbf{u} \in \mathcal{P}_\mathbf{c}$, then this means that $\bigl(\tau_{-\mathbf{p}}(I)\bigr)^\Phi = \phi$ for all $\mathbf{u} \in \mathcal{P}_\mathbf{c}$; thus the preceding paragraph with $\tau_{-\mathbf{p}}(I)$ instead of I gives $\bigl(\tau_{-\mathbf{p}}(\Omega)\bigr)^\Phi = 2\phi$ and $\bigl(\tau_{-\mathbf{p}}(\Psi)\bigr)^\Phi = 2\phi - \pi/2$, that is $\Omega^\Phi(\mathbf{u},\mathbf{p}) = 2\phi$ and $\Psi^\Phi(\mathbf{u},\mathbf{p}) = 2\phi - \pi/2$. Therefore (*i*) holds.

By (3.15) we have

$$\Delta(\mathbf{p}) = 2 \int_{\mathcal{P}_\mathbf{c}} I^\mathcal{A}(\mathbf{u})A(\mathbf{u}) \exp\bigl[i\, I^\Phi(\mathbf{u},\mathbf{p})\bigr]\, d\mathbf{u},$$

from which we derive that

$$\begin{aligned} \Omega + i\,\Psi = \Delta^2 &= 4 \iint_{\mathcal{P}_\mathbf{c}\times\mathcal{P}_\mathbf{c}} I^\mathcal{A}(\mathbf{u})A(\mathbf{u})I^\mathcal{A}(\mathbf{v})A(\mathbf{v}) \exp\bigl[i\, I^\Phi(\mathbf{u},\mathbf{p})\bigr] \exp\bigl[i\, I^\Phi(\mathbf{v},\mathbf{p})\bigr]\, d\mathbf{u}d\mathbf{v} \\ &= 4 \iint_{\mathcal{P}_\mathbf{c}\times\mathcal{P}_\mathbf{c}} I^\mathcal{A}(\mathbf{u})A(\mathbf{u})I^\mathcal{A}(\mathbf{v})A(\mathbf{v}) \Bigl(\cos\bigl[I^\Phi(\mathbf{u},\mathbf{p}) + I^\Phi(\mathbf{v},\mathbf{p})\bigr] + i\, \sin\bigl[I^\Phi(\mathbf{u},\mathbf{p}) + I^\Phi(\mathbf{v},\mathbf{p})\bigr]\Bigr)\, d\mathbf{u}d\mathbf{v}. \end{aligned}$$

Taking the real and imaginary part of both sides, (4.17) results. ∎

Note that changing the sign of I does not modify Ω and Ψ; thus the latter two functions are invariant to a shift of all phases by π.

It follows from (i) that if for a given $\mathbf{p} \in \mathcal{E}$ we have $I^\Phi(\mathbf{u}, \mathbf{p}) = \phi$ for all $\mathbf{u} \in \mathcal{P}_\mathbf{c}$, then:

— For $\phi = 0$ or $\phi = \pi$, we have $[\tau_{-\mathbf{p}}(\Omega)]^\wedge \geq 0$.

— For $\phi = \pi/4$ or $\phi = -3\pi/4$, we have $[\tau_{-\mathbf{p}}(\Psi)]^\wedge \geq 0$.

— For $\phi = \pm\pi/2$, we have $[\tau_{-\mathbf{p}}(\Omega)]^\wedge \leq 0$.

— For $\phi = 3\pi/4$ or $\phi = -\pi/4$, we have $[\tau_{-\mathbf{p}}(\Psi)]^\wedge \leq 0$.

Alternately, using (ii), we see that $\Omega(\mathbf{p})$ and $\Psi(\mathbf{p})$ measure the extent to which all phases $I^\Phi(\mathbf{u}, \mathbf{p})$ are clustered around certain multiples of $\pi/4$:

(i) As all $I^\Phi(\mathbf{u}, \mathbf{p})$ get close to 0 (resp. π), all $I^\Phi(\mathbf{u}, \mathbf{p}) + I^\Phi(\mathbf{v}, \mathbf{p})$ will approach 0, and so their cosines and sines will tend to 1 and 0 respectively; thus $\Omega(\mathbf{p})$ will increase towards $E(\mathbf{p})$ and $\Psi(\mathbf{p})$ will approach 0.

(ii) As all $I^\Phi(\mathbf{u}, \mathbf{p})$ get close to $\pi/4$ (resp. $-3\pi/4$), all $I^\Phi(\mathbf{u}, \mathbf{p}) + I^\Phi(\mathbf{v}, \mathbf{p})$ will approach $\pi/2$, and so their cosines and sines will tend to 0 and 1 respectively; thus $\Omega(\mathbf{p})$ will approach 0 and $\Psi(\mathbf{p})$ will increase towards $E(\mathbf{p})$.

(iii) As all $I^\Phi(\mathbf{u}, \mathbf{p})$ get close to $\pi/2$ (resp. $-\pi/2$), all $I^\Phi(\mathbf{u}, \mathbf{p}) + I^\Phi(\mathbf{v}, \mathbf{p})$ will approach π, and so their cosines and sines will tend to -1 and 0 respectively; thus $\Omega(\mathbf{p})$ will decrease towards $-E(\mathbf{p})$ and $\Psi(\mathbf{p})$ will approach 0.

(iv) As all $I^\Phi(\mathbf{u}, \mathbf{p})$ get close to $3\pi/4$ (resp. $-\pi/4$), all $I^\Phi(\mathbf{u}, \mathbf{p}) + I^\Phi(\mathbf{v}, \mathbf{p})$ will approach $-\pi/2$, and so their cosines and sines will tend to 0 and -1 respectively; thus $\Omega(\mathbf{p})$ will approach 0 and $\Psi(\mathbf{p})$ will decrease towards $-E(\mathbf{p})$.

More generally, we can link $\Omega(\mathbf{p})$ and $\Psi(\mathbf{p})$ with $\varphi(\mathbf{p})$, the average phase at $\mathbf{p}$. Combining (4.8) and (4.16), we see that

$$\Omega(\mathbf{p}) + i\,\Psi(\mathbf{p}) = \Delta^2(\mathbf{p}) = E(\mathbf{p})e^{2i\varphi(\mathbf{p})},$$

in other words

$$\Omega(\mathbf{p}) = E(\mathbf{p})\cos[2\varphi(\mathbf{p})] \qquad \text{and} \qquad \Psi(\mathbf{p}) = E(\mathbf{p})\sin[2\varphi(\mathbf{p})]. \tag{4.18}$$

While $E(\mathbf{p})$ measures the *extent* to which there is a feature at a point $\mathbf{p}$, the additional information provided by $\Omega(\mathbf{p})$ and $\Psi(\mathbf{p})$ allows us to give the average phase modulo π at that point, and so to describe the *type* of feature encountered, where each type includes both the positive and negative feature. For example a line edge at $\mathbf{p}$ (either dark or light) has $\Omega(\mathbf{p})$ close to $E(\mathbf{p})$ and $\Psi(\mathbf{p})$ close to 0, while a step edge at $\mathbf{p}$ (either dark to light or light to dark) has $\Omega(\mathbf{p})$ close to $-E(\mathbf{p})$ and $\Psi(\mathbf{p})$ close to 0. On the other hand a compound line plus step edge as in Figure 2 (a), having phases close to $-\pi/4$, will give $\Omega(\mathbf{p})$ close to 0 and $\Psi(\mathbf{p})$ close to $-E(\mathbf{p})$; a left-right symmetry of this profile would have phases close to $\pi/4$, and so $\Psi(\mathbf{p})$ close to $E(\mathbf{p})$. As we consider lines and steps as basic edges, we will give more importance to the function Ω than to its counterpart Ψ.

Another classification of the type of a feature is given in [63]; it relies on an examination of maxima, minima, and zero-crossings of Γ and Σ rather than Ω and Ψ; in other words (cfr. (4.9) and (4.18)), it is based on $\varphi(\mathbf{p})$ rather than on $2\varphi(\mathbf{p})$.

An interesting fact is that any quadratic combination of $\Gamma = I * C$ and $\Sigma = I * S$, being of the form $a\Gamma^2 + b\Sigma^2 + c\Gamma\Sigma$, will be a linear combination of $E = \Gamma^2 + \Sigma^2$, $\Omega = \Gamma^2 - \Sigma^2$, and $\Psi = 2\Gamma\Sigma$. Thus every quadratic combination of Γ and Σ can be interpreted in terms of Fourier phases.

As an illustration, we consider Γ^2 and Σ^2, which are associated to traditional approaches to edge detection. Usually one convolves the image I with a single filter G, which can be a derivative of a Gaussian, a Gabor cosine or sine function, etc.; generally one of the following holds:

(a) With the aim of detecting line edges, G is even-symmetric, and has constant zero Fourier phase; thus one can consider that $G = C$.

(b) With the aim of detecting step edges, G is odd-symmetric, and has constant Fourier phase $\pi/2$ (or $-\pi/2$); thus one can consider that $G = \pm S$.

Then edges are localized at local maxima of $|I * G|$, or equivalently of $(I * G)^2$. Thus we have only to consider maxima of Γ^2 (in (a)) or Σ^2 (in (b)). Since $\Gamma^2 = (E + \Omega)/2$ and $\Sigma^2 = (E - \Omega)/2$, (4.7) and (4.17) give:

$$\begin{aligned}\Gamma^2(\mathbf{p}) = 2 \iint_{\mathcal{P}_\mathbf{c}\times\mathcal{P}_\mathbf{c}} & d\mathbf{u}d\mathbf{v}\, I^\mathcal{A}(\mathbf{u})A(\mathbf{u})I^\mathcal{A}(\mathbf{v})A(\mathbf{v}) \\ & \Big(\cos\big[I^\Phi(\mathbf{u},\mathbf{p}) - I^\Phi(\mathbf{v},\mathbf{p})\big] + \cos\big[I^\Phi(\mathbf{u},\mathbf{p}) + I^\Phi(\mathbf{v},\mathbf{p})\big]\Big); \\ \Sigma^2(\mathbf{p}) = 2 \iint_{\mathcal{P}_\mathbf{c}\times\mathcal{P}_\mathbf{c}} & d\mathbf{u}d\mathbf{v})\, I^\mathcal{A}(\mathbf{u})A(\mathbf{u})I^\mathcal{A}(\mathbf{v})A(\mathbf{v}) \\ & \Big(\cos\big[I^\Phi(\mathbf{u},\mathbf{p}) - I^\Phi(\mathbf{v},\mathbf{p})\big] - \cos\big[I^\Phi(\mathbf{u},\mathbf{p}) + I^\Phi(\mathbf{v},\mathbf{p})\big]\Big).\end{aligned} \tag{4.19}$$

Thus Γ^2 and Σ^2 measure mixed aspects of phase congruence. Both phase congruence in general and closeness of phases with 0 (or π) contribute to maxima of Γ^2; thus congruence of phases in the interval $[-\pi/8, \pi/8]$ will give a relatively high value for Γ^2. Similarly, both phase congruence in general and closeness of phases with $\pi/2$ (or $-\pi/2$) contribute to maxima of Σ^2; thus congruence of phases in the interval $[3\pi/8, 5\pi/8]$ will give a relatively high value for Σ^2. We have also the following analogue of item (ii) of Proposition 4.6:

(a) If for a given $\mathbf{p} \in \mathcal{E}$ we have $I^\Phi(\mathbf{u},\mathbf{p}) = 0$ for all $\mathbf{u} \in \mathcal{P}_\mathbf{c}$, then $\Gamma^\Phi(\mathbf{u},\mathbf{p}) = (\Gamma^2)^\Phi(\mathbf{u},\mathbf{p}) = 0$ for all $\mathbf{u} \in \mathcal{P}_\mathbf{c}$, and $\Gamma^2(\mathbf{p}) > \Gamma^2(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{E}$ such that $\mathbf{x} \neq \mathbf{p}$.

(b) If for a given $\mathbf{p} \in \mathcal{E}$ we have $I^\Phi(\mathbf{u},\mathbf{p}) = \pi/2$ for all $\mathbf{u} \in \mathcal{P}_\mathbf{c}$, then $\Sigma^\Phi(\mathbf{u},\mathbf{p}) = (\Sigma^2)^\Phi(\mathbf{u},\mathbf{p}) = 0$ for all $\mathbf{u} \in \mathcal{P}_\mathbf{c}$, and $\Sigma^2(\mathbf{p}) > \Sigma^2(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{E}$ such that $\mathbf{x} \neq \mathbf{p}$.

Every edge detector has (at least implicitly) an underlying model of an "ideal edge", which is always detected and localized at its exact position, even if the scale of the edge detector is changed by a modification of the size of the filters. From what we have seen, an ideal edge at the origin should be an image I such that the corresponding "energy function" obtained after application of the edge operator (namely $E = \Gamma^2 + \Sigma^2$, Γ^2, or Σ^2) has constant zero Fourier phase, so that it gives an absolute maximum at the origin, whatever the scale of the filter. We see thus that perfect edges for the single filter line detector using Γ^2 are given by signals having constant 0 (or π) phase, while perfect edges for the single filter step detector using Σ^2 are given by signals having constant $\pi/2$ (or $-\pi/2$) phase. On the other hand, perfect edges in the phase congruence model are given by all constant phase signals, whatever the value of that constant phase. In other words the phase congruence model generalizes previous approaches.

Some authors [1,13,17,26,27,45,46,47,54] have considered quadratic edge detectors using two filters having constant Fourier phases 0 and $-\pi/2$ respectively, but having different amplitude spectra, for example: the Gabor cosine and sine functions, the first and second derivatives of a Gaussian, etc. What kind of edges do they detect? We have the following result:

PROPOSITION 4.9. *Let G and Z be two real-valued continuous integrable functions such that $G^\Phi(\mathbf{u}) = 0$ and $Z^\Phi(\mathbf{u}) = -\pi/2$ for all $\mathbf{u} \in \mathcal{P}_\mathbf{c}$. Let I be in $L^1 + L^2$, and set $Q = (I * G)^2 + (I * Z)^2$. Suppose that there is some point $\mathbf{p} \in \mathcal{E}$ and some angle θ such that $I^\Phi(\mathbf{u}, \mathbf{p}) = \theta$ for all $\mathbf{u} \in \mathcal{P}_\mathbf{c}$, and one of the following holds:*

(*i*) $G^\mathcal{A} \geq Z^\mathcal{A}$ *and* $\theta = 0$ *or* π.

(*ii*) $G^\mathcal{A} \leq Z^\mathcal{A}$ *and* $\theta = \pm\pi/2$.

Then $Q^\Phi(\mathbf{u}, \mathbf{p}) = 0$ for all $\mathbf{u} \in \mathcal{E}$, and $Q(\mathbf{p}) > Q(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{E}$ such that $\mathbf{x} \neq \mathbf{p}$.

PROOF. As in Proposition 4.2, $I * G$ and $I * Z$ are uniformly continuous and square-integrable, with $(I * G)^\wedge = \widehat{I}\widehat{G}$ and $(I * Z)^\wedge = \widehat{I}\widehat{Z}$. We have $[\tau_{-\mathbf{p}}(I)]^\Phi(\mathbf{u}) = \theta$ for all $\mathbf{u} \in \mathcal{P}_\mathbf{c}$, so that $[\tau_{-\mathbf{p}}(I * G)]^\Phi(\mathbf{u}) = \theta$ and $[\tau_{-\mathbf{p}}(I * Z)]^\Phi(\mathbf{u}) = \theta - \pi/2$. Using Lemma 3.12 as in the proof of Proposition 4.6, we obtain that $[\tau_{-\mathbf{p}}(I * G)^2]^\wedge = \tau_{-\mathbf{p}}(I * G)^\wedge * \tau_{-\mathbf{p}}(I * G)^\wedge$ and $[\tau_{-\mathbf{p}}(I * Z)^2]^\wedge = \tau_{-\mathbf{p}}(I * Z)^\wedge * \tau_{-\mathbf{p}}(I * Z)^\wedge$.

If (*i*) holds, then $\tau_{-\mathbf{p}}(I * G)$ has constant phase 0 or π on $\mathcal{E}$, and $\tau_{-\mathbf{p}}(I * Z)$ has constant phase $\pm\pi/2$ on $\mathcal{P}_\mathbf{c}$, in other words $[\tau_{-\mathbf{p}}(I * G)]^\wedge = \pm(I * G)^\mathcal{A}$ and $[\tau_{-\mathbf{p}}(I * Z)]^\wedge = \pm i \operatorname{sgn}_\mathbf{c} \cdot (I * Z)^\mathcal{A}$. Hence

$$\begin{aligned}
[\tau_{-\mathbf{p}}(Q)]^\wedge &= \left[\tau_{-\mathbf{p}}[(I * G)^2 + \tau_{-\mathbf{p}}(I * Z)^2\right]^\wedge \\
&= [\tau_{-\mathbf{p}}(I * G)]^\wedge * [\tau_{-\mathbf{p}}(I * G)]^\wedge + [\tau_{-\mathbf{p}}(I * Z)]^\wedge * [\tau_{-\mathbf{p}}(I * Z)]^\wedge \\
&= (\pm 1)^2 (I * G)^\mathcal{A} * (I * G)^\mathcal{A} + (\pm i \operatorname{sgn}_\mathbf{c})^2 (I * Z)^\mathcal{A} * (I * Z)^\mathcal{A} \\
&= (I * G)^\mathcal{A} * (I * G)^\mathcal{A} - (I * Z)^\mathcal{A} * (I * Z)^\mathcal{A}.
\end{aligned}$$

As $(I * G)^\mathcal{A} \geq (I * Z)^\mathcal{A}$, $[\tau_{-\mathbf{p}}(Q)]^\wedge$ is a non-negative function.

If (*ii*) holds then $\tau_{-\mathbf{p}}(I * Z)$ has constant phase 0 or π on $\mathcal{E}$, and $\tau_{-\mathbf{p}}(I * G)$ has constant phase $\pm\pi/2$ on $\mathcal{P}_\mathbf{c}$. We intervert $I * G$ and $I * Z$ in the above argument, and obtain again that $[\tau_{-\mathbf{p}}(Q)]^\wedge$ is a non-negative function.

Thus in both cases $\tau_{-\mathbf{p}}(Q)$ has constant zero Fourier phase. As Q is continuous, $\widehat{Q}$ is integrable by Proposition 3.4. By Corollary 3.6, $Q(\mathbf{p}) > Q(\mathbf{x})$ for $\mathbf{x} \neq \mathbf{p}$. ∎

For example, if G and Z are the Gabor cosine and sine functions, as $G^\mathcal{A} \geq Z^\mathcal{A}$, the edge detector will detect perfect line edges, which have a constant phase 0. For signals having constant phase $\pm\pi/2$, the detection of the corresponding edge is not guaranteed.

Another way of using two filters is to build two edge detectors, one for line edges using the even-symmetric filter, and one for step edges using the odd-symmetric filter, and to apply both detectors in parallel to the image. We show below that for a compound edge formed by the superposition of two signals with constant phases 0 and $\pi/2$ respectively, this leads to edge duplication, in other words the two edge detectors localize the edge on opposite sides of its true location:

PROPOSITION 4.10. *Let I be given by $I(\mathbf{x}) = P(x_c)$ for $x_c = \mathbf{x} \cdot \mathbf{c}$ (cfr. (4.3)), where $P \in L^1 + L^2$, P is neither even-symmetric nor odd-symmetric, and $P^\Phi(\nu) \in [0, \pi/2]$ for all $\nu > 0$. Consider two integrable functions $G, Z : \mathcal{E} \to \mathrm{I\!R}$, with $g = G_{/\mathbf{c}}$ and $z = Z_{/\mathbf{c}}$, such that:*

(*i*) *$P * g$ and $P * z$ are continuous;*

(*ii*) *g is even-symmetric and has constant zero Fourier phase;*

(*iii*) *z is odd-symmetric and has constant Fourier phase $-\pi/2$;*

(*iv*) *$\widehat{P}\widehat{g}$, $\widehat{P}\widehat{z}$, $\xi\widehat{P}\widehat{g}$, and $\xi\widehat{P}\widehat{z}$ are integrable, in other words*

$$\int_{\mathrm{I\!R}} (1 + |x|)|\widehat{P}(x)|(|\widehat{g}(x)| + |\widehat{z}(x)|)\, dx < \infty.$$

Then $I*G$ and $I*Z$ are derivable in x_c, and for $\mathbf{y}\in\mathcal{E}_\mathbf{c}$ we have

$$\frac{\partial[(I*G)^2]}{\partial x_c}(\mathbf{y})\cdot\frac{\partial[(I*Z)^2]}{\partial x_c}(\mathbf{y})<0.$$

Proof. By Proposition 4.5, $(I*G)(\mathbf{x})=(P*g)(x_c)$ and $(I*Z)(\mathbf{x})=(P*z)(x_c)$ (see (4.4)). Since $P*g$ and $P*z$ are continuous and their Fourier transforms $\widehat{P}\widehat{g}$ and $\widehat{P}\widehat{z}$ are integrable, we can apply Lemma 3.1, and so $P*g=(\widehat{P}\widehat{g})^\vee$ and $P*z=(\widehat{P}\widehat{z})^\vee$.

Let $P_e=(P+P_\rho)/2$ and $P_o=(P-P_\rho)/2$ be the even-symmetric and odd-symmetric parts of P, in other words $P=P_e+P_o$, where P_e is even-symmetric and P_o is odd-symmetric. Since P is neither even-symmetric nor odd-symmetric, P_e and P_o are non-zero. As $P^\Phi(\nu)\in[0,\pi/2]$ for all $\nu>0$, we deduce that $P_e^\Phi(\nu)=0$ and $P_o^\Phi(\nu)=\pi/2$ for all $\nu>0$. Since the Fourier transform is linear and commutes with the reflection ρ, we have $P_e*g=(\widehat{P}_e\widehat{g})^\vee$, $P_o*g=(\widehat{P}_o\widehat{g})^\vee$, $P_e*z=(\widehat{P}_e\widehat{z})^\vee$ and $P_o*z=(\widehat{P}_o\widehat{z})^\vee$. Moreover, as $\xi\widehat{P}\widehat{g}$ and $\xi\widehat{P}\widehat{z}$ are integrable, the same will be true for $\xi\widehat{P}_e\widehat{g}$, $\xi\widehat{P}_o\widehat{g}$, $\xi\widehat{P}_e\widehat{z}$, and $\xi\widehat{P}_o\widehat{z}$; by the L^1 Fourier derivative formula, P_e*g, P_o*g, P_e*z, and P_o*z will be derivable, with $(P_e*g)'=2\pi i\,(\xi\widehat{P}_e\widehat{g})^\vee$, $(P_o*g)'=2\pi i\,(\xi\widehat{P}_o\widehat{g})^\vee$, $(P_e*z)'=2\pi i\,(\xi\widehat{P}_e\widehat{z})^\vee$ and $(P_o*z)'=2\pi i\,(\xi\widehat{P}_o\widehat{z})^\vee$.

As P_e and g are even-symmetric while P_o and z are odd-symmetric, we deduce that P_o*g, P_e*z, $(P_e*g)'$, and $(P_o*z)'$ are odd-symmetric; in particular

$$(P_o*g)(0)=(P_e*z)(0)=(P_e*g)'(0)=(P_o*z)'(0)=0.$$

We obtain then:

$$\begin{aligned}[(P*g)^2]'(0)&=2(P*g)(0)(P*g)'(0)\\&=2\big[(P_o*g)(0)+(P_e*g)(0)\big]\big[(P_o*g)'(0)+(P_e*g)'(0)\big]\\&=2(P_e*g)(0)(P_o*g)'(0)\end{aligned}\tag{4.20}$$

and

$$\begin{aligned}[(P*z)^2]'(0)&=2(P*z)(0)(P*z)'(0)\\&=2\big[(P_o*z)(0)+(P_e*z)(0)\big]\big[(P_o*z)'(0)+(P_e*z)'(0)\big]\\&=2(P_o*z)(0)(P_e*z)'(0).\end{aligned}\tag{4.21}$$

Now (3.13) gives:

$$\begin{aligned}(P_e*g)(0)&=\int_\mathcal{E}\widehat{P}_e(\nu)\widehat{g}(\nu)\,d\nu,\\(P_o*g)'(0)&=\int_\mathcal{E}2\pi i\,\nu\widehat{P}_o(\nu)\widehat{g}(\nu)\,d\nu,\\(P_o*z)(0)&=\int_\mathcal{E}\widehat{P}_o(\nu)\widehat{z}(\nu)\,d\nu,\\\text{and}\qquad(P_e*z)'(0)&=\int_\mathcal{E}2\pi i\,\nu\widehat{P}_e(\nu)\widehat{z}(\nu)\,d\nu.\end{aligned}\tag{4.22}$$

Now since $P_e^\Phi(\nu)=g^\Phi(\nu)=0$, $P_o^\Phi(\nu)=\pi/2$, and $z^\Phi(\nu)=-\pi/2$ for all $\nu>0$, we have thus for all $\nu\neq0$: $\operatorname{sgn}(\widehat{P}_e(\nu))=\operatorname{sgn}(\widehat{g}(\nu))=1$, $\operatorname{sgn}(\widehat{P}_o(\nu))=i\operatorname{sgn}(\nu)$, and $\operatorname{sgn}(\widehat{z}(\nu))=-i\operatorname{sgn}(\nu)$, so that:

$$\begin{aligned}\operatorname{sgn}\big(\widehat{P}_e(\nu)\widehat{g}(\nu)\big)&=1\cdot1=1,\\\operatorname{sgn}\big(2\pi i\,\nu\widehat{P}_o(\nu)\widehat{g}(\nu)\big)&=i\cdot\operatorname{sgn}(\nu)\cdot i\operatorname{sgn}(\nu)\cdot1=-1,\\\operatorname{sgn}\big(\widehat{P}_o(\nu)\widehat{z}(\nu)\big)&=i\operatorname{sgn}(\nu)\cdot[-i\operatorname{sgn}(\nu)]=1,\\\text{and}\qquad\operatorname{sgn}\big(2\pi i\,\nu\widehat{P}_e(\nu)\widehat{z}(\nu)\big)&=i\cdot\operatorname{sgn}(\nu)\cdot1\cdot[-i\operatorname{sgn}(\nu)]=1.\end{aligned}$$

Putting this inside (4.22), we obtain:

$$\operatorname{sgn}\big((P_e * g)(0)\big) = 1, \quad \operatorname{sgn}\big((P_o * g)'(0)\big) = -1, \quad \operatorname{sgn}\big((P_o * z)(0)\big) = 1, \quad \text{and} \quad \operatorname{sgn}\big((P_e * z)'(0)\big) = 1.$$

Combining this with (4.20) and (4.21), we get:

$$\begin{aligned} &\operatorname{sgn}\big([(P * g)^2]'(0) \cdot [(P * z)^2]'(0)\big) \\ =&\operatorname{sgn}\big((P_e * g)(0)\big) \cdot \operatorname{sgn}\big((P_o * g)'(0)\big) \cdot \operatorname{sgn}\big((P_o * z)(0)\big) \cdot \operatorname{sgn}\big((P_e * z)'(0)\big) \\ =&1 \cdot (-1) \cdot 1 \cdot 1 = -1. \end{aligned}$$

Since $(I * G)(\mathbf{x}) = (P * g)(x_c)$ and $(I * Z)(\mathbf{x}) = (P * z)(x_c)$, where $P * g$ and $P * z$ are derivable, $I * G$ and $I * Z$ are derivable in x_c, and for $\mathbf{y} \in \mathcal{E}_\mathbf{c}$ we have $\partial[(I * G)^2]/\partial x_c(\mathbf{y}) = [(P * g)^2]'(0)$ and $\partial[(I * Z)^2]/\partial x_c(\mathbf{y}) = [(P * z)^2]'(0)$. Therefore the result follows. ∎

Thus at the hyperspace $\mathcal{E}_\mathbf{c}$ given by $x_c = 0$ in the profile P, one of $|I * G|$ and $|I * Z|$ is strictly increasing in x_c, while the other is strictly decreasing in x_c. Therefore the maxima of $|I * G|$ and $|I * Z|$ in the normal direction $\mathbf{c}$ lie on both sides of the line $x_c = 0$. This will be the case for an edge which is the linear superposition of an ideal step edge and a line edge as in Figure 2 (a).

4.4. Orientation selectivity

Many previous studies of quadratic models for edge detection, in particular of the phase congruence model [26,27,37,38,39,45,47,51,63] assumed one-dimensional signals and filters. Here we will consider two-dimensional signals and the problems associated with the choice of the filter orientation. Up to now, we have supposed that the filters C and S have a fixed orientation $\mathbf{c}$, and all one-dimensional edge profiles were chosen to have their normal orientation parallel to it. We will now examine what happens when the orientation of the filters is allowed to vary.

We take thus $d \geq 2$. Let R be a rotation of $\mathcal{E}$; for every unit vector $\mathbf{n}$ we have $R(\mathcal{E}_\mathbf{n}) = \mathcal{E}_{R(\mathbf{n})}$ and $R(\mathcal{P}_\mathbf{n}) = \mathcal{P}_{R(\mathbf{n})}$. We can also apply R to filters and signals, and so for every function $F : \mathcal{E} \to \mathbb{C}$ we define $R(F)$ by

$$R(F)\big(R(\mathbf{x})\big) = F(\mathbf{x}),$$

in other words

$$R(F)(\mathbf{x}) = F\big(R^{-1}(\mathbf{x})\big). \tag{4.23}$$

We know that the rotation R commutes with all algebraic operations on functions, that it distributes the convolution, namely

$$R(F * G) = R(F) * R(G),$$

and that it commutes with the Fourier transform, in other words

$$[R(F)]^\wedge = R(\widehat{F}).$$

Note that $R(\operatorname{sgn}_\mathbf{n}) = \operatorname{sgn}_{R(\mathbf{n})}$, so that for two functions F and G in $\mathbf{n}$-quadrature,

$$[R(G)]^\wedge = R(\widehat{G}) = R(-i \operatorname{sgn}_\mathbf{n} \cdot \widehat{F}) = -i\, R(\operatorname{sgn}_\mathbf{n}) \cdot R(\widehat{F}) = -i \operatorname{sgn}_{R(\mathbf{n})} \cdot [R(G)]^\wedge,$$

in other words $R(F)$ and $R(G)$ will be in $R(\mathbf{n})$-quadrature. From (3.4) and (4.23) we obtain for all $x \in \mathbb{R}$:

$$\begin{aligned} R(F)_{/\mathbf{n}}(t) &= \int_{\mathcal{E}_\mathbf{n}} R(F)(\mathbf{y} + t\mathbf{n})\, d\mathbf{y} = \int_{\mathcal{E}_\mathbf{n}} F\big(R^{-1}(\mathbf{y} + t\mathbf{n})\big)\, d\mathbf{y} \\ &= \int_{\mathcal{E}_{R^{-1}(\mathbf{n})}} F\big(\mathbf{z} + tR^{-1}(\mathbf{n})\big)\, d\mathbf{z} = F_{/R^{-1}(\mathbf{n})}(t) \end{aligned},$$

that is

$$R(F)_{/\mathbf{n}} = F_{/R^{-1}(\mathbf{n})}. \tag{4.24}$$

(3.7) gives then

$$\big(R(F)_{/\mathbf{n}}\big)^\wedge(\nu) = \big(F_{/R^{-1}(\mathbf{n})}\big)^\wedge(\nu) = \widehat{F}(\nu R^{-1}(\mathbf{n})). \tag{4.25}$$

We will now consider the behaviour of the rotated filters $R(C)$, $R(S)$, and $R(D) = R(C) + i\,R(S)$ on an image I forming a one-dimensional profile P along a normal direction $\mathbf{n}$ (cfr. (4.3)):

$$\forall \mathbf{y} \in \mathcal{E}_{\mathbf{n}}, \forall x \in \mathbb{R}, \qquad I(\mathbf{y} + x\mathbf{n}) = P(x). \tag{4.26}$$

By Proposition 4.5 (see (4.4)), we have for every integrable function F:

$$\forall \mathbf{y} \in \mathcal{E}_{\mathbf{n}}, \forall x \in \mathbb{R}, \qquad (I * F)(\mathbf{y} + x\mathbf{n}) = (P * F_{/\mathbf{n}})(x). \tag{4.27}$$

By (4.24) we get:

$$\forall \mathbf{y} \in \mathcal{E}_{\mathbf{n}}, \forall x \in \mathbb{R}, \qquad \big(I * R(F)\big)(\mathbf{y} + x\mathbf{n}) = \big(P * R(F)_{/\mathbf{n}}\big)(x) = \big(P * F_{/R^{-1}(\mathbf{n})}\big)(x). \tag{4.28}$$

For $P \in L^1 + L^2$ we have then by (4.25):

$$\big[P * F_{/R^{-1}(\mathbf{n})}\big]^\wedge(\nu) = \widehat{P}(\nu)\big(F_{/R^{-1}(\mathbf{n})}\big)^\wedge(\nu) = \widehat{P}(\nu)\widehat{F}(\nu R^{-1}(\mathbf{n})). \tag{4.29}$$

Since the filters C, S, and D are oriented along the unit vector $\mathbf{c}$, the rotated filters $R(C)$, $R(S)$, and $R(D)$ will be oriented along $R(\mathbf{c})$; this orientation is not necessarily the same as that of the profile of I, given by the unit vector $\mathbf{n}$. It is to be expected that the result of applying these rotated filters to I will depend on the angle between $\mathbf{n}$ and $R(\mathbf{c})$; we obtain indeed the following result:

PROPOSITION 4.11. *Let $d \geq 2$ and I be given by (4.26) for a one-dimensional profile $P \in L^1 + L^2$. Let R be a rotation of $\mathcal{E}$ such that $R^{-1}(\mathbf{n}) \cdot \mathbf{c} = \mathbf{n} \cdot R(\mathbf{c}) > 0$. Then the Fourier phases of the one-dimensional profile $P * D_{/R^{-1}(\mathbf{n})}$ of $I * R(D)$ are the same as those of $P * D_{/\mathbf{c}}$, and the same holds with C and S instead of D. The one-dimensional phase congruence model applies to the one-dimensional profiles of $I * R(D)$, $I * R(C)$, and $I * R(S)$. In particular if P has local phase $P^\Phi(\nu, p)$ constant at point p for all $\nu > 0$, then the energy function will have an absolute maximum at the hyperplane $\mathbf{x} \cdot \mathbf{n} = p$.*

PROOF. As $R^{-1}(\mathbf{n}) \cdot \mathbf{c} = \mathbf{n} \cdot R(\mathbf{c}) > 0$, $R^{-1}(\mathbf{n}) \in \mathcal{P}_{\mathbf{c}}$, so by (4.1, 4.2) and (4.24, 4.25), $C_{/R^{-1}(\mathbf{n})}$, $S_{/R^{-1}(\mathbf{n})}$, and $D_{/R^{-1}(\mathbf{n})}$ have the same phases as $C_{/\mathbf{c}}$, $S_{/\mathbf{c}}$, and $D_{/\mathbf{c}}$ respectively; by Proposition 4.4, they satisfy Axioms 1, 2, and 3 for $d = 1$. Combining this fact with (4.27, 4.28, 4.29), it follows that the phase congruence model applies to the one-dimensional profile of $I * R(D)$. By Proposition 4.6, the constancy of the local phase $P^\Phi(\nu, p)$ at p for $\nu > 0$ leads to an absolute maximum of $|P * D_{/R^{-1}(\mathbf{n})}|^2$ at p, in other words by (4.28), the energy function $|I * R(D)|^2$ has an absolute maximum at the hyperplane $\mathbf{x} \cdot \mathbf{n} = p$. ∎

For $R^{-1}(\mathbf{n}) \cdot \mathbf{c} = \mathbf{n} \cdot R(\mathbf{c}) < 0$, that is $R^{-1}(\mathbf{n}) \in \mathcal{N}_{\mathbf{c}}$, we have by (4.2):

$$\big(C_{/R^{-1}(\mathbf{n})}\big)^\wedge(\nu) = A(\nu R^{-1}(\mathbf{n})) \qquad \text{and} \qquad \big(S_{/R^{-1}(\mathbf{n})}\big)^\wedge(\nu) = +i\,\mathrm{sgn}\nu\, A(\nu R^{-1}(\mathbf{n})),$$

which means that the one-dimensional phase congruence model (Axioms 1, 2, and 3 for $d = 1$) are verified if we replace S by $-S$. Since the energy function E takes the sum of squares of convolutions by $R(C)$ and $R(S)$, Proposition 4.11 remains thus essentially true for $R^{-1}(\mathbf{n})\cdot\mathbf{c}=\mathbf{n}\cdot R(\mathbf{c})<0$. Note that for $R^{-1}(\mathbf{n})\cdot\mathbf{c}=\mathbf{n}\cdot R(\mathbf{c})=0$, that is $R^{-1}(\mathbf{n})\in\mathcal{E}_{\mathbf{c}}$, we have $C_{/R^{-1}(\mathbf{n})}$ and $S_{/R^{-1}(\mathbf{n})}$ identically zero by Lemma 4.3.

The concrete meaning of this result is that a one-dimensional feature can be correctly localized even when the normal orientation of the filters does not match that of the feature, provided that they are not perpendicular. In practice, as the angle between the normal orientations of the feature and of the filters tends to $\pi/2$, the amplitude of $D_{/R^{-1}(\mathbf{n})}$ will diminish, and quantization errors will prevent the localization of maxima of the energy function.

After the localization of the edge, the next problem is the determination of its orientation. Traditional approaches based on the convolution of the image with a single mask rotated into several orientations, select at every point the orientation for which the absolute value of the convolution is the highest. The early rationale behind this procedure was that the grey-level profile of the mask was chosen to represent a local template of an ideal (step or line) edge, and edge detection could thus be achieved as a form of template matching: the higher the correlation with a template with a certain orientation, the higher the likelihood of having there such an edge template with that orientation.

Since we are using convolution kernels specified by analytic properties, in particular in the Fourier domain, and do not consider them as templates for an edge profile, the approach derived from template matching is not guaranteed to work properly. We will show it through three simple examples with two-dimensional images, involving some peculiar filters applied to an ideal step edge.

We assume temporarily that $d=2$. We choose $\mathbf{n}=\mathbf{c}$ and take a unit vector $\mathbf{t}$ perpendicular to $\mathbf{n}$; these two vectors will represent the *tangential* and *normal* directions to that edge; we express every vector $\mathbf{x}\in\mathcal{E}$ as a pair (x_t,x_n) of coordinates in the basis $\{\mathbf{t},\mathbf{n}\}$. The rotation R is characterized by its oriented angle θ; it is the angle from $\mathbf{n}=\mathbf{c}$ to $R(\mathbf{c})$ (or equivalently, from $R^{-1}(\mathbf{n})$ to $\mathbf{c}=\mathbf{n}$). Then the condition $R^{-1}(\mathbf{n})\cdot\mathbf{c}=\mathbf{n}\cdot R(\mathbf{c})>0$ means simply that $|\theta|<\pi/2$. Given a filter $F:\mathcal{E}\to\mathbb{R}$, we can write F_θ for $R(F)$.

Let the image profile be given by a two-dimensional Heaviside step edge:

$$I(x_t,x_n)=P(x_n)=\begin{cases}0 & \text{if } x_n<0,\\ 1 & \text{if } x_n>0,\end{cases}\tag{4.30}$$

as illustrated in Figure 12, top left.

Let us consider first a filter F whose support is restricted to the normal direction, having no extent in the tangential direction. The filter can be considered as a generalized function $(x_t,x_n)\mapsto\delta(x_t)\cdot f(x_n)$, where δ is the Dirac impulse and f is an integrable function $\mathbb{R}\to\mathbb{R}$; in fact, F is a tempered distribution $J\mapsto\int_{\mathbb{R}}J(0,x_n)f(x_n)\,dx$. Take an angle θ such that $|\theta|<\pi/2$; the convolution $I*F_\theta$ of the step I with the rotated filter $R(F)=F_\theta$ gives at every point $\mathbf{p}=(p_t,p_n)$:

$$\begin{aligned}(I*F_\theta)(\mathbf{p}) &= \int_{\mathbb{R}^2}I(\mathbf{p}-\mathbf{y})F_\theta(\mathbf{y})\,d\mathbf{y}=\int_{\mathbb{R}^2}I\big(\mathbf{p}-R(\mathbf{x})\big)R(F)\big(R(\mathbf{x})\big)\,d\mathbf{x} \qquad (\mathbf{y}=R(\mathbf{x}))\\ &= \int_{\mathbb{R}^2}I\big(\mathbf{p}-R(\mathbf{x})\big)F(\mathbf{x})\,d\mathbf{x}=\int_{\mathbb{R}^2}I\big(\mathbf{p}-R(x_t,x_n)\big)\delta(x_t)f(x_n)\,dx_t dx_n\\ &= \int_{\mathbb{R}}I\big(\mathbf{p}-R(0,x_n)\big)f(x_n)\,dx_n=\int_{\mathbb{R}}I(p_t-y\sin\theta,p_n-y\cos\theta)f(y)\,dy\\ &= \int_{\mathbb{R}}P(p_n-y\cos\theta)f(y)\,dy.\end{aligned}$$

Since the profile P is given by the Heaviside step function (4.30), we have $P(p_n - y\cos\theta) = 1$ for $y < p_n/\cos\theta$, and 0 for $y > p_n/\cos\theta$, so that we get

$$(I * F_\theta)(p_t, p_n) = \int_{-\infty}^{p_n/\cos\theta} f(y)\,dy. \tag{4.31}$$

Geometrically speaking, this convolution by F_θ is constructed as follows: at every point $\mathbf{p} = (p_t, p_n)$, draw a line making an angle θ with the normal direction; we integrate on this line the reflected function f_ρ multipled by the edge profile, which means in fact that we integrate f_ρ on the portion of this line lying on the right side of the Heaviside step edge. This is illustrated in Figure 12, top right.

There is no reason for having a maximum of the absolute value of (4.31) for $\theta = 0$. Indeed for $p_n = 0$ the value of (4.31) does not depend on θ. Furthermore, we show below how for $p_n \neq 0$ it is possible to have for the absolute value of (4.31) an absolute maximum at $\theta \neq 0$, but no local maximum at $\theta = 0$. We illustrate in Figure 12 two possible profiles for the function f, one even-symmetric and the other odd-symmetric. We make the simple assumption that the zero-crossings of f are $-a, a$ in the even-symmetric case, and $-a, 0, a$ in the odd-symmetric case; here $f(x) \geq 0$ for $-a \leq x \leq 0$ and $f(x) < 0$ for $x < -a$. Let

$$\alpha = \int_{-a}^{0} f(y)\,dy \qquad \text{and} \qquad \beta = -\int_{-\infty}^{-a} f(y)\,dy,$$

so that $\alpha, \beta > 0$. Take p_n such that $-a < p_n < 0$. Note that (4.31) is symetric in θ: $(I * F_\theta)(p_t, p_n) = (I * F_{-\theta})(p_t, p_n)$; we have thus only to examine its evolution as a function of $|\theta|$. As $|\theta|$ decreases from $\pi/2$ to $\arccos[p_n/(-a)]$, $p_n/\cos\theta$ increases from $-\infty$ to $-a$, so that (4.31) decreases from 0 to $-\beta$; as $|\theta|$ decreases further from $\arccos[p_n/(-a)]$ to 0, $p_n/\cos\theta$ increases from $-a$ to p_n, so that (4.31) increases to $-\beta + \int_{-a}^{p_n} f$, remaining smaller than $\alpha - \beta$. This means thus that $\left|(I * F_\theta)(p_t, p_n)\right|$ has a maximum at $\theta = \pm\arccos[p_n/(-a)]$ and an extremum at $\theta = 0$; this extremum is a minimum if $\int_{-a}^{p_n} f \leq \beta$, in particular if $\alpha \leq \beta$, but it is a maximum otherwise; it will be an absolute maximum (i.e., greater than the maximum at $\theta = \pm\arccos[p_n/(-a)]$) if $\int_{-a}^{p_n} f > 2\beta$, which requires that $\alpha > 2\beta$.

Therefore it is possible for $\left|(I * F_\theta)(p_t, p_n)\right|$, as a function of θ to have an absolute minimum at $\theta = 0$ for some $p_n \neq 0$, while for $p_n = 0$ it will be constant. Note that (4.31) as a function of p_n and θ is discontinuous at $\theta = \pm\pi/2$; as θ increases from 0 to $\pi/2$, its evolution w.r.t. p_n becomes faster: the energy profile around $p_n = 0$ becomes steeper. We have thus two reasons for requiring the filter to have some width in the tangential direction: continuity of the convolution w.r.t. position and orientation, and orientation selectivity, in other words the possibility to determine the edge orientation as the one giving the highest result for the energy function. In practice, if the filter has a wide support in the normal direction but a narrow one in the tangential direction, orientation selectivity will not be achieved, and for $\theta \neq 0$ the edge in the filtered image will be sharper than for $\theta = 0$. In fact, it has been verified experimentally [46] that orientation selectivity increases with the ratio of tangential width over normal width, and that it is even necessary to take the width in the tangential direction equal to three times the width in the normal direction.

Let us now consider a second example with a step edge detector using a separable odd-symmetric filter defined as the product of a Gabor sine function in the normal direction and a Gabor cosine function in the tangential direction:

$$F(x_t, x_n) = G_\sigma(x_t)\cos(2\pi\alpha x_t) \cdot G_\sigma(x_n)\sin(2\pi\alpha x_n), \tag{4.32}$$

where G_σ is the Gaussian with standard deviation σ (cfr. (3.25)), and α is the frequency of the cosine and sine modulations. We show in Figure 13 the sign and zero-crossings of F. Since the Gaussian has a Gaussian-type Fourier transform, it is easy to check that F has constant phase $-\pi/2$ on $\mathcal{P}_\mathbf{n}$, in other words that $\mathrm{sgn}(\widehat{F}(u_t, u_n)) = -i\mathrm{sgn}(u_n)$. Let $F_{\pi/4}$ be the filter rotated by an angle of $\pi/4$ radians (i.e., 45 degrees). Calculations made on pp. 51,52 of [52] show that for $\sigma\alpha$ large enough, convolution with the step edge I of (4.30) gives:

$$\left|(I * [F_{\pi/4}])(x_t, 0)\right| \gg \left|(I * F)(x_t, 0)\right|.$$

This means in practice that the larger we take σ and α, the more will show the tendency of the filter to detect edges making a an angle of 45 degrees with the normal orientation. If we refer to Figure 13, as σ and α increase, among the square regions enclosed by the zero-crossings around the origin, the number of those in which the absolute value of F is non-negligible will increase, so that the grey-level profile of F becomes dominated by an alternation of diagonal bands made from square regions of respectively positive and negative sign.

One could raise an objection against the latter example, that the Gabor cosine function is not a smoothing function like the Gaussian, but rather a feature detector; thus the filter being the product of feature detectors in the normal and tangential directions, should in fact detect features in the intermediate diagonal directions.

We now give a third example to show that even with a standard edge detector formed as the product of of a Gaussian in the tangential direction and a Gaussian derivative in the normal direction (cfr. [7]), the detection of the edge orientation is sometimes possible only in a close neighbourhood of the edge position. Let the filter F be defined by

$$F(x_t, x_n) = x_n G_\nu(x_n) \cdot G_\tau(x_t), \qquad \text{where} \quad \tau > \nu. \tag{4.33}$$

Here G_τ and G_ν are Gaussians of respective standard deviations τ and ν (cfr. (3.25)), and we have assumed (as in [46]) that the support of the filter is wider in the tangential direction than in the normal one. Take an angle θ with $|\theta| < \pi/2$, and let F_θ be the filter rotated by an angle of θ. Calculations made on pp. 53,54 of [52] show that convolution with the step edge I of (4.30) gives:

$$(I * F_\theta)(x_t, x_n) = -\frac{\nu^2 \cos\theta}{\beta\sqrt{2\pi}} \exp[-x_n^2/2\beta^2], \qquad \text{where} \qquad \beta = \left(\tau^2 \sin^2\theta + \nu^2 \cos^2\theta\right)^{1/2}. \tag{4.34}$$

The energy function at point (x_t, x_n) for the filter F_θ is the square of this expression; it is a function of x_n and θ:

$$G(x_n, \theta) = \frac{\nu^4 \cos^2\theta}{2\pi\beta^2} \exp[-x_n^2/\beta^2] = \frac{\nu^4 \cos^2\theta}{2\pi\left(\tau^2 \sin^2\theta + \nu^2 \cos^2\theta\right)} \exp\left[\frac{-x_n^2}{\tau^2 \sin^2\theta + \nu^2 \cos^2\theta}\right].$$

Note that G is symmetric in both x_n and θ. Further calculations on p. 54 of [52] give the sign of $\partial G(x_n, \theta)/\partial\theta$ and the maxima of G as a function of θ for a fixed x_n. Let us fix $|x_n|$; there are two cases:

(*a*) $|x_n| \leq \tau\nu/\sqrt{\tau^2 - \nu^2}$, that is $(\tau^2 + x_n^2)(\tau^2 - \nu^2) \leq \tau^4$.
Then for every $\theta \neq 0$, $G(x_n, \theta)$ increases for $\theta < 0$, reaches a maximum at $\theta = 0$, and then decreases for $\theta > 0$.

(*b*) $|x_n| > \tau\nu/\sqrt{\tau^2 - \nu^2}$, that is $(\tau^2 + x_n^2)(\tau^2 - \nu^2) > \tau^4$.

Let

$$\theta[x_n] = \arccos\Big(\frac{\tau^2}{\sqrt{(\tau^2 + x_n^2)(\tau^2 - \nu^2)}}\Big). \tag{4.35}$$

We have $0 < \theta[x_n] < \pi/2$, and $\theta[x_n]$ increases from 0 to $\pi/2$ as $|x_n|$ increases from $\tau\nu/\sqrt{\tau^2 - \nu^2}$ to ∞. Then $G(x_n, \theta)$ increases for $-\pi/2 < \theta < -\theta[x_n]$, reaches a maximum at $\theta = -\theta[x_n]$, decreases for $-\theta[x_n] < \theta < 0$, reaches a local minimum at $\theta = 0$, increases again for $0 < \theta < \theta[x_n]$, reaches again a maximum at $\theta = \theta[x_n]$, and finally decreases for $\theta[x_n] < \theta < \pi/2$.

This shows that the correct orientation of the step edge is obtained only at points whose distance to the edge position does not exceed $\tau\nu/\sqrt{\tau^2 - \nu^2}$. We illustrate this fact in Figure 14.

Note that the whole argument relies on the fact that $\tau > \nu$. In fact, when ν increases towards τ, $\tau\nu/\sqrt{\tau^2 - \nu^2}$ increases towards $+\infty$, so that case (*a*) grows towards excluding case (*b*). Thus, following [46] and the result of the first example, we took a wider extent of the filter in the tangential direction in order to improve orientation selectivity, but we remark that precisely this wider extent restricts orientation selectivity to the neighbourhood of the edge position. We will thus seek criteria in order to guarantee correct orientation selection at the edge position only, knowing from Proposition 4.11 that the edge position can be found even when its orientation is not known.

If we look back at the failure of orientation selectivity in the first two examples, we remarked in the first one that the filter has no extent in the tangential direction, while in the second one, the Gabor cosine function, although it has constant zero phase, is not appropriate for the tangential direction; a Gaussian (as in the third example) would be preferable. Now considering the Fourier amplitude spectrum of these filters in the tangential direction, in the first example the Dirac impulse has constant 1 Fourier amplitude, while in the second example the Gabor cosine function has a Fourier amplitude spectrum with a minimum at the origin, and two maxima on both sides of the origin. On the other hand, the Fourier amplitude of a Gaussian is proportional to a Gaussian, having a unique maximum, located at the origin; the standard deviation of the Gaussian amplitude spectrum is inversely proportional to the standard deviation τ of that Gaussian in the spatial domain, so that when we increase the tangential width of the filter, the Fourier amplitude spectrum will become narrower in the tangential direction. This indicates a pattern necessary for the amplitude spectrum of the filter: to have its maxima on the normal axis, and to decrease in the tangential direction.

Our Axiom 7 is a way of satisfying such a requirement, and we will see that it garantees orientation selectivity at the edge position. We do not restrict ourselves to $d = 2$ as in the examples, we assume generally that $d \geq 2$.

Proposition 4.12. *Let $d \geq 2$ and I be given by (4.26) for a one-dimensional profile $P \in L^1 + L^2$. Let $p \in \mathbb{R}$ with local phase $P^\Phi(\nu, p) = \theta$ constant for all $\nu > 0$. Let C, S, and $D = C + i\,S$ satisfy the 7 axioms of the phase congruence model. Let the filter F be given by one of the following:*

(*a*) *$F = D$.*

(*b*) *$F = C$, with $\theta = 0$ or π.*

(*c*) *$F = S$, with $\theta = \pm\pi/2$.*

Let $\mathbf{t} \in \mathcal{E}_\mathbf{n}$, and consider a family of rotations R_θ for $\theta \in [0, \pi/2]$, such that $R_\theta(\mathbf{c}) = \cos\theta\mathbf{n} + \sin\theta\mathbf{t}$. Then for $\mathbf{y} \in \mathcal{E}_\mathbf{n}$ and $\theta \in [0, \pi/2]$, the energy function $|I * R_\theta(F)|^2(\mathbf{y}, p)$ at $(\mathbf{y}, p)$, as a function of θ, is: strictly decreasing for $\theta \in [0, \pi/2]$ (for the strong version of Axiom 7), or: decreasing on $[0, \pi/2]$, with a unique maximum at 0 (for the weak version of Axiom 7).

PROOF. For $\theta \in [0, \pi/2]$, we set

$$\begin{aligned} \mathbf{n}_\theta &= +\cos\theta\,\mathbf{n} + \sin\theta\,\mathbf{t}. \\ \mathbf{t}_\theta &= -\sin\theta\,\mathbf{n} + \cos\theta\,\mathbf{t}, \end{aligned} \tag{4.36}$$

Let $\mathbf{d} = R_0^{-1}(\mathbf{t})$; thus $R_0(\mathbf{c}) = \mathbf{n}$ and $R_0(\mathbf{d}) = \mathbf{t}$. By hypothesis, we have $R_\theta(\mathbf{c}) = \mathbf{n}_\theta$ for $\theta \in [0, \pi/2]$. If $d = 2$ we have necessarily $R_\theta(\mathbf{d}) = \mathbf{t}_\theta$. For $d > 2$ there is (for each θ) a rotation R'_θ such that $R'_\theta(\mathbf{c}) = \mathbf{n}_\theta$ and $R'_\theta(\mathbf{d}) = \mathbf{t}_\theta$; now since $R'_\theta(\mathbf{c}) = R_\theta(\mathbf{c})$, we have $R'_\theta = R_\theta R^*_\theta$ for a rotation R^*_θ such that $R^*_\theta(\mathbf{c}) = \mathbf{c}$, and by Axiom 4 we have $F = R^*_\theta(F)$, so that $R'_\theta(F) = R_\theta(R^*_\theta(F)) = R_\theta(F)$. As we are considering the convolution of I by $R_\theta(F)$, we can without loss of generality take R'_θ instead of R_θ and suppose that $R_\theta(\mathbf{d}) = \mathbf{t}_\theta$ for $\theta \in [0, \pi/2]$. Now (4.36) gives

$$\mathbf{n} = \cos\theta\,\mathbf{n}_\theta - \sin\theta\,\mathbf{t}_\theta = \cos\theta\,R_\theta(\mathbf{c}) - \sin\theta\,R_\theta(\mathbf{d}),$$

so that

$$R_\theta^{-1}(\mathbf{n}) = \cos\theta\,\mathbf{c} - \sin\theta\,\mathbf{d}.$$

In particular, $R_\theta^{-1}(\mathbf{n}) \in \mathcal{P}_\mathbf{c}$ when $\theta < \pi/2$. By (4.28) we have

$$\forall \mathbf{y} \in \mathcal{E}_\mathbf{n}, \forall x \in \mathbb{R}, \qquad \big(I * R_\theta(F)\big)(\mathbf{y} + x\mathbf{n}) = \big(P * F_{/R_\theta^{-1}(\mathbf{n})}\big)(x) = \big(P * F_{/\cos\theta\,\mathbf{c} - \sin\theta\,\mathbf{d}}\big)(x)$$

and (4.29) gives

$$\big[P * F_{/R_\theta^{-1}(\mathbf{n})}\big]^\wedge(\nu) = \widehat{P}(\nu)\big[F_{/R_\theta^{-1}(\mathbf{n})}\big]^\wedge(\nu) = \widehat{P}(\nu)\widehat{F}(\nu R_\theta^{-1}(\mathbf{n})) = \widehat{P}(\nu)\widehat{F}\big(\nu(\cos\theta\,\mathbf{c} - \sin\theta\,\mathbf{d})\big).$$

By Proposition 4.4, $C_{/R_\theta^{-1}(\mathbf{n})}$, $S_{/R_\theta^{-1}(\mathbf{n})}$, and $D_{/R_\theta^{-1}(\mathbf{n})}$ satisfy Axioms 1, 2, and 3 with $d = 1$ when $\theta < \pi/2$, and are zero when$\theta = \pi/2$.

We consider case (a), that is $F = D$. By Proposition 4.6, in particular (4.7), we obtain for every $\mathbf{y} \in \mathcal{E}_\mathbf{n}$:

$$\begin{aligned} \big|\big(I * R_\theta(D)\big)(\mathbf{y} + p\mathbf{n})\big|^2 &= \big|\big(P * D_{/R_\theta^{-1}(\mathbf{n})}\big)(p)\big|^2 \\ &= 4 \iint_{\mathbb{R}^+ \times \mathbb{R}^+} P^\mathcal{A}(u)\big[C_{/R_\theta^{-1}(\mathbf{n})}\big]^\mathcal{A}(u) P^\mathcal{A}(v)\big[C_{/R_\theta^{-1}(\mathbf{n})}\big]^\mathcal{A}(v)\, \cos\big[P^\Phi(u,p) - P^\Phi(v,p)\big]\, du dv \\ &= 4 \iint_{\mathbb{R}^+ \times \mathbb{R}^+} P^\mathcal{A}(u) A\big(u(\cos\theta\,\mathbf{c} - \sin\theta\,\mathbf{d})\big) P^\mathcal{A}(v) A\big(v(\cos\theta\,\mathbf{c} - \sin\theta\,\mathbf{d})\big)\, du dv, \end{aligned}$$

where $A = C^\mathcal{A}$ (see (4.1)). The integrand is positive; with the strong version of Axiom 7, it is a strictly decreasing function of θ for $\theta \in [0, \pi/2]$; with the weak version of Axiom 7, it is a decreasing function of θ having a unique maximum at 0. Hence the integral, namely $\big(I * R_\theta(F)\big)(\mathbf{y} + p\mathbf{n})$, has the same property (i.e., is a strictly decreasing function of θ, resp., a decreasing function of θ having a unique maximum at 0). From (4.9) and Proposition 4.7 it follows that

$$\big|\big(P * C_{/R_\theta^{-1}(\mathbf{n})}\big)(p)\big|^2 = \big|\big(P * D_{/R_\theta^{-1}(\mathbf{n})}\big)(p)\big|^2$$

in the case (b) and

$$\big|\big(P * S_{/R_\theta^{-1}(\mathbf{n})}\big)(p)\big|^2 = \big|\big(P * D_{/R_\theta^{-1}(\mathbf{n})}\big)(p)\big|^2$$

in the case (c), so that the result holds also in these two cases. ∎

In order to have an idea of the restriction imposed upon D by Axiom 7, let us consider the case where $d = 2$; here a vector $\mathbf{x} \in \mathcal{E}$ can be written (x_b, x_c), where x_c is the component of $\mathbf{x}$ in the standard normal direction of D given by the unit vector $\mathbf{c}$, and x_b is the component in the tangential direction, given by a unit vector $\mathbf{b}$ perpendicular to $\mathbf{c}$.

In [12] the phase congruence model was studied in the case where the filters D is separable, in other words $D(x_b, x_c) = b(x_b) \cdot f(x_c)$, and symmetric w.r.t. the tangential direction, that is $D(x_b, x_c) = D(-x_b, x_c)$, in other words b is even-symmetric; we have then $C(x_b, x_c) = b(x_b) \cdot c(x_c)$ and $S(x_b, x_c) = b(x_b) \cdot s(x_c)$, where $f = c + i\, s$. Here Axioms 1, 2, and 3 mean that b, c, and s are continuous and integrable, c is even-symmetric, s is odd-symmetric, $\widehat{b}$ and $\widehat{c}$ have real non-negative values, and (c, s) is a quadrature pair. Axiom 4 is void for $d = 2$; Axioms 5 and 6 were not considered in [12], but it was shown that Axiom 7 leads here to very strong conditions on $\beta = \widehat{b}$ and $\gamma = \widehat{c}$. The following is an improvement of a result of [12]:

PROPOSITION 4.13. *Let C, S, and $D = C + i\,S$ satisfy Axioms 1, 2, and 3 of the phase congruence model. Assume further that $A(x_b, x_c) = \beta(x_b)\gamma(x_c)$, where β and γ are non-negative, even-symmetric, and continuously derivable functions $\mathbb{R} \to \mathbb{R}$. Then the following weakened version of Axiom 7:*

($*$) For every unit vector $\mathbf{d} \in \mathcal{E}_{\mathbf{c}}$ and for every $\nu > 0$, the function of variable $\theta \in [0, \pi/2]$ given by $A\big(\nu(\cos\theta \mathbf{c} + \sin\theta \mathbf{d})\big)$, is decreasing on $[0, \pi/2]$.

is equivalent to each one of the following three statements:

(i) *For every $u > 0$, $\beta(u \sin\theta)\gamma(u\cos\theta)$ is decreasing on $\theta \in [0, \pi/2]$.*

(ii) *For every $x, y > 0$ such that $\gamma(x) > 0$ and $\beta(y) > 0$, we have:*

$$\frac{\beta'(y)}{y\beta(y)} \leq \frac{\gamma'(x)}{x\gamma(x)}.$$

(iii) *β is monotonously decreasing on $\mathbb{R}^+$ (the set of reals $x \geq 0$), and there is some $K > 0$ such that*

(a) *For every $y > 0$ having $\beta(y) > 0$,*

$$\frac{|\beta'(y)|}{y\beta(y)} \geq K.$$

(b) *For every $x > 0$ having $\gamma(x) > 0$ and $\gamma'(x) < 0$,*

$$\frac{|\gamma'(x)|}{x\gamma(x)} \leq K.$$

Assume further that there is no open set $V \subseteq (\mathbb{R}^+)^2$ and no $\alpha, \lambda > 0$ such that for all $(x_b, x_c) \in V$ we have $A(x_b, x_c) = \lambda \exp\big[-\alpha(x_b^2 + x_c^2)\big]$. Then the weak version of Axiom 7 holds iff for all $x > 0$ we have $\gamma(x) > 0$, while the strong version of Axiom 7 holds iff for all $x, y > 0$ we have $\beta(y) > 0$ and $\gamma(x) > 0$.

PROOF. In (x_b, x_c) coordinates, $\mathbf{c} = (0, 1)$ and $\mathbf{d} = (1, 0)$ or $(-1, 0)$; this gives thus $A\big(\nu(\cos\theta \mathbf{c} + \sin\theta \mathbf{d})\big)$ equal to $\beta(u \sin\theta)\gamma(u\cos\theta)$ or $\beta(-u \sin\theta)\gamma(u\cos\theta)$; but the two are the same by the even symmetry of β, so that

$$A\big(\nu(\cos\theta \mathbf{c} + \sin\theta \mathbf{d})\big) = \beta(u \sin\theta)\gamma(u\cos\theta)$$

in any case. Hence $(*)$ is equivalent to (i).

Note that for every $x > 0$, if $\gamma(x) = 0$, as γ has only non-negative values, γ has a minimum at x and so $\gamma'(x) = 0$. Similarly for every $y > 0$, if $\beta(y) = 0$, then $\beta'(y) = 0$.

We show now the equivalence between (i) and (ii). By the finite increment theorem, (i) is equivalent to:

$$\forall u > 0,\ \forall \theta \in]0, \pi/2[, \quad \frac{\partial}{\partial \theta}\big(\beta(u \sin\theta)\gamma(u\cos\theta)\big) \leq 0.$$

Now

$$\frac{\partial}{\partial \theta}\big(\beta(u \sin\theta)\gamma(u\cos\theta)\big) = u\cos\theta\beta'(u\sin\theta)\gamma(u\cos\theta) - u\sin\theta\beta(u\sin\theta)\gamma'(u\cos\theta),$$

so that the condition becomes

$$u\cos\theta\beta'(u\sin\theta)\gamma(u\cos\theta) \leq u\sin\theta\beta(u\sin\theta)\gamma'(u\cos\theta) \qquad \text{for} \quad u > 0 \text{ and } 0 < \theta < \pi/2.$$

But the set of pairs $(u\cos\theta, u\sin\theta)$ for $u > 0$ and $0 < \theta < \pi/2$ coincides with the set of pairs (x, y) for $x, y > 0$. Thus (i) is equivalent to

$$\forall x, y > 0, \quad x\beta'(y)\gamma(x) \leq y\beta(y)\gamma'(x). \tag{4.37}$$

Two special cases arise:

— $\gamma(x) = 0$, so that $\gamma'(x) = 0$.

— $\beta(y) = 0$, so that $\beta'(y) = 0$.

In both cases the inequality of (4.37) is trivially verified as $0 \leq 0$. We have thus only to consider the remaining case where $\gamma(x) > 0$ and $\beta(y) > 0$, and dividing each member of the inequality $x\gamma(x)\beta'(y) \leq y\gamma'(x)\beta(y)$ by the positive factor $x\gamma(x)y\beta(y)$, (4.37) becomes (ii).

We show next that (iii) implies (ii). If (iii) is verified, as β is decreasing, for every $y > 0$ we have $\beta'(y) \leq 0$; if $\beta(y) > 0$, then (a) gives

$$-\frac{\beta'(y)}{y\beta(y)} = \frac{|\beta'(y)|}{y\beta(y)} \geq K. \tag{4.38}$$

Taking $x > 0$ with $\gamma(x) > 0$, either $\gamma'(x) \geq 0$ and so

$$-\frac{\gamma'(x)}{x\gamma(x)} \leq 0 \leq K,$$

or $\gamma'(x) < 0$ and (b) gives

$$-\frac{\gamma'(x)}{x\gamma(x)} = \frac{|\gamma'(x)|}{x\gamma(x)} \leq K;$$

Thus

$$-\frac{\gamma'(x)}{x\gamma(x)} \leq K \tag{4.39}$$

anyway. Combining both inequalities (4.38) and (4.39), we get

$$-\frac{\beta'(y)}{y\beta(y)} \geq K \geq -\frac{\gamma'(x)}{x\gamma(x)},$$

which gives (ii) by a change of sign.

We show then that (ii) implies (iii). Since A is integrable by Proposition 4.1, γ is integrable, and so it is not monotonously increasing on $\mathbb{R}^+$. Thus there exist a_1, a_2 with $0 \leq a_1 < a_2$ and

$\gamma(a_1) > \gamma(a_2)$. By the finite increment theorem, there is thus some $\xi > 0$ (with $a_1 < \xi < a_2$) such that $\gamma'(\xi) < 0$; we must then have $\gamma(\xi) > 0$. By (ii), for every $y > 0$ such that $\beta(y) > 0$, we have:

$$\frac{\beta'(y)}{y\beta(y)} \leq \frac{\gamma'(\xi)}{\xi\gamma(\xi)} < 0,$$

in other words $\beta'(y) < 0$; on the other hand for $\beta(y) = 0$ we have $\beta'(y) = 0$. Thus β is monotonously decreasing on $\mathbb{R}^+$. Let

$$K = \sup\bigl\{-\frac{\gamma'(z)}{z\gamma(z)} \mid z > 0 \text{ and } \gamma(z) > 0\bigr\}.$$

As $-\gamma'(\xi)/\xi\gamma(\xi) > 0$, we have $K > 0$. By (ii) we have for every $y > 0$ with $\beta(y) > 0$:

$$\frac{|\beta'(y)|}{y\beta(y)} = -\frac{\beta'(y)}{y\beta(y)} \geq \sup\bigl\{-\frac{\gamma'(z)}{z\gamma(z)} \mid z > 0 \text{ and } \gamma(z) > 0\bigr\} = K,$$

so that (a) holds. On the other hand for $x > 0$ with $\gamma(x) > 0$ and $\gamma'(x) < 0$ we have trivially

$$\frac{|\gamma'(x)|}{x\gamma(x)} = -\frac{\gamma'(x)}{x\gamma(x)} \leq \sup\bigl\{-\frac{\gamma'(z)}{z\gamma(z)} \mid z > 0 \text{ and } \gamma(z) > 0\bigr\} = K,$$

so that (b) holds.

Suppose finally that there is no open subset of $(\mathbb{R}^+)^2$ where $A(x_b, x_c) = \lambda \exp\bigl[-\alpha(x_b^2 + x_c^2)\bigr]$ (with $\alpha, \lambda > 0$). It is easily seen that if

$$\frac{\beta'(y)}{y\beta(y)} = \frac{\gamma'(x)}{x\gamma(x)} = -K \tag{4.40}$$

holds with $\gamma(x) > 0$ and $\beta(y) > 0$ on a square interval $x \in]\xi_1, \xi_2[$, $y \in]\eta_1, \eta_2[$, then we obtain $\beta(y)\gamma(x) = \lambda \exp\bigl[-K(x_b^2 + x_c^2)/2\bigr]$ (with $\lambda > 0$) in that interval, a contradiction.

Given $u > 0$ and $0 \leq \theta_1 < \theta_2 \leq \pi/2$, by (i) we have always $\beta(u\sin\theta_1)\gamma(u\cos\theta_1) \geq \beta(u\sin\theta_2)\gamma(u\cos\theta_2)$, so that $\beta(u\sin\theta_1)\gamma(u\cos\theta_1) = \beta(u\sin\theta_2)\gamma(u\cos\theta_2)$ iff $\beta(u\sin\theta)\gamma(u\cos\theta)$ is constant for $\theta \in [\theta_1, \theta_2]$; using the same argument as in the above proof of the equivalence between $(*)$ and (i), this holds iff for every $\theta \in]\theta_1, \theta_2[$ we have either $\gamma(u\cos\theta) = 0$, $\beta(u\sin\theta) = 0$, or $\beta'(u\sin\theta)/u\sin\theta\beta(u\sin\theta) = \gamma'(u\cos\theta)/u\cos\theta\gamma(u\cos\theta)$; from the proof that (ii) implies (iii) it follows then that in this case we have

$$\frac{\beta'(u\sin\theta)}{u\sin\theta\beta(u\sin\theta)} = -K = \frac{\gamma'(u\cos\theta)}{u\cos\theta\gamma(u\cos\theta)}.$$

Thus for all $x \in]u\cos\theta_2, u\cos\theta_1[$ and $y \in]u\sin\theta_1, \sin\theta_2[$, such that $\gamma(x) > 0$ and $\beta(y) > 0$, we will have (4.40).

If $\gamma(x) = 0$ for some $x > 0$, then for $0 < \theta \leq \pi/2$ we have

$$\beta(x\sin 0)\gamma(x\cos 0) = \beta(0)\gamma(x) = 0 \leq \beta(x\sin\theta)\gamma(x\cos\theta),$$

and the weak version of Axiom 7 is not satisfied. Conversely, suppose that $\gamma(x) > 0$ for every $x > 0$; as β is decreasing, continuous, and not identically zero on R^+, there is some $\eta > 0$ such that $\beta(y) > 0$ for $0 \leq y \leq \eta$. Take any $u > 0$, and let $\theta^* \in]0, \pi/2[$ such that $u\sin\theta^* \leq \eta$. Then for $0 < \theta \leq \theta^*$ we have $\gamma(x) > 0$ and $\beta(y) > 0$ for $x \in]u\cos\theta, u\cos 0[$ and $y \in]u\sin 0, u\sin\theta[$, and (4.40) cannot hold for all such x, y. Hence $\beta(u\sin 0)\gamma(u\cos 0) > \beta(u\sin\theta)\gamma(u\cos\theta)$, and the weak version of Axiom 7 is satisfied.

If $\beta(y)=0$ for some $y>0$, then taking $u>y$, there is some $\theta^*\in]0,\pi/2[$ such that $y=u\cos\theta^*$, and for $\theta^*<\theta\leq\pi/2$ we have

$$\beta(u\sin\theta^*)\gamma(u\cos\theta^*)=\beta(u\sin\theta^*)\gamma(y)=0\leq\beta(u\sin\theta)\gamma(u\cos\theta),$$

so that the strong version of Axiom 7 is not satisfied. If $\gamma(x)=0$ for some $x>0$, we have seen above that the weak version of Axiom 7 does not hold, so that the strong version is also contradicted. Conversely, suppose that $\gamma(x)>0$ and $\beta(y)>0$ for every $x,y>0$; given $u>0$ and $0\leq\theta_1<\theta_2\leq\pi/2$, in the square interval made of all $x\in]u\cos\theta_2,u\cos\theta_1[$ and $y\in]u\sin\theta_1,\sin\theta_2[$, we have $\gamma(x)>0$ and $\beta(y)>0$ and (4.40) cannot hold for all such x,y. Hence $\beta(u\sin\theta_1)\gamma(u\cos\theta_1)>\beta(u\sin\theta_2)\gamma(u\cos\theta_2)$ and the strong version of Axiom 7 is satisfied. ∎

Note that when $A(x_b,x_c)=\lambda\exp\bigl[-\alpha(x_b^2+x_c^2)\bigr]$ on an open set V, then A is rotationally symmetric on it, which is incompatible with the strong version of Axiom 7.

The conditions (*ii*) and (*iii*) were found by Fousse in [12], where the equivalence between (*i*) and (*ii*) and the sufficiency of (*iii*) were shown. If we return to our above examples where orientation selectivity fails, in the first one we had $b=\delta$, the Dirac impulse, whose Fourier transform is constant 1, so that condition (*ii*) gives $\gamma'(x)\geq 0$ whenever $\gamma(x)>0$, in other words $\gamma=\widehat{c}$ is increasing on $\mathbb{R}^+$, which is impossible for c in L^1+L^2. On the other hand in the second example, b was a Gabor cosine function, whose Fourier transform is not decreasing on $\mathbb{R}^+$, contradicting condition (*iii*).

We mentioned above the observation in [46] that orientation selectivity is improved when the spatial extent of the edge detection filter is greater in the tangential direction than in the normal one. Axiom 7 and especially Proposition 4.13 provide a rationale for it. Indeed, the wider the spatial extent of D in the tangential direction, the narrower the spatial extent of its Fourier transform $\widehat{D}$ in that direction; more precisely, if we replace $D(x_b,x_c)=b(x_b)f(x_c)$ by $D(x_b/w,x_c)=b(x_b/w)f(x_c)$ (where $w>1$), then $\widehat{D}(u_b,u_c)=\beta(u_b)\gamma(u_c)$ will be replaced by $w\widehat{D}(wu_b,u_c)=w\beta(wu_b)\gamma(u_c)$, and as β is decreasing on $\mathbb{R}^+$, for $u>0$ the ratio $\widehat{D}(wu\sin\theta,u\cos\theta)/\widehat{D}(0,u)$ will be lower than $\widehat{D}(u\sin\theta,u\cos\theta)/\widehat{D}(0,u)$ for $0<\theta<\pi/2$.

The traditional method for localizing edges with an edge-detecting filter D consists in two steps:

(1°) First compute at every point $\mathbf{p}$ the angle θ for which the energy function $|(I*D_\theta)(\mathbf{p})|^2$ is the greatest; write it $\theta(\mathbf{p})$.

(2°) Second select as edge position the set all points $\mathbf{p}$ such that $|(I*D_{\theta(\mathbf{p})})(\mathbf{p})|^2\geq|(I*D_{\theta(\mathbf{q})})(\mathbf{q})|^2$ for all points $\mathbf{q}$ in a neighbourhood of $\mathbf{p}$ in the normal direction.

The neighbourhood in (2°) can be purely local, or have some extent depending on the width of the filter D (we will return to this question in the next subsection). This method, although introduced in an empirical framework based on template matching, is justified in the phase congruence model by Propositions 4.11 and 4.12. Indeed, assume as above an image I forming a one-dimensional edge profile P in the normal direction: $I(x_t,x_n)=P(x_n)$. Then we know that the phase of $|(I*D_{\theta(\mathbf{p})})|^2$ does not depend on $\theta(\mathbf{p})$ (provided that $\theta(\mathbf{p})$ is not oriented in the edge tangential direction), so that if $\mathbf{p}$ is close to the edge, but not on it, we will have

$$|(I*D_{\theta(\mathbf{p})})(\mathbf{p})|^2<|(I*D_{\theta(\mathbf{p})})(\mathbf{q})|^2\leq|(I*D_{\theta(\mathbf{q})})(\mathbf{q})|^2$$

for some neighbouring point $\mathbf{q}$ in the normal direction; on the other hand if $\mathbf{p}$ is on the edge, then we know that $\theta(\mathbf{p})$ will be along the normal direction of the edge. Thus (2°) will eliminate points which do not lie on the edge, while (1°) will give the edge orientation on edge points.

Of courses, this argument based on an ideal one-dimensional edge is justified for straight edges, and does not hold when we have strongly curved edges (in other words edges whose radius of curvature is comparable to the spatial extent of the filter).

4.5. Authentication of edges, and scale-space behaviour of the energy function

A priori, any local maximum of the energy function in the normal direction could be selected as an edge point. But this could produce spurious edges, for example a point where the local phases of the image are only slightly less discordant than in its neighbourhood. Moreover, some filters satisfying our seven axioms may respond to a pure edge (an ideal step, line, or roof) with an energy function having one global maximum at the edge location and also other local maxima which do not correspond to any perceptually meaningful feature in the image. We must thus find methods for avoiding (or eliminating) such spurious maxima in the energy function. In the first place, we can select the filters carefully in order to have a unique maximum on an ideal edge (step, line, or roof); some steps towards a mathematical formalization of this requirement were given on pp. 35-37 of [52]. However this will generally be unsufficient and we need also to eliminate some of the maxima obtained in the energy function. Two general types of criteria can be envisaged:

(i) The peak in the energy function must be high enough.

(ii) The peak in the energy function must be wide enough.

If we follow (i) and require high peaks, we must first calibrate the energy function in order to make it independent of the average image contrast: if the image grey-levels are all multiplied by a positive constant, the energy function should not change. We can for example use as measure of the energy at a point $\mathbf{p}$ the *phase congruence function*

$$\frac{|\Delta(\mathbf{p})|}{\int_{\mathcal{E}} I^{\mathcal{A}} A} \tag{4.41}$$

introduced in (4.14), whose values range in the interval $[0,1]$ and is equal to 1 only when all phases $I^{\Phi}(\mathbf{u},\mathbf{p})$ at $\mathbf{p}$ for all frequencies $\mathbf{u} \in \mathcal{P}_{\mathbf{n}}$ are equal to a constant. We can also take some variants such as

$$\frac{|\Delta(\mathbf{p})|}{\|\widehat{I}\|_{\infty} \cdot \|A\|_1} \tag{4.42}$$

when I is integrable, or

$$\frac{|\Delta(\mathbf{p})|}{\|\widehat{I}\|_2 \cdot \|A\|_2} = \frac{|\Delta(\mathbf{p})|}{\|I\|_2 \cdot \|C\|_2} \tag{4.43}$$

when I is square-integrable; note that $\|\widehat{I}\|_{\infty} \cdot \|A\|_1$ and $\|\widehat{I}\|_2 \cdot \|A\|_2$ are both $\geq \int_{\mathcal{E}} I^{\mathcal{A}} A$, thanks to Hölder's inequality, so that the values of (4.42) and (4.43) range also in the interval $[0,1]$. Then, using one of the measures $(4.41, 4.42, 4.43)$ of phase congruence, we may select edge points among maxima of the energy function using Canny's [7] *hysteresis* technique: we take two thresholds, an upper one and a lower one (say, 1/2 and 1/4), and we select the two sets U and L consisting of all maxima of the energy function for which the measure of phase congruence exceeds the upper or the lower threshold respectively; then the set of edge points consists of all connected components of L having a non-void intersection with U.

On the other hand, requiring wide peaks as in (ii) does not necessitate a calibration of the energy function, but rather a standard width to compare the peaks with. We can consider that this must be the width of the grey-level profile of D in the normal direction, because $|D|^2$ is the energy function for the input signal given by a Dirac impulse. This width can be measured as

$$\frac{\|\xi_{\mathbf{c}} D\|_1}{\|D\|_1} = \frac{\int_{\mathcal{E}} |x_c D(\mathbf{y}, x_c)|\, d\mathbf{y} dx_c}{\int_{\mathcal{E}} |D(\mathbf{y}, x_c)|\, d\mathbf{y} dx_c}$$

or

$$\frac{\|\xi_{\mathbf{c}} D\|_2}{\|D\|_2} = \left(\frac{\int_{\mathcal{E}} x_c^2 |D(\mathbf{y}, x_c)|^2\, d\mathbf{y} dx_c}{\int_{\mathcal{E}} |D(\mathbf{y}, x_c)|^2\, d\mathbf{y} dx_c} \right)^{\frac{1}{2}},$$

or the least w such that for $|x_c| > w$ we have $|D(\mathbf{y}, x_c)| < \varepsilon \|D\|_p$ (where $p = 1$, 2, or ∞), or such that for all $\mathbf{y} \in \mathcal{E}_{\mathbf{c}}$ we have $\int_{|x_c|>w} |D|^p < \varepsilon \int_{\mathcal{E}} |D|^p$ (where $p = 1$ or 2), with ε being chosen very small (say, 1/100). We can also take as standard input signal a hyperplane constant in the tangential direction and making a Dirac impulse in the normal one, and so we replace D by $D_{/\mathbf{c}}$ in the above formulas.

Then we might select as edge points those for which the energy function is greater than that of all points at distance at most kw of it in the normal direction, where k is some threshold smaller than 1 (say, 1/2). In other words we admit edges separated by a distance between kw and w, but for closer edges one of them must be condidered as spurious. We might also require of an edge point $\mathbf{p}$ that for all points $\mathbf{q}$ in the normal direction w.r.t. $\mathbf{p}$ and at distance less than kw from it, the ratio of energies $E(\mathbf{p})/E(\mathbf{q})$ must increase with the distance between $\mathbf{p}$ and $\mathbf{q}$ according to some function of that distance:

$$\frac{E(\mathbf{p})}{E(\mathbf{q})} \geq \psi\big(d(\mathbf{p}, \mathbf{q})\big);$$

this function ψ could be selected from the what happens with the Dirac impulse (or Dirac impulse hyperplane) as input signal, in other words the grey-level profile of $|D|^2$ (or $|D_{/\mathbf{c}}|^2$) around the origin. The effect of this stronger criterion would be to consider a close succession of edges as texture or noise, which should not appear in the edge map.

The two approaches could be combined, so that we might require peaks to be both wide and high, or that the product of the peak height and width should exceed a given threshold. These considerations are speculative, we have no mathematical result justifying them, and the criteria suggested above should be experimented with natural images. Morphological operators [58] could also be applied in order to eliminate spurious peaks, and the watershed transformation [2] could be used instead of non-maxima deletion in order to produce closed contours.

We said in Section 2 that each feature corresponds to a certain scale, that a change of scale can lead to modifications of the edges; this was illustrated with Figure 9. Now it should be clear that the scale corresponding to an edge is proportional to the above-mentioned width w of the filter D detecting it. Consider for example the slanted ridge profile shown in Figure 9. When the filter D is very narrow (say, w is 1/40-th the width of the ridge), the energy function will have four peaks corresponding to the four Mach bands; these peaks will be well separated and will not interfere. As the filter width increases (say, w going up to to 1/6-th the width of the ridge), the two energy peaks at the extremities of each step will interfere, and will at a certain scale merge into a single peak; thus two steps are detected. Increasing further the filter width (say, until w becomes larger than the width of the ridge), the two energy peaks located at each step will interfere and finally merge into a single peak; thus a single bar edge is detected.

In this example we see that as the scale (i.e., the width) of the filter increases, features can merge into other ones whose nature is different; ideally there should here not be any new feature arising at a certain scale from nothing, or a feature dividing into several ones at a wider scale. This is the principle of *causality*: any feature existing at a certain scale which does not arise from one at a smaller scale must be considered as spurious. As we said at the end of Section 2, Kube and Perona have shown in [27] that when features are localized at *local* maxima of the energy function, the phase congruence model (i.e., a quadratic edge detector using pairs of filters in Fourier quadrature) does not satisfy this principle; this was illustrated in concrete examples with the pair of filters consisting of a Gaussian derivative and its Hilbert transform. For quadratic operators where one of the filters is the derivative of the other, only for a certain class of filters which includes the Gaussian and its derivatives, is the causality property verified. Thus the phase congruence model introduces spurious non-causal features which are not detected by quadratic edge detectors using Gaussian derivatives. A typical pattern of causality failure, where a new feature arises from nothing as scale increases, is shown in Figure 10 (see also Figures 2 and 3 of [27]); in this example the new maximum of the energy function is local, but if we compare its value with a wider neighbourhood, it ceases to be a maximum; thus it could be eliminated by taking as edge points only regional maxima of the energy function, as explained above. Thus we may hope that our above requirement (*ii*) for authenticating edges could guarantee causality. To our knowledge, there have been no studies of causality for features localized at *regional* maxima of the energy function, in other words where at scale s the selected feature points are maxima of energy on a range proportional to s.

Note that points where the Fourier phases of the image are equal to a constant lead to an absolute maximum of the energy function at all scales. This is true either if we scale the filter D (taking at scale s the filter D_s defined by $D_s(\mathbf{x}) = D(\mathbf{x}/s)$), or if we smooth the image with a function W having constant zero phase (say, a Gaussian of increasing scale), because the phases of $I * W$ are the same as those of I (cfr. [Ronse]).

5. Related questions and conclusion

We will discuss here miscellaneous problems concerning the phase congruence model, in particular: (*i*) how to adapt it to a digital framework and digitize the filters specified by our requirements in the Euclidean framework, and in this respect the property of uniform continuity will play an important role; (*ii*) applications to other vision tasks. We end then with the conclusion.

5.1. Digitization of the filters

Any practical implementation must assume that images and filters are sampled, and that filters have bounded support. The classical signal processing approach to digitization, based on the Shannon sampling theorem, is unsuitable for the analysis of visual features, because it assumes unnecessarily that the signal can be band-limited, while it gives a secondary role to the spatial localization of masks, which is however crucial to to the spatial localization of features; furthermore it does not guarantee that our mathematical results will extend to the digital case. It is better to take the approach suggested by Hummel and Lowe [21], which we describe here in general mathematical terms.

We already have the Euclidean space $\mathcal{E} = \mathbb{R}^d$; consider now the corresponding digital space $\mathcal{D} = \mathbb{Z}^d$. Let $\mathcal{I}(\mathcal{E})$, $\mathcal{I}(\mathcal{D})$, $\mathcal{F}(\mathcal{E})$, $\mathcal{F}(\mathcal{D})$ be the families of respectively Euclidean images, digital images, Euclidean filters, and digital filters. Theory gives us a Euclidean filter $D = C + i\,S$; practice gives us a digital input image I; we must be able to apply D to I, and for this we must digitize

D. The basic idea underlying the method of Hummel and Lowe is that the sampling of filters must be considered as corresponding to an extrapolation of digital images into Euclidean ones. Thus the filter sampling $\Sigma : \mathcal{F}(\mathcal{E}) \to \mathcal{F}(\mathcal{D})$ corresponds to the image extrapolation $\Xi : \mathcal{I}(\mathcal{D}) \to \mathcal{I}(\mathcal{E})$ in such a way that for every $F \in \mathcal{F}(\mathcal{E})$ and $J \in \mathcal{I}(\mathcal{D})$ we have

$$\int_{\mathcal{D}} J \cdot \Sigma[F] = \int_{\mathcal{E}} \Xi[J] \cdot F. \tag{4.44}$$

Note that in this equation, integration on $\mathcal{E}$ is done w.r.t. the Lebesgue measure, while integration on $\mathcal{D}$ is done w.r.t. the discrete measure, in other words $\int_{\mathcal{D}} J \cdot \Sigma[F]$ must be read as $\sum_{\mathbf{z}\in\mathcal{D}} J(\mathbf{z}) \cdot \Sigma[F](\mathbf{z})$. Assuming that Ξ and Σ commute with the reflection ρ and with all translations by points in $\mathcal{D}$, (4.44) gives:

$$\forall \mathbf{p} \in \mathcal{D}, \qquad \big(J * \Sigma[F]\big)(\mathbf{p}) = \big(\Xi[J] * D\big)(\mathbf{p}). \tag{4.45}$$

Here the first convolution is made on $\mathcal{D}$ (with the integral becoming a series), and the second one on $\mathcal{E}$. Given a bounded integrable function $W : \mathcal{E} \to \mathbb{R}$, we define Ξ and Σ by

$$\begin{aligned} \Xi[J] &= \sum_{\mathbf{z}\in\mathcal{D}} J(\mathbf{z})\tau_{\mathbf{z}}(W) : \ \mathbf{x} \mapsto \sum_{\mathbf{z}\in\mathcal{D}} J(\mathbf{z})W(\mathbf{x}-\mathbf{z}) \\ \text{and} \qquad \Sigma[F] &: \ \mathbf{z} \mapsto (W * J)(\mathbf{z}) = \int_{\mathcal{E}} d\mathbf{x}\, W(\mathbf{z}-\mathbf{x})F(\mathbf{x}). \end{aligned} \tag{4.46}$$

It is easily checked that Ξ and Σ commute with the reflection and with all translations by points in $\mathcal{D}$, and (4.45) is verified as follows:

$$\begin{aligned} \big(J * \Sigma[F]\big)(\mathbf{p}) &= \sum_{\mathbf{z}\in\mathcal{D}} \int_{\mathcal{E}} d\mathbf{x}\, J(\mathbf{z})W(\mathbf{p}-\mathbf{z}-\mathbf{x})F(\mathbf{x}) \\ &= \int_{\mathcal{E}} d\mathbf{x} \sum_{\mathbf{z}\in\mathcal{D}} J(\mathbf{z})W(\mathbf{p}-\mathbf{z}-\mathbf{x})F(\mathbf{x}) = \big(\Xi[J] * F\big)(\mathbf{p}). \end{aligned}$$

This equality holds indeed if we assume J in $\ell^1 + \ell^2$ (i.e., to be the sum of a summable image and a square-summable one). This was the choice for Ξ and Σ in [21], where several examples of functions W were considered, and the advantages of this new sampling method over the classical one was experimentally demonstrated.

Note that by (4.45) the digital image $J * \Sigma[F]$ resulting from the application of the sampled filter $\Sigma[F]$ to the original digital image J, is equal to the classical sampling (in other words, the digital trace) of the Euclidean image $\Xi[J] * F$ resulting from the application of the original Euclidean filter F to the extrapolated image $\Xi[J]$.

Now consider our digital image I, and the Euclidean filter D specified in the phase congruence model. The mathematical properties of that model can be applied to $\Xi[I] * D$, of which $I * \Sigma[D]$ is a sampling. We can thus expect that some properties of $\Xi[I] * D$ will be inherited by $I * \Sigma[D]$. For example if I is a digital edge profile, with a proper choice of W, $\Xi[I]$ will be a Euclidean edge profile, so that $|\Xi[I] * D|$ will have a peak at the edge location $\mathbf{x}$. Now assuming I to be in $\ell^1 + \ell^2$, $\Xi[I]$ will be in $L^1 + L^2$, hence $\Xi[I] * D$ will be uniformly continuous (by Proposition 4.2), so that we know that at the digital point $\mathbf{z}$ closest to the peak $\mathbf{x}$ of $|\Xi[I] * D|$, we will have

$$\Big|\big(I * \Sigma[D]\big)(\mathbf{z}) - \big(\Xi[I] * D\big)(\mathbf{x})\Big| = \Big|\big(\Xi[I] * D\big)(\mathbf{z}) - \big(\Xi[I] * D\big)(\mathbf{x})\Big| \leq \psi(\mathbf{z}-\mathbf{x}), \tag{4.47}$$

where ψ is the modulus of continuity of $\Xi[I] * D$, so that this difference can become arbitrary small, provided that the digitization step is taken small enough. Thus we can expect a peak in $|I * \Sigma[D]|$ at some digital point close to $\mathbf{x}$. Moreover, as $\mathcal{D}$ "tends" to $\mathcal{E}$ (in the sense that the distance of $\mathcal{E}$ to $\mathcal{D}$ tends to 0, in other words that the resolution of the sampling grid tends to 0), then $I * \Sigma[D]$ will "tend" to $\Xi[I] * D$ (in the sense that the distance of the graph of $\Xi[I] * D$ to the graph of $I * \Sigma[D]$ will tend to 0).

One takes often W to be the characteristic function of the Euclidean cell corresponding to the origin in $\mathcal{D}$; for example if $\mathcal{E} = \mathbb{R}^d$ and $\mathcal{D} = \mathbb{Z}^d$, W will have value 1 on the hypercube made of all $(x_1, \ldots, x_d)$ having $-1/2 \leq x_i < 1/2$ for $i = 1, \ldots, d$, and value 0 outside that hypercube. Now given a uniformly continuous image I^* on $\mathcal{E}$, consider the digital image I on $\mathcal{D}$ obtained by classical sampling (that is $I(\mathbf{p}) = I^*(\mathbf{p})$ for every $\mathbf{p} \in \mathcal{D}$); for every $\mathbf{x} \in \mathcal{E}$, let $\mathbf{z} \in \mathcal{D}$ such that $\mathbf{x}$ is in the cell of $\mathbf{z}$ (in other words, $W(\mathbf{x} - \mathbf{z}) = 1$), we have $\Xi[I](\mathbf{x}) = I(\mathbf{z}) = I^*(\mathbf{z})$, so that

$$\big|\Xi[I](\mathbf{x}) - I^*(\mathbf{x})\big| = \big|I^*(\mathbf{z}) - I^*(\mathbf{x})\big| \leq \vartheta(\mathbf{z} - \mathbf{x}), \tag{4.48}$$

where ϑ is the modulus of continuity of I^*. As D is integrable, for every $\mathbf{x} \in \mathcal{E}$ we will have

$$\Big|\big(\Xi[I] * D\big)(\mathbf{x}) - \big(I^* * D\big)(\mathbf{x})\Big| \leq \vartheta \cdot \|D\|_1,$$

where ϑ is the supremum of all $\vartheta(\mathbf{p})$ for $\mathbf{p}$ in the Euclidean cell of the origin in $\mathcal{D}$. Thus when $\mathcal{D}$ "tends" to $\mathcal{E}$, $\Xi[I]$ and $\Xi[I] * D$ will tend uniformly to I^* and $I^* * D$ respectively.

The above two arguments (see (4.47) and (4.48)) show the importance of uniform continuity for digitization, in particular for the convergence of the filtered digital image to its Euclidean counterpart when the digital grid tends to the Euclidean space (i.e., the grid resolution tends to zero).

Of courses, it is also possible to make a theory of phase congruence for digital images. Note that the Fourier spectrum of digital signals is periodic, so that the distinction of "positive" and "negative" frequencies becomes arbitrary.

5.2. *Some applications, and conclusion*

The phase congruence model has been devised in order to be used for the detection and localization of edges. Here the precision of the localization of edge points demands that the filters C and S have a narrow support in the spatial domain.

On the other hand this model can also be used for other tasks than feature localization, but in order to make measurements on whole regions, and this time with C and S having a narrow support in the Fourier domain. Let us mention the model proposed by [49] for the measurement of stereo disparity. Here $d = 2$ and we assume that the normal orientation $\mathbf{n} = \mathbf{c}$ of C and S is horizontal; we suppose further that $\widehat{C}$ and $\widehat{S}$ have their energy concentrated in two narrow strips around the two lines $u_n = \pm d$ for some distance $d > 0$, where u_n represents the component of vector $\mathbf{u}$ along the normal orientation of C and S. Thus the spectrum of $D = C + i\,S$ is concentrated around the line $u_n = d$. We have two images I_1 and I_2. Suppose first that there is a uniform horizontal disparity h between them, in other words $I_1(x_n, x_t) = I_2(x_n + h, x_t)$ for all points (x_n, x_t); then we have $\widehat{I_1}(u_n, u_t) = \widehat{I_2}(u_n, u_t) \cdot \exp(2\pi i\, u_n h)$ for all frequency vectors (u_n, u_t). Hence we get $\widehat{I_1}(u_n, u_t)\widehat{D}(u_n, u_t)$ and $\widehat{I_2}(u_n, u_t)\widehat{D}(u_n, u_t)$ both very weak for u_n far from d, while for u_n in the vicinity of d we will have $\widehat{I_1}(u_n, u_t)\widehat{D}(u_n, u_t) \approx \widehat{I_2}(u_n, u_t)\widehat{D}(u_n, u_t)\exp(2\pi i\, dh)$; therefore $\widehat{I_1}\widehat{D} \approx \widehat{I_2}\widehat{D}\exp(2\pi i\, dh)$, in other words $I_1 * D$ will be "close" to $\exp(2\pi i\, dh)(I_2 * D)$. Removing the assumption of uniform disparity, the local disparity between I_1 and I_2 at a point (p_n, p_t) will be the value h such that $I_1(x_n, x_t)$ is "close" to $I_2(x_n + h, x_t)$ in the neighbourhood of of (p_n, p_t); it can be estimated as the argument of the complex number $(I_1 * D)(p_n, p_t)/(I_2 * D)(p_n, p_t)$.

Such an approach for computing stereo disparity can also be used for measuring region motion between two images. In [49] a three-dimensional model of disparity measurement using filters in quadrature is given for the integration of stereo and motion.

Let us now conclude. The phase congruence model for edge detection comprises several aspects:

(i) Edges are characterized as points of maximum Fourier phase congruence in the image. This type of definition is objective and to some extent scale-independent; in particular points where all Fourier phases coincide give an absolute maximum of the energy function at all scales. However there is no precise qualification of the notion of maximum phase congruence, since purely local maxima can be meaningless and do not respect the principle of causality in scale-space [27].

(ii) Edge detection is achieved by looking for maxima of the sum of squares of convolutions of the image with two filters C and S satisfying the requirements given in Section 4, or equivalently of the absolute value of the convolution of the image with the complex-valued filter $D = C + i\, S$. The use of two filters C and S, respectively even- and odd-symmetric, rather than a single one, is justified by the existence of several types of edges (see Figure 1), but one could envisage using more than two filters [47] if necessary. The Fourier amplitude and phase characteristics of C and S are justified from mathematical considerations, since interesting facts result from the nature of the Fourier spectrum of D (with positive values on $\mathcal{P}_{\mathbf{n}}$, and vanishing on $\mathcal{N}_{\mathbf{n}}$); in particular we could characterize orientation selectivity in this framework. Their phase characteristics are also justified from a physiological point of view [48]. However the equality of Fourier amplitudes of C and S can be challenged, since derivative pairs have some advantage over Hilbert transform pairs, for example causality in scale space [27].

Our study does not attempt to justify this model as the one corresponding to the functioning of human or animal visual detection of edges, nor does it claim any validity w.r.t. the photometric relevance of the edges that it gives. Furthermore the practical effectiveness of this model compared to other ones, in particular with methods using only one filter for each orientation, or using several filters separately, should be ascertained through experimental work. Probably this model would be most advantageously exploited in combination with radically different approaches, for example region-based segmentation (in particular watersheds). It should also be interesting to extend the model to multidirectional features and keypoints, such as curved edges, corners, junctions, or terminations.

The notion of Fourier phase congruence is not spatially localized, which contradicts the local nature of visual edges. A refined theory is needed, using concepts of localized Fourier phases, which could for example be based on wavelets.

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