Philips Research Laboratory Brussels
Av. E. Van Becelaere 2, Box 8
B-1170 Brussels, Belgium

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Lattices of inf-overfilters

C. Ronse
Philips Research Laboratory Brussels
Avenue Van Becelaere 2, Box 8, B-1170 Brussels, Belgium

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In an algebraic framework for mathematical morphology [8], Matheron defined a new class of operators called inf-overfilters, which generalize openings. They allow the design of new types of openings, some of them having a meaningful practical interpretation. Their algebraic properties and characterizations have been considered in [6,8]. We study here some complete lattices formed by such operators or by their associated openings.

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1. Introduction

The concept of an inf-overfilter is due to Matheron (see Chapter 6 of [8]). Some of its properties are found in Sections 6.3, 6.4, and 9.9 of [8], and in Section 4 of [6]. The purpose of this document is to deepen this study.

We adopt the notation and terminology of [1] for lattices, and of [3,6] for mathematical morphology and its expression in the framework of complete lattices. The object space (set of images on which we work) is a complete lattice $\mathcal{L}$ with universal bounds O and I, and the set $\mathcal{O}$ of operators $\mathcal{L} \to \mathcal{L}$ inherits that complete lattice structure, having universal bounds $\mathbf{O}$ and $\mathbf{I}$. The identity operator is written $\mathbf{id}$. We also consider a group $\mathbf{T}$ of automorphisms of the object space $\mathcal{L}$, and call a $\mathbf{T}$-operator an operator which is $\mathbf{T}$-invariant, in other words which commutes with every element of $\mathbf{T}$. We assume that the reader is acquainted with the basic properties of dilations, erosions, openings, and closings (see Section 2 of [3] and Subsections 2.1 and 2.2 of [6], or Chapters 1 and 5 of [8]).

Definition 1. An *inf-overfilter* is an increasing operator η such that $\eta(\mathbf{id} \wedge \eta) = \eta$. Dually, a *sup-underfilter* is an increasing operator ζ such that $\zeta(\mathbf{id} \vee \zeta) = \zeta$.

By duality, we can restrict our analysis to inf-overfilters and openings, the corresponding results for sup-underfilters and closings following immediately. Note that for an increasing operator η we always have $\eta(\mathbf{id} \wedge \eta) \leq \eta\,\mathbf{id} = \eta$; hence η will be an inf-overfilter if $\eta(\mathbf{id} \wedge \eta) \geq \eta$. We call a $\mathbf{T}$-*inf-overfilter* a $\mathbf{T}$-invariant inf-overfilter. When $\mathbf{T}$-invariance is not necessary, one can set $\mathbf{T} = \{\mathbf{id}\}$ and drop the prefix '$\mathbf{T}$-'.

The following elementary result (see [6], Proposition 4.1) highlights the meaning of the concept of an inf-overfilter:

Proposition 1. *Given an inf-overfilter η, $\eta \leq \eta^2$ and $\mathbf{id} \wedge \eta$ is an opening.*

An operator ψ such that $\psi^2 \geq \psi$ is called by Matheron an *overfilter*, and this explains the origin of the term 'inf-overfilter'. Any opening is an inf-overfilter as it corresponds to the particular case where $\eta = \mathbf{id} \wedge \eta$.

An inf-overfilter can be interpreted as an increasing operator η applying to $X \in \mathcal{L}$ an opening $\mathbf{id} \wedge \eta$, but adding to it something more (the difference between $\eta(X)$ and $(\mathbf{id} \wedge \eta)(X)$), which does not depend on X, but only on the result $(\mathbf{id} \wedge \eta)(X)$ of that opening.

A practical example of inf-overfilters is given by the operators introduced in [5] for digital grey-level images. By composing a rank filter ρ_B^k associated to a rank k and a structuring element B with the dilation δ_B by B (in other words the max filter associated to the reflected structuring element $\check{B}$), one obtains an inf-overfilter (see Subsection 4.2 of [6]); the corresponding opening $\mathbf{id} \wedge \delta_B \rho_B^k$ has the following interpretation: it transforms an image X into the supremum of all portions of it which consist of a sufficiently large subset of a translate of B. It is thus a generalization of the morphological opening by B, obtained for $k = 1$, which associates to image X the supremum of all portions of it equal to a translate of B.

Matheron and Serra (see [8], Sections 6.3, 6.4, and 9.9) sudied basic properties and characterizations of inf-overfilters. This work was extended in Section 4 of [6], with the assumption of $\mathbf{T}$-invariance. We recall some of these results in Section 2. In Section 3 we study the complete lattice of inf-overfilters associated to any given opening, and in Section 4 we give decomposition formulas for such inf-overfilters.

2. Basic properties of inf-overfilters

The following five results are proved in Subsection 4.1 of [6], and generalize some results of Matheron and Serra.

Although the converse of Proposition 1 is not true (see Subsection 4.1 of [6] for a counterexample), nevertheless an opening of the form $\mathbf{id} \wedge \psi$, where ψ is increasing, arises in fact from an inf-overfilter:

Proposition 2. *Let ψ be an increasing operator such that $\mathbf{id} \wedge \psi$ is an opening. Let $\eta = \psi(\mathbf{id} \wedge \psi)$. Then η is an inf-overfilter and $\mathbf{id} \wedge \eta = \mathbf{id} \wedge \psi$.*

Proposition 3. *The set of $\mathbf{T}$-inf-overfilters is sup-closed and has $\mathbf{I}$ as greatest element.*

Proposition 4. *Let η be a $\mathbf{T}$-inf-overfilter, α a $\mathbf{T}$-opening, ε a $\mathbf{T}$-erosion, and ψ an increasing $\mathbf{T}$-operator. Then the following operators are $\mathbf{T}$-inf-overfilters:*

(*i*) *$\psi\eta$, if $\psi \geq \mathbf{id} \wedge \eta$.*
(*ii*) *η^2.*
(*iii*) *$\psi\alpha$, if $\psi \geq \alpha$.*
(*iv*) *$\psi\varepsilon$, if $\psi \geq \varepsilon$.*

Note that any constant operator $\gamma_A : X \mapsto A$ is an inf-overfilter. It is $\mathbf{T}$-invariant if A is fixed by $\mathbf{T}$.

Corollary 5. *Given an increasing $\mathbf{T}$-operator η, the following three statements are equivalent:*

(*i*) *η is a $\mathbf{T}$-inf-overfilter.*
(*ii*) *If α_η is the greatest $\mathbf{T}$-opening $\leq \eta$, then $\eta\alpha_\eta = \eta$.*
(*iii*) *There is a $\mathbf{T}$-opening α and an increasing $\mathbf{T}$-operator θ such that $\theta \geq \alpha$ and $\eta = \theta\alpha$.*

3. The complete lattice of T-inf-overfilters associated to a T-opening

From Corollary 5 we know that $\mathbf{T}$-inf-overfilters can be characterized as operators of the form $\theta\alpha$ for a $\mathbf{T}$-opening α and an increasing $\mathbf{T}$-operator $\geq \alpha$. We make thus the following:

Definition 2. Given a $\mathbf{T}$-opening α, write $\mathcal{H}_{\mathbf{T}}(\alpha)$ for the set of all $\mathbf{T}$-inf-overfilters $\theta\alpha$, where θ is an increasing $\mathbf{T}$-operator $\geq \alpha$.

Let now α be a fixed $\mathbf{T}$-opening.

Lemma 6. *Given a $\mathbf{T}$-operator η, $\eta \in \mathcal{H}_{\mathbf{T}}(\alpha)$ if and only if η is increasing, $\eta \geq \alpha$, and $\eta\alpha = \eta$.*

Proof. If $\eta \in \mathcal{H}_{\mathbf{T}}(\alpha)$, that is $\eta = \theta\alpha$ for an increasing $\theta \geq \alpha$, then η is increasing, $\eta = \theta\alpha \geq \alpha\alpha = \alpha$, and $\eta\alpha = \theta\alpha\alpha = \theta\alpha = \eta$. If η is increasing, $\eta \geq \alpha$, and $\eta\alpha = \eta$, then we take $\theta = \eta$. ∎

Corollary 7. *α is the unique $\mathbf{T}$-opening in $\mathcal{H}_{\mathbf{T}}(\alpha)$.*

Proof. For an opening $\alpha' \in \mathcal{H}_{\mathbf{T}}(\alpha)$, by Lemma 6 we have $\alpha' \geq \alpha$ and $\alpha'\alpha = \alpha'$; now the latter equality implies $\alpha' \leq \alpha$ (see Proposition 2.3 of [6]), so that $\alpha' = \alpha$. ∎

Theorem 8. *$\mathcal{H}_{\mathbf{T}}(\alpha) \cup \{\mathbf{O}\}$ is a complete sublattice of $\mathcal{O}$.*

Proof. Consider a non-empty family of elements η_j of $\mathcal{H}_{\mathbf{T}}(\alpha)$ ($j \in J \neq \emptyset$). As $\eta_j \geq \alpha$ and $\eta_j\alpha = \alpha$ for each $j \in J$ (by Lemma 6), we get $\bigwedge_{j\in J} \eta_j \geq \alpha$ and $(\bigwedge_{j\in J} \eta_j)\alpha = \bigwedge_{j\in J}(\eta_j\alpha) = \bigwedge_{j\in J} \eta_j$; now $\bigwedge_{j\in J} \eta_j$ is $\mathbf{T}$-invariant, hence it belongs to $\mathcal{H}_{\mathbf{T}}(\alpha)$. Similarly $\bigvee_{j\in J} \eta_j \in \mathcal{H}_{\mathbf{T}}(\alpha)$. Thus $\mathcal{H}_{\mathbf{T}}(\alpha)$ is closed under non-empty suprema and infima. Now $\mathbf{I} \in \mathcal{H}_{\mathbf{T}}(\alpha)$, and so $\mathcal{H}_{\mathbf{T}}(\alpha) \cup \{\mathbf{O}\}$ is both sup- and inf-closed. ∎

Note that $\mathcal{H}_{\mathbf{T}}(\alpha)$ is itself a complete lattice, with the same supremum and infimum operations as in $\mathcal{O}$, except that $\sup \emptyset = \alpha$ instead of $\mathbf{O}$.

Proposition 9. *Given an increasing $\mathbf{T}$-operator $\theta \geq \alpha$, for any $\eta \in \mathcal{H}_{\mathbf{T}}(\alpha)$, $\theta\eta \in \mathcal{H}_{\mathbf{T}}(\alpha)$. In particular $\mathcal{H}_{\mathbf{T}}(\alpha)$ is closed under composition.*

Proof. By Lemma 6 we have $\eta\alpha = \eta$ and $\eta \geq \alpha$, so that $\theta\eta\alpha = \theta\eta$ and $\theta\eta \geq \alpha\alpha = \alpha$, that is $\theta\eta \in \mathcal{H}_{\mathbf{T}}(\alpha)$. ∎

Lemma 10. *Let α' be an opening such that $\alpha' \geq \alpha$. Then $\eta\alpha' = \eta$ for every $\eta \in \mathcal{H}_{\mathbf{T}}(\alpha)$; in particular when $\eta \geq \alpha'$ we have $\eta \in \mathcal{H}_{\mathbf{T}}(\alpha')$.*

Proof. As $\mathbf{id} \geq \alpha' \geq \alpha$ and $\eta\alpha = \eta$, we get $\eta \geq \eta\alpha' \geq \eta\alpha = \eta$, that is $\eta\alpha' = \eta$. If $\eta \geq \alpha'$, then $\eta \in \mathcal{H}_{\mathbf{T}}(\alpha')$ by Lemma 6. ∎

Corollary 11. *For every $\eta, \eta' \in \mathcal{H}_{\mathbf{T}}(\alpha)$, $\eta(\mathbf{id} \wedge \eta') = \eta$ and $(\mathbf{id} \wedge \eta)(\mathbf{id} \wedge \eta') = \mathbf{id} \wedge \eta \wedge \eta'$.*

Proof. As $\mathbf{id} \wedge \eta'$ is an opening $\geq \alpha$, $\eta(\mathbf{id} \wedge \eta') = \eta$ by Lemma 11. Thus

$$(\mathbf{id} \wedge \eta)(\mathbf{id} \wedge \eta') = \mathbf{id}(\mathbf{id} \wedge \eta') \wedge \eta(\mathbf{id} \wedge \eta') = (\mathbf{id} \wedge \eta') \wedge \eta = \mathbf{id} \wedge \eta \wedge \eta'. \;\blacksquare$$

Definition 3. Write $\mathcal{A}_{\mathbf{T}}(\alpha)$ for the set of all $\mathbf{T}$-openings of the form $\mathbf{id} \wedge \eta$, where $\eta \in \mathcal{H}_{\mathbf{T}}(\alpha)$.

Proposition 12. *$\mathcal{A}_{\mathbf{T}}(\alpha) \cup \{\mathbf{I}\}$ is inf-closed, and for any $\alpha_1, \ldots, \alpha_n \in \mathcal{A}_T(\alpha)$ ($n \geq 2$),*

$$\alpha_1 \cdots \alpha_n = \alpha_1 \wedge \ldots \wedge \alpha_n.$$

Moreover, if $\mathcal{L}$ satisfies the infinite supremum distributy condition

$$X \wedge (\bigvee_{j\in J} Y_j) = \bigvee_{j\in J} (X \wedge Y_j), \tag{ISD}$$

then $\mathcal{A}_{\mathbf{T}}(\alpha) \cup \{\mathbf{O}\}$ is sup-closed and $\mathcal{A}_{\mathbf{T}}(\alpha) \cup \{\mathbf{O}, \mathbf{I}\}$ is a complete sublattice of $\mathcal{O}$.

Proof. For $\alpha_1 = \mathbf{id} \wedge \eta_1$ and $\alpha_2 = \mathbf{id} \wedge \eta_2$, where $\eta_1, \eta_2 \in \mathcal{H}_{\mathbf{T}}(\alpha)$, Corollary 11 says that

$$\alpha_1\alpha_2 = (\mathbf{id} \wedge \eta_1)(\mathbf{id} \wedge \eta_2) = \mathbf{id} \wedge \eta_1 \wedge \eta_2 = \alpha_1 \wedge \alpha_2.$$

For $\alpha_1, \ldots, \alpha_n \in \mathcal{A}_T(\alpha)$, where $n > 2$, the equality $\alpha_1 \cdots \alpha_n = \alpha_1 \wedge \ldots \wedge \alpha_n$ follows by induction:

$$\begin{aligned} \alpha_1 \cdots \alpha_n &= (\alpha_1 \cdots \alpha_{n-1})\alpha_n = (\alpha_1 \wedge \ldots \wedge \alpha_{n-1})\alpha_n \\ &= \alpha_1\alpha_n \wedge \ldots \wedge \alpha_{n-1}\alpha_n = (\alpha_1 \wedge \alpha_n) \wedge \ldots \wedge (\alpha_{n-1} \wedge \alpha_n) = \alpha_1 \wedge \ldots \wedge \alpha_n. \end{aligned}$$

Given a non-empty family of elements η_j of $\mathcal{H}_{\mathbf{T}}(\alpha)$ $(j \in J \neq \emptyset)$, by Theorem 8 we have $\bigwedge_{j\in J} \eta_j \in \mathcal{H}_{\mathbf{T}}(\alpha)$. Hence $\bigwedge_{j\in J}(\mathbf{id} \wedge \eta_j) = \mathbf{id} \wedge \bigwedge_{j\in J} \eta_j \in \mathcal{A}_{\mathbf{T}}(\alpha)$. Thus $\mathcal{A}_{\mathbf{T}}(\alpha)$ is closed under non-empty infima, and so $\mathcal{A}_{\mathbf{T}}(\alpha) \cup \{\mathbf{I}\}$ is inf-closed.

By Theorem 8 again, $\bigvee_{j\in J} \eta_j \in \mathcal{H}_{\mathbf{T}}(\alpha)$; now if $\mathcal{L}$ satisfies the condition (ISD), then $\mathcal{O}$ satisfies it also and we get $\bigvee_{j\in J}(\mathbf{id} \wedge \eta_j) = \mathbf{id} \wedge (\bigvee_{j\in J} \eta_j) \in \mathcal{A}_{\mathbf{T}}(\alpha)$. Thus $\mathcal{A}_{\mathbf{T}}(\alpha)$ is closed under non-empty suprema, and so $\mathcal{A}_{\mathbf{T}}(\alpha) \cup \{\mathbf{O}\}$ is sup-closed. Therefore $\mathcal{A}_{\mathbf{T}}(\alpha) \cup \{\mathbf{O}, \mathbf{I}\}$ is both sup- and inf-closed. ∎

This result is very interesting, because in general an infimum or composition of openings is not an opening. Some examples will be given at the end of the next section.

Proposition 13. *Given openings α_j and $\eta_j \in \mathcal{H}_{\mathbf{T}}(\alpha_j)$ $(j \in J)$, we have $\bigvee_{j\in J} \eta_j \in \mathcal{H}_{\mathbf{T}}(\bigvee_{j\in J} \alpha_j)$.*

Proof. For $J = \emptyset$, this reduces to $\mathbf{O} \in \mathcal{H}_{\mathbf{T}}(\mathbf{O})$. Assume thus that $J \neq \emptyset$. Clearly $\eta = \bigvee_{j\in J} \eta_j$ is increasing and $\mathbf{T}$-invariant. Now $\alpha = \bigvee_{j\in J} \alpha_j$ is a $\mathbf{T}$-opening, and for each $j \in J$ we have $\eta \geq \eta_j = \eta_j\alpha_j$ and $\mathbf{id} \geq \alpha \geq \alpha_j$; hence $\eta \geq \eta\alpha \geq \eta_j\alpha_j = \eta_j$, that is $\eta\alpha = \eta$. As $\eta_j \geq \alpha_j$ for each $j \in J$, we get $\eta \geq \alpha$. ∎

4. Decomposition formulas

We will give formulas representing elements of $\mathcal{H}_{\mathbf{T}}(\alpha)$. We recall that in a $\mathbf{T}$-adjunction (ε, δ), δ is a $\mathbf{T}$-dilation, ε is a $\mathbf{T}$-erosion, $\delta = \delta\varepsilon\delta$, $\varepsilon = \varepsilon\delta\varepsilon$, and $\delta\varepsilon$ is a $\mathbf{T}$-opening, called a *morphological* $\mathbf{T}$*-opening*.

Proposition 14. *For any $\mathbf{T}$-adjunction (ε, δ), the lattice $\mathcal{H}_{\mathbf{T}}(\delta\varepsilon)$ is the set of all $\psi\varepsilon$, where ψ is an increasing $\mathbf{T}$-operator $\geq \delta$.*

Proof. If $\eta \in \mathcal{H}_{\mathbf{T}}(\delta\varepsilon)$, then $\eta = \eta\delta\varepsilon$ and $\eta \geq \delta\varepsilon$. Setting $\psi = \eta\delta$, ψ is an increasing $\mathbf{T}$-operator, $\eta = \eta\delta\varepsilon = \psi\varepsilon$, and $\psi = \eta\delta \geq \delta\varepsilon\delta = \delta$. Conversely, if $\eta = \psi\varepsilon$ for an increasing $\mathbf{T}$-operator $\psi \geq \delta$, then $\eta = \psi\varepsilon \geq \delta\varepsilon$ and $\eta\delta\varepsilon = \psi\varepsilon\delta\varepsilon = \psi\varepsilon = \eta$, that is $\eta \in \mathcal{H}_{\mathbf{T}}(\delta\varepsilon)$. ∎

Proposition 15. *Given two non-empty index sets J, K, let $(\varepsilon_j, \delta_j)$ be a $\mathbf{T}$-adjunction for $j \in J$, let ψ_{kj} be an increasing $\mathbf{T}$-operator for $j \in J$ and $k \in K$, and assume that $\psi_{kj} \geq \delta_j$ for every j, k. Then the operator*

$$\eta = \bigwedge_{k\in K} \bigvee_{j\in J} \psi_{kj}\varepsilon_j \tag{1}$$

belongs to $\mathcal{H}_{\mathbf{T}}(\bigvee_{j\in J} \delta_j\varepsilon_j)$.

Proof. By Proposition 14 we have $\psi_{kj}\varepsilon_j \in \mathcal{H}_{\mathbf{T}}(\delta_j\varepsilon_j)$ for every $j \in J$, $k \in K$. Proposition 13 implies that $\eta_k = \bigvee_{j\in J} \psi_{kj}\varepsilon_j \in \mathcal{H}_{\mathbf{T}}(\bigvee_{j\in J} \delta_j\varepsilon_j)$ for every $k \in K$. As $\mathcal{H}_{\mathbf{T}}(\bigvee_{j\in J} \delta_j\varepsilon_j)$ is closed under non-empty infima (by Theorem 8), $\eta = \bigwedge_{k\in K} \eta_k \in \mathcal{H}_{\mathbf{T}}(\bigvee_{j\in J} \delta_j\varepsilon_j)$. ∎

The invariants of the opening $\mathbf{id} \wedge \eta$ for η as in (1) are characterized in Theorem 4.6 and Corollary 4.7 of [6].

In order to give a converse of Proposition 15, we use a decomposition of an increasing $\mathbf{T}$-operator θ as an infimum of $\mathbf{T}$-dilations. This requires of course that $\theta(O) = O$, but even then such a decomposition is not always possible (see for example Subsection 4.2 of [3]).

THEOREM 16. *Suppose that in $\mathcal{L}$ every increasing $\mathbf{T}$-operator fixing O is an infimum of $\mathbf{T}$-dilations. Let $\alpha = \bigvee_{j\in J} \delta_j \varepsilon_j$, where $J \neq \emptyset$ and each $(\varepsilon_j, \delta_j)$ is a $\mathbf{T}$-adjunction. Let η be a $\mathbf{T}$-operator. Then $\eta \in \mathcal{H}_{\mathbf{T}}(\alpha)$ if and only if there exists a non-empty index set K, a family of increasing $\mathbf{T}$-operators ψ_{kj} $(j \in J,\ k \in K)$, such that $\psi_{kj} \geq \delta_j$ for every j, k, and η takes the form* (1). *Moreover, if $\eta(O) = O$, then we can choose the operators ψ_{kj} to be $\mathbf{T}$-dilations.*

PROOF. By Proposition 15 we have only to prove that every $\eta \in \mathcal{H}_{\mathbf{T}}(\alpha)$ takes this form. Define the operators γ and θ by $\gamma(X) = \eta(O)$ for $X \in \mathcal{L}$, $\theta(O) = O$, and $\theta(X) = \eta(X)$ for $X \neq O$. It is easy to check that θ is an increasing $\mathbf{T}$-operator and that $\eta = \theta \vee \gamma$. By our assumption we have the decomposition $\theta = \bigwedge_{k\in K} \delta'_k$, where δ'_k is a dilation for each $k \in K$. As $\theta(O) = O$, $\theta \neq \mathbf{I}$, and so $K \neq \emptyset$. Let $\eta' = \bigwedge_{k\in K}(\delta'_k \vee \gamma)$; we will show that $\eta' = \eta$. First, given $X \neq O$, for each $k \in K$ we have $\delta'_k(X) \geq \theta(X) = \eta(X) \geq \eta(O) = \gamma(X)$, and so $(\delta'_k \vee \gamma)(X) = \delta'_k(X)$; then

$$\eta'(X) = \bigwedge_{k\in K} (\delta'_k \vee \gamma)(X) = \bigwedge_{k\in K} \delta'_k(X) = \theta(X) = \eta(X).$$

Next, for each $k \in K$ we have $\delta'_k(O) = O$ and so $(\delta'_k \vee \gamma)(O) = \gamma(O) = \eta(O)$; then

$$\eta'(O) = \bigwedge_{k\in K} (\delta'_k \vee \gamma)(O) = \bigwedge_{k\in K} \eta(O) = \eta(O).$$

Thus $\eta'(X) = \eta(X)$ for each $X \in \mathcal{L}$, that is $\eta' = \eta$. Now we have

$$\begin{aligned}
\eta = \eta\alpha = \eta'\alpha &= \Big[\bigwedge_{k\in K} (\delta'_k \vee \gamma)\Big]\Big(\bigvee_{j\in J} \delta_j\varepsilon_j\Big) = \bigwedge_{k\in K} \Big[(\delta'_k \vee \gamma)\Big(\bigvee_{j\in J} \delta_j\varepsilon_j\Big)\Big] \\
&= \bigwedge_{k\in K} \Big[\delta'_k\Big(\bigvee_{j\in J} \delta_j\varepsilon_j\Big) \vee \gamma\Big(\bigvee_{j\in J} \delta_j\varepsilon_j\Big)\Big] = \bigwedge_{k\in K} \Big[\Big(\bigvee_{j\in J} \delta'_k\delta_j\varepsilon_j\Big) \vee \gamma\Big] \\
&= \bigwedge_{k\in K} \bigvee_{j\in J} (\delta'_k\delta_j\varepsilon_j \vee \gamma) = \bigwedge_{k\in K} \bigvee_{j\in J} (\delta'_k \vee \gamma)\delta_j\varepsilon_j.
\end{aligned}$$

We used above the two facts that each δ'_k commutes with the supremum and that $\gamma\beta = \gamma$ for every operator β. Finally $\delta'_k \vee \gamma \geq \delta'_k \geq \theta \geq \alpha \geq \delta_j\varepsilon_j$ and so $(\delta'_k \vee \gamma)\delta_j \geq \delta_j\varepsilon_j\delta_j = \delta_j$. We take thus $\psi_{kj} = (\delta'_k \vee \gamma)\delta_j$ for $j \in J,\ k \in K$. If $\eta(O) = O$, then $\gamma = \mathbf{O}$ and so $\psi_{kj} = \delta'_k\delta_j$, a dilation. ∎

REMARK. (i) For $\mathbf{T} = \{\mathbf{id}\}$, every increasing operator fixing O is an infimum of dilations (see [3], Theorem 2.4), and every opening is a supremum of morphological openings (see [6], Proposition 2.9). In this case Theorem 16 characterizes $\mathcal{H}_{\{\mathbf{id}\}}(\alpha)$ for any opening α.
(ii) If $\mathbf{T} \neq \{\mathbf{id}\}$, then we have not always such decompositions. However:

— if $\mathcal{L}$ satisfies the so-called *Basic Assumption* ($\mathbf{T}$ is abelian and transitive on a sup-generating family of $\mathcal{L}$), then every $\mathbf{T}$-opening is a supremum of morphological $\mathbf{T}$-openings (see [6], Theorem 2.11);

— if $\mathcal{L}$ satisfies the *dual* of that Basic Assumption ($\mathbf{T}$ is abelian and transitive on an inf-generating family of $\mathcal{L}$), then every increasing $\mathbf{T}$-operator is an infimum of $\mathbf{T}$-dilations (see [3], Theorem 3.11 and Remark 3.2 (iv)).

Note that if $\mathcal{L}$ satisfies the Basic Assumption or its dual, $\mathbf{I}$ is the only increasing $\mathbf{T}$-operator which does not fix O. Thus in the case of Boolean or grey-level images on a Euclidean or digital space, Theorem 16 characterizes $\mathcal{H}_{\mathbf{T}}(\alpha)$ for any $\mathbf{T}$-opening α, and the operators ψ_{kj} will be dilations, except for $\eta = \mathbf{I}$.

(iii) Taking θ, γ as defined in the above proof, if we set $\eta_0 = \theta\alpha$, then $\eta_0 \in \mathcal{H}_{\mathbf{T}}(\alpha)$, $\eta_0(O) = O$, $\eta = \eta_0 \vee \gamma$, and

$$\eta_0 = \bigwedge_{k\in K} \bigvee_{j\in J} \delta'_k \delta_j \varepsilon_j, \qquad \text{while} \qquad \eta = \bigwedge_{k\in K} \bigvee_{j\in J} (\delta'_k \vee \gamma)\delta_j \varepsilon_j.$$

The real difficulty in the proof is in showing that $\gamma \vee \bigwedge_{k\in K} \delta'_k = \bigwedge_{k\in K} (\delta'_k \vee \gamma)$ and

$$\gamma \vee \bigwedge_{k\in K} \bigvee_{j\in J} \delta'_k \delta_j \varepsilon_j = \bigwedge_{k\in K} \bigvee_{j\in J} (\delta'_k \vee \gamma)\delta_j \varepsilon_j$$

without assuming (ISD) (if we assume it, this is trivial).

Example. The rank-max opening for Boolean or grey-level images on a digital space described in [5] is one example in a larger class of openings. The basic idea, which is explained in Subsection 4.2 of [6], is to take an opening which transforms an image X into the supremum of all portions of it which contain 'most' of a translate of a finite structuring element B. If $C_1, \ldots, C_m$ are the smallest subsets of B which contain 'most' of B, then the resulting $\mathbf{T}$-opening takes the form

$$\alpha = \mathbf{id} \wedge \delta_B(\bigvee_{i=1}^{m} \varepsilon_{C^i}), \qquad \text{where} \quad B \supseteq \bigcup_{j=1}^{m} C_j. \tag{2}$$

If we take for sets C^i all subsets of B having size t ($1 \leq t \leq |B|$), then we get the rank-max opening for the rank $k = |B| - t + 1$. For $t = |B|$, this gives $k = 1$, $m = 1$ and $C^1 = B$, and then α reduces to α_B. Thus the opening by B is a particular case of (2).

Clearly the $\mathbf{T}$-opening (2) is of the form $\mathbf{id} \wedge \eta$, where η has the form $\delta(\bigvee_{j\in J} \varepsilon_j) = \bigvee_{j\in J} \delta\varepsilon_j$, and for each $j \in J$, $(\varepsilon_j, \delta_j)$ is a $\mathbf{T}$-adjunction and $\delta \geq \delta_j$. Thus it is a particular form of (1), with $\psi_{kj} = \delta$ for each $k \in K$ and $j \in J$, and so $\eta \in \mathcal{H}_{\mathbf{T}}(\alpha)$ for $\alpha = \bigvee_{j\in J} \delta_j \varepsilon_j$.

While keeping the C_j constant ($j \in J$), we can modify δ_B (with the constraint $B \supseteq \bigcup_{j=1}^{m} C_j$, in other words $\delta_B \geq \bigvee_{i=1}^{m} \delta_{C_j}$), and we obtain thus different openings in $\mathcal{A}_{\mathbf{T}}(\alpha)$, where $\alpha = \bigvee_{j=1}^{m} \delta_{C_j} \varepsilon_{C_j}$. They satisfy the property of Proposition 12: given a non-empty family of such openings, we can take their composition or equivalently their infimum, and also their supremum, and we still get an opening in $\mathcal{A}_{\mathbf{T}}(\alpha)$. In other words openings of the form (2), together with $\mathbf{O}, \mathbf{I}$, generate a complete sublattice $\mathcal{A}^*_{\mathbf{T}}(\alpha) \cup \{\mathbf{O}, \mathbf{I}\}$ of $\mathcal{A}_{\mathbf{T}}(\alpha) \cup \{\mathbf{O}, \mathbf{I}\}$, which is also closed under composition; clearly element of $\mathcal{A}^*_{\mathbf{T}}(\alpha)$ take the form

$$\mathbf{id} \wedge \psi(\bigvee_{i=1}^{m} \varepsilon_{C^i}), \qquad \text{where} \quad \psi(O) = O \quad \text{and} \quad \psi \geq \bigvee_{i=1}^{m} \delta_{C^i}. \tag{3}$$

Conversely every such ψ is an infimum of a non-empty family of dilations $\delta_B \geq \bigvee_{i=1}^{m} \delta_{C^i}$, and so an operator of the form (3) can be written as

$$\mathbf{id} \wedge (\bigwedge_{k \in K} \delta_{B_k})(\bigvee_{i=1}^{m} \varepsilon_{C^i}), \qquad \text{where} \quad K \neq \emptyset \quad \text{and} \quad B_k \supseteq \bigcup_{j=1}^{m} C_j \quad \text{for} \quad k \in K. \tag{4}$$

Thus (3) and (4) are equivalent characterizations of the openings in $\mathcal{A}^*_{\mathbf{T}}(\alpha)$, and so such openings are non-empty infima of openings of the form (2).

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