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Openings: Main Properties, and How to Construct Them

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Abstract: Openings (and dually closings) are an important class of transformations used in image analysis. We review their main algebraic properties, in particular their characterization in terms of the invariance domain, and give several methods for constructing them. A particular emphasis is laid on inf-overfilters as a class of operators generating new types of openings. In conformity with the new algebraic framework for mathematical morphology, we assume that the object space under study is any complete lattice, optionally provided with an arbitrary group of automorphisms.

1. Opening and Closing: Two Simple but Useful Notions

Suppose that we have an object space $\mathcal{L}$ whose elements (written $A, B, \ldots, Y, Z$, etc.) can be compared thanks to a partial order relation $\leq$. For example $\mathcal{L}$ can be:

(a) The set $\mathcal{P}(\mathcal{E})$ of all subsets of a Euclidean or digital space $\mathcal{E}$, or the set $Conv(\mathcal{E})$ of all convex subsets of $\mathcal{E}$; here $\leq$ is the inclusion relation $\subseteq$.

(b) The set of grey-level images on $\mathcal{E}$, where for two images X and Y, $X \leq Y$ if for any point $p \in \mathcal{E}$, its respective grey-levels $X(p)$ and $Y(p)$ satisfy $X(p) \leq Y(p)$.

(c) The set of all partitions of $\mathcal{E}$, or the set of those partitions whose classes are connected; they arise in particular when one attempts to segment an image on $\mathcal{E}$ into meaningful components; for two partitions X and Y of $\mathcal{E}$, $X \leq Y$ means that every class of X is contained in a class of Y, and we say then that Y is *coarser* than X, or that X is *finer* than Y. See also [22], pp. 15, 32, and 94–98 for more details.

Many other choices for $\mathcal{L}$ can be envisaged. The reader is referred in particular to [6,19,22].

Assume that we want to filter an object $X \in \mathcal{L}$ in order to remove from it some aspects that we don't need, or to extract from X some particular type of information. For this purpose we apply some "filtering" operator ψ to X. Let us examine some desirable properties for ψ:

(i) *Idempotence*: $\psi^2 = \psi$, that is $\psi(\psi(X)) = \psi(X)$ for any $X \in \mathcal{L}$.

This corresponds to the usual notion of a perfect filter: it removes completely what is unwanted, and so needs not be applied a second time. This is for example the case with ideal band-pass filters in signal processing. In contrast the median filter is not idempotent, and it is not even guaranteed that a repeated application of it to an image converges to a stable result. It has been shown [26] that the repetition of a median filter with symmetric windows on a digital image having a finite number of pixels with non-zero grey-level, leads after a finite number of steps either to a stable solution, or to an oscillation of period 2. Such an oscillation between two images is illustrated for example in Figure 8.1 on p. 160 of [22].

In [12] considerations about human low-level vision lead to the postulate that edge detection must be idempotent. It has similarly been argued in [21] that idempotence is a normal requirement of each stage in a sequence of operations in image analysis. See also Subsection 1.1 of [19].

Now let us relate the behaviour of ψ to the partial ordering of $\mathcal{L}$ by $\leq$. We require that if ψ removes something, then it removes anything smaller. In particular if $X \leq Y \leq Z$ and $\psi(X) = \psi(Z)$ (ψ removes the difference between X and Z), then we have also $\psi(X) = \psi(Y)$ (ψ removes the difference between X and Y, and that between Y and Z). A sufficient condition for this is:

(ii) *Growth*: For any $X, Y \in \mathcal{L}$, $X \leq Y$ implies $\psi(X) \leq \psi(Y)$.

We say then that ψ is *increasing*. An increasing idempotent operator is called a *morphological filter*. In the case where $(\mathcal{L}, \leq)$ is a complete lattice, the properties of such operators have been analysed by Matheron and Serra in Chapters 5 to 10 of [22]. We recall in the Appendix some of Matheron's results. It appears that an analytic decomposition of morphological filters into simple building blocks is a very hard task, even in the particular case where $\mathcal{L}$ consists of the set of parts of a digital space $\mathcal{E}$. However, this is easily achieved if we assume one of the following two constraints, which are dual w.r.t. the partial order $\leq$:

(iii) *Extensivity*: For any $X \in \mathcal{L}$, $\psi(X) \geq X$.

(iii') *Anti-extensivity*: For any $X \in \mathcal{L}$, $\psi(X) \leq X$.

An extensive morphological filter is called a *closing*, while an anti-extensive morphological filter is called an *opening*. In the case of images, an opening will usually remove small positive features, such as peaks and narrow ridges, while a closing will remove small negative features, such as holes and narrow valleys. As explained in Chapter 10 of [20], openings are related to size distributions: given $\lambda > 0$, the operator ψ_λ extracting from a population the subset consisting of all elements of size at least λ must clearly be an opening (for example, in an army take all soldiers at least six feet tall).

We end with an optional property: invariance under a symmetry group. In Euclidean morphology [20], operators are generally required to be translation-invariant, which means that they commute with any Euclidean translation. Some studies in image analysis consider the stronger assumption of isotropy, that is invariance under both translations and rotations. Note that these symmetries preserve the inclusion relation on sets. We generalize symmetries by automorphisms. An *automorphism* of $(\mathcal{L}, \leq)$ is a permutation τ of $\mathcal{L}$ which preserves the partial order $\leq$, in other words such that for $X, Y \in \mathcal{L}$, $X \leq Y \iff \tau(X) \leq \tau(Y)$. We consider a group $\mathbf{T}$ (that is, $\mathbf{T}$ is closed under composition and inversion), consisting of some automorphisms of $\mathcal{L}$; in fact $\mathbf{T}$ is an arbitrary subgroup of the group $Aut(\mathcal{L})$ of all automorphisms of $\mathcal{L}$. We assume then:

(iv) **T**-*invariance*: For any $\tau \in \mathbf{T}$, $\tau\psi = \psi\tau$.

We will use the prefix "**T**-" for "**T**-invariant"; we will speak thus of **T**-operators, **T**-dilations, **T**-erosions, **T**-openings, **T**-closings, etc. (see Section 3 of [6]). As we said above, the requirement of **T**-invariance is optional, for example it is absent from the characterizations given in [22]. All results that we will give in the framework of **T**-invariance can directly be applied to the case where it is not assumed: one has just to take **T** consisting of only the identity operator **id**.

Conditions (i) to (iv) were presented for a set $\mathcal{L}$ provided with a partial order relation $\leq$. In order to give a decompositions of **T**-openings and **T**-closings in terms of this structure, $\mathcal{L}$ itself must admit decompositions related to that partial order. In other words we will assume that $(\mathcal{L}, \leq)$ is a *complete lattice*, which means that every non-void subset $\mathcal{K}$ of $\mathcal{L}$ has a least upper bound in $\mathcal{L}$ or *supremum*, as well as a greatest lower bound in $\mathcal{L}$ or *infimum*; the supremum and infimum of $\mathcal{K}$ are necessarily unique, and we write them respectively $\sup \mathcal{K}$ or $\bigvee \mathcal{K}$, and $\inf \mathcal{K}$ or $\bigwedge \mathcal{K}$. As we will see in Section 2, **T**-openings and **T**-closings on a complete lattice are easily characterized by their domain of invariance, and can be decomposed in terms of elementary operators called *structural* **T**-openings and **T**-closings.

Clearly opening and closing are dual concepts w.r.t. the partial order $\leq$. Therefore every statement concerning openings can be translated into a similar statement concerning closings by interverting $\leq$ and $\geq$, $\bigvee$ and $\bigwedge$, extensivity and anti-extensivity, dilations and erosions, etc. We may thus to a great extent restrict ourselves to openings.

Most practitioners of mathematical morphology know the opening by a structuring element, defined on subsets of a Euclidean or digital space $\mathcal{E}$, which can be built as a composition of the erosion and dilation by that structuring element. They know sometimes the opening by a grey-level structuring function, defined on grey-level images on that space $\mathcal{E}$. Both are examples of structural **T**-openings (for sets, **T** is the group of translations of $\mathcal{E}$, while for grey-level functions, it includes also grey-level translations). However there are many other openings besides these. In the remainder of this section, we will describe a few unconventional openings on some well-known complete lattices. Then Section 2 will be devoted to recalling the algebraic theory of **T**-openings [19]. Finally Section 3 will describe *inf-overfilters*, a class of operators containing openings as particular cases, and which allow the construction of new families of openings.

1.1. Connectivity classes

Much of what we will say here is based on [22], Section 2.6. Connectivity is defined in the Euclidean space $\mathbb{R}^d$ in terms of the topology, but in the digital space $\mathbb{Z}^d$ it is defined in terms of paths built from neighbouring pixels. In both cases the family $\mathcal{C}$ of connected subsets of the space $\mathcal{E}$ satisfies the following two requirements: the empty set and a point are connected, and a union of connected sets containing a given point is connected. Formally:

(i) *$\emptyset \in \mathcal{C}$ and for any $x \in \mathcal{E}$, $\{x\} \in \mathcal{C}$.*

(ii) *For any subset $\mathcal{B}$ of $\mathcal{C}$, $\bigcap \mathcal{B} \neq \emptyset$ implies that $\bigcup \mathcal{B} \in \mathcal{C}$*

These two conditions characterize any family $\mathcal{C}$ of sets as a *connectivity class* on $\mathcal{P}(\mathcal{E})$ (the set of parts of $\mathcal{E}$). Now connectivity can also be defined in terms of the operator associating to each set and each point the connected component of that set containing that point; for a fixed point, it acts as an opening on sets. We postulate thus the existence for each $x \in \mathcal{E}$ of an opening γ_x on the complete lattice $(\mathcal{P}(\mathcal{E}), \subseteq)$ of subsets of $\mathcal{E}$, such that:

(iii) *For any* $x \in \mathcal{E}$, $\gamma_x(\{x\}) = \{x\}$.

(iv) *For any* $x, y \in \mathcal{E}$ *and* $A \subseteq \mathcal{E}$, $\gamma_x(A) \cap \gamma_y(A) = \emptyset$ *or* $\gamma_x(A) = \gamma_y(A)$.

(v) *For any* $x \in \mathcal{E}$ *and* $A \subseteq \mathcal{E}$, $x \in A$ *or* $\gamma_x(A) = \emptyset$.

Clearly these properties are verified when $\gamma_x(A)$ is the connected component of A containing x, in the usual Euclidean or digital sense. We call a family of openings γ_x, $x \in \mathcal{E}$, satisfying these three conditions a *system of connectivity openings* on $\mathcal{P}(\mathcal{E})$. It is shown in Theorem 2.8 of [22] that the above two definitions of connectivity are equivalent:

PROPOSITION 1.1. *There is a one-to-one correspondence between connectivity classes on* $\mathcal{P}(\mathcal{E})$ *given by* (i) *and* (ii), *and systems of connectivity openings on* $\mathcal{P}(\mathcal{E})$ *satisfying* (iii), (iv), *and* (v). *A connectivity class* $\mathcal{C}$ *and the corresponding family of connectivity openings* γ_x *define each other by the following two equivalent relations:*

— *For* $A \subseteq \mathcal{E}$, $\gamma_x(A)$ *is the union of all* $C \in \mathcal{C}$ *such that* $x \in C \subseteq A$; *in other words, it is* $\emptyset$ *for* $x \notin A$, *while for* $x \in A$ *it is the greatest* $C \in \mathcal{C}$ *such that* $x \in C \subseteq A$.

— $\mathcal{C}$ *is the set of all* $\gamma_x(A)$ *for* $x \in \mathcal{E}$ *and* $A \subseteq \mathcal{E}$.

It is also possible to characterize connectivity classes in terms of an opening on the complete lattice of partitions of $\mathcal{E}$ described in the example (c) on the first page of this chapter: to the connectivity class $\mathcal{C}$ correponds the opening which associates to each partition P a finer one made by splitting each class of P into its connected components. Note finally that for a group $\mathbf{T}$ of permutations of $\mathcal{E}$, $\mathcal{C}$ is $\mathbf{T}$-invariant if and only if $\tau\gamma_x = \gamma_{\tau(x)}\tau$ for every $\tau \in \mathbf{T}$.

Thanks to that general definition, one can define new types of connectivity from the known digital and Euclidean ones. Two examples are given on pp. 54–56 of [22]; the second one is particularly interesting, since it allows to regroup together any two connected components which are close to one another, leading to a formalization of the concept of "nearly connected" (as illustrated in Figure 2.8 there). It is based on Serra's Proposition 2.9, which we state here in a slightly more detailed form:

PROPOSITION 1.2. *Let* γ_x, $x \in \mathcal{E}$, *be a system of connectivity openings on* $\mathcal{P}(\mathcal{E})$ *corresponding to a connectivity class* $\mathcal{C}$. *Consider a map* $W : \mathcal{E} \to \mathcal{P}(\mathcal{E})$ *such that* $x \in W(x) \in \mathcal{C}$ *for each* $x \in \mathcal{E}$, *and let* δ_W *be the dilation given by* $\delta_W(A) = \bigcup_{x \in A} W(x)$. *For each* $x \in \mathcal{E}$ *define the operator* ν_x *on* $\mathcal{P}(\mathcal{E})$ *by:*

$$\nu_x(A) = \begin{cases} A \cap \gamma_x \delta_W(A) & \text{if } x \in A; \\ \emptyset & \text{if } x \notin A. \end{cases}$$

Then ν_x, $x \in \mathcal{E}$, *is a a system of connectivity openings on* $\mathcal{P}(\mathcal{E})$, *and the corresponding connectivity class* $\mathcal{N}$ *consists of all subsets* A *of* $\mathcal{E}$ *such that* $\delta_W(A) \in \mathcal{C}$. *We have* $\mathcal{C} \subseteq \mathcal{N}$.

The characterization of the connectivity class $\mathcal{N}$ is new. Note that this result cannot be improved by generalizing the dilation δ_W to an extensive and increasing operator ψ such that $\psi(\{x\}) \in \mathcal{C}$ for every $x \in \mathcal{E}$. A counterexample can be found for $\mathcal{E} = \mathbb{R}^2$ by taking the usual connectivity, an increasing operator ψ satisfying $\psi(X) = X$ for any set X included in a line, but such that there is a non-collinear triple x, y, z of points with $\psi(\{x, y, z\}) = [x, y] \cap \{z\}$, where $[x, y]$ is the closed segment spanned by x and y; then for $X = \{x, y, z\}$ we have $\nu_x(X) = X \cap \gamma_x \psi(X) = X \cap [x, y] = \{x, y\}$, and as $\psi(\{x, y\}) = \{x, y\}$, we get $\nu_x^2(X) = \nu_x(\{x, y\}) = \{x, y\} \cap \gamma_x \psi(\{x, y\}) = \{x, y\} \cap \{x\} = \{x\}$, so that ν_x is not idempotent.

No proof of Proposition 1.2 was given in [22]: the growth and anti-extensivity of each ν_x, as well as conditions *(iii)*, *(iv)*, and *(v)* are straightforward, but for the idempotence of ν_x, the reader was referred to another source. We will thus show here the idempotence of ν_x and the characterization of $\mathcal{N}$:

Proof. Note that as $p \in W(p)$ for each $p \in \mathcal{E}$, δ_W is extensive. Take a non-void $A \subseteq \mathcal{E}$ and $x \in A$. We will first prove that for any $a \in A$ such that $W(a) \cap \gamma_x \delta_W(A) \neq \emptyset$, we have $W(a) \subseteq \gamma_x \delta_W(A)$. Indeed, $\gamma_x \delta_W(A)$ and $W(a)$ are both in $\mathcal{C}$, and as $W(a) \cap \gamma_x \delta_W(A) \neq \emptyset$, $\gamma_x \delta_W(A) \cup W(a) \in \mathcal{C}$ by *(ii)*. As $W(a) \subseteq \delta_W(A)$, we get

$$x \in \gamma_x \delta_W(A) \subseteq \gamma_x \delta_W(A) \cup W(a) \subseteq \delta_W(A).$$

But $\gamma_x \delta_W(A)$ is by definition the greatest element of $\mathcal{C}$ which contains x and is contained in $\delta_W(A)$, and as $\gamma_x \delta_W(A) \cup W(a) \in \mathcal{C}$, we must have $\gamma_x \delta_W(A) \cup W(a) \subseteq \gamma_x \delta_W(A)$, that is $W(a) \subseteq \gamma_x \delta_W(A)$.

Let us now show that $\delta_W \nu_x(A) = \gamma_x \delta_W(A)$. Take first $p \in \gamma_x \delta_W(A)$. As γ_x is anti-extensive, we have $p \in \delta_W(A)$, and by definition of δ_W we get $p \in W(q)$ for some $q \in A$. As $p \in W(q) \cap \gamma_x \delta_W(A)$, the preceding paragraph gives $W(q) \subseteq \gamma_x \delta_W(A)$. Now $q \in A$ and $q \in W(q)$, so that we get $q \in A \cap W(q) \subseteq A \cap \gamma_x \delta_W(A) = \nu_x(A)$. As $p \in W(q)$ and $q \in \nu_x(A)$, we obtain $p \in \delta_W(\nu_x(A))$. Thus $\gamma_x \delta_W(A) \subseteq \delta_W \nu_x(A)$. Take next $p \in \delta_W \nu_x(A)$. By definition of δ_W, $p \in W(q)$ for some $q \in \nu_x(A)$. Now $\nu_x(A) = A \cap \gamma_x \delta_W(A) \subseteq \gamma_x \delta_W(A)$; hence $q \in \gamma_x \delta_W(A)$, and as $q \in W(q)$, the preceding paragraph gives $W(q) \subseteq \gamma_x \delta_W(A)$. As $p \in W(q)$, we get $p \in \gamma_x \delta_W(A)$, and so $\delta_W \nu_x(A) \subseteq \gamma_x \delta_W(A)$. The equality follows.

We can now prove that each ν_x is idempotent. Take $A \subseteq \mathcal{E}$ and $x \in \mathcal{E}$. If $x \notin A$, we have $\nu_x(A) = \emptyset$, and as ν_x is anti-extensive, we get $\nu_x \nu_x(A) = \emptyset$. If $x \in A$, then $\nu_x(A) = A \cap \gamma_x \delta_W(A)$. Applying γ_x to both sides of $\delta_W \nu_x(A) = \gamma_x \delta_W(A)$, the fact that γ_x is idempotent gives $\gamma_x \delta_W \nu_x(A) = \gamma_x \gamma_x \delta_W(A) = \gamma_x \delta_W(A)$. As $x \in \nu_x(A)$, we get

$$\nu_x \nu_x(A) = \nu_x(A) \cap \gamma_x \delta_W \nu_x(A) = \big(A \cap \gamma_x \delta_W(A)\big) \cap \gamma_x \delta_W(A) = A \cap \gamma_x \delta_W(A) = \nu_x(A).$$

Let us finally characterize the connectivity class $\mathcal{N}$. Take first a non-void $A \subseteq \mathcal{E}$ and $x \in A$. If $A \in \mathcal{N}$, then $A = \nu_x(A)$, and so $\delta_W(A) = \delta_W \nu_x(A) = \gamma_x \delta_W(A)$, so that $\delta_W(A) \in \mathcal{C}$. Conversely, if $\delta_W(A) \in \mathcal{C}$, then $\delta_W(A) = \gamma_x \delta_W(A)$, so that $\nu_x(A) = A \cap \gamma_x \delta_W(A) = A \cap \delta_W(A)$; but have $A \subseteq \delta_W(A)$, and so $\nu_x(A) = A$ and $A \in \mathcal{N}$. Second, $\emptyset \in \mathcal{N}$ and $\delta_W(\emptyset) = \emptyset \in \mathcal{C}$. Thus for any $A \subseteq \mathcal{E}$, $A \in \mathcal{N} \iff \delta_W(A) \in \mathcal{C}$.

Finally $\mathcal{C} \subseteq \mathcal{N}$, because for a non-void $C \in \mathcal{C}$ and $x \in C$, we have $C = \gamma_x(C)$ and $C \subseteq \delta_W(C)$, so that $C = C \cap \gamma_x(C) \subseteq C \cap \gamma_x \delta_W(C) = \nu_x(C)$, that is $C = \nu_x(C)$ and $C \in \mathcal{N}$. ∎

We illustrate this result in the case where $\mathcal{E} = \mathbb{Z}^2$ (the digital plane), $\mathcal{C}$ is the set of 4-connected subsets of $\mathcal{E}$, and δ_W is the translation-invariant dilation δ_B by a connected structuring element B centered about the origin (that is, each $W(p)$ is the translate B_p of B by p). We derive from $\mathcal{C}$ the new connectivity class $\mathcal{N}$ consisting of all sets which are 4-connected or which join in a 4-connected set under the dilation δ_W. We show in Figure 1.1 the decomposition of a digital set into its $\mathcal{N}$-connected components when B is the 5-pixel cross.

$$\begin{array}{ccc} & B & \\ B & B & B \\ & B & \end{array} \qquad\qquad \begin{array}{ccccccccccccc} \cdot & 1 & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 2 & 2 \\ 1 & 1 & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 2 & 2 & 2 \\ \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & 2 \end{array}$$

FIGURE 1.1. *The $\mathcal{N}$-connected components of the set are numbered 1 and 2, they consist of maximal unions of 4-connected sets which join under dilation by B.*

We will now give a second method for constructing a new connectivity class from an old one, which derives from a suggestion by H. Heijmans. Anticipating on Section 2, for any operator ψ on $\mathcal{P}(\mathcal{E})$, we call an *invariant of* ψ any $X \subseteq \mathcal{E}$ such that $\psi(X) = X$, and write $\mathrm{Inv}(\psi)$ for the set of invariants of ψ. Given an opening α on $\mathcal{P}(\mathcal{E})$, we will say that α is *connected for the connectivity class* $\mathcal{C}$ if for every invariant A of α, all connected components of A for $\mathcal{C}$ are also invariants of α. Clearly this is equivalent to the requirement that $\alpha\gamma_x\alpha = \gamma_x\alpha$ for every $x \in \mathcal{E}$, where all γ_x, $x \in \mathcal{E}$, form the sytem of connectivity openings associated to $\mathcal{C}$. It is easy to show that for any two openings α_0 and α_1 we have $\alpha_0\alpha_1\alpha_0 = \alpha_1\alpha_0 \iff \alpha_1\alpha_0\alpha_1 = \alpha_1\alpha_0 \iff (\alpha_1\alpha_0)^2 = \alpha_1\alpha_0$. We have thus three equivalent formulations for α being connected for $\mathcal{C}$: for every $x \in \mathcal{E}$,

$$\alpha\gamma_x\alpha = \gamma_x\alpha \iff \gamma_x\alpha\gamma_x = \gamma_x\alpha \iff (\gamma_x\alpha)^2 = \gamma_x\alpha. \tag{1.1}$$

For example, given $B \subseteq \mathcal{E}$, the operator $\alpha : X \mapsto X \cap B$ on $\mathcal{P}(\mathcal{E})$ is an opening connected for any connectivity class, because the invariants of α are the subsets of B.

For any subset $\mathcal{B}$ of $\mathcal{P}(\mathcal{E})$, write $\mathbf{A}(\mathcal{B})$ for the operator mapping any $X \subseteq \mathcal{E}$ to the union of all elements of $\mathcal{B}$ included in X. It is straightforward (see again Section 2) that $\mathbf{A}(\mathcal{B})$ is an opening. We have the following characterization:

LEMMA 1.3. *An operator α is an opening connected for the connectivity class $\mathcal{C}$ if and only if there is a subset $\mathcal{B}$ of $\mathcal{C}$ such that $\alpha = \mathbf{A}(\mathcal{B})$. The set of such openings is closed under arbitrary union.*

PROOF. Suppose that α is connected for $\mathcal{C}$, and let $\mathcal{B} = \mathcal{C} \cap \mathrm{Inv}(\alpha)$. Let $X \subseteq \mathcal{E}$. For any $x \in \alpha(X)$, let B be the connected component of $\alpha(X)$ containing x; as $\alpha(X) \in \mathrm{Inv}(\alpha)$ (by idempotence) and α is connected, $B \in \mathrm{Inv}(\alpha)$, and as $B \in \mathcal{C}$, we have $B \in \mathcal{B}$; thus $x \in B \subseteq \alpha(X) \subseteq X$ for $B \in \mathcal{B}$, and hence $x \in \mathbf{A}(\mathcal{B})(X)$. Thus $\alpha(X) \subseteq \mathbf{A}(\mathcal{B})(X)$. Conversely for any $B \in \mathcal{B}$ such that $B \subseteq X$, as α is increasing we have $B = \alpha(B) \subseteq \alpha(X)$, and so $\mathbf{A}(\mathcal{B})(X) \subseteq \alpha(X)$. The equality $\alpha = \mathbf{A}(\mathcal{B})$ follows.

Suppose now that $\alpha = \mathbf{A}(\mathcal{B})$ for some $\mathcal{B} \subseteq \mathcal{C}$. Let $A \in \mathrm{Inv}(\mathbf{A}(\mathcal{B}))$, and let B be a connected component of A for $\mathcal{C}$. For any $x \in B$, as $x \in A = \mathbf{A}(\mathcal{B})(A)$, there is some

$B' \in \mathcal{B}$ such that $x \in B' \subseteq A$; as $B' \in \mathcal{C}$ and B is the connected component of x in A, this means that $B' \subseteq B$, and so $x \in \mathbf{A}(\mathcal{B})(B)$. Hence $B \subseteq \mathbf{A}(\mathcal{B})(B)$, and as $\mathbf{A}(\mathcal{B})(B) \subseteq B$ (by anti-extensivity), we get $B = \mathbf{A}(\mathcal{B})(B)$. Therefore α is connected for $\mathcal{C}$.

For any family $\mathcal{B}_j, j \in J$ of subsets of $\mathcal{C}$, we have $\bigcup_{j\in J} \mathbf{A}(\mathcal{B}_j) = \mathbf{A}(\bigcup_{j\in J} \mathcal{B}_j)$ (see Section 2). Thus the set of openings connected for $\mathcal{C}$ is closed under union. ∎

The openings $\mathbf{A}(\mathcal{B})$, where $\mathcal{B} \subseteq \mathcal{P}(\mathcal{E})$, will play an important role in Section 2 for the structural analysis of openings, but that time we will consider any complete lattice, not only $\mathcal{P}(\mathcal{E})$. Now we give the method for constructing a new connectivity class from an existing one and a connected opening:

PROPOSITION 1.4. *Let $\mathcal{C}$ be a connectivity class and α an opening connected for $\mathcal{C}$. Let $\mathcal{S}$ the subset of $\mathcal{C}$ consisting of $\emptyset$, all singletons $\{x\}$ for $x \in \mathcal{E}$, and all $C \in \mathcal{C} \cap \mathrm{Inv}(\alpha)$. Then $\mathcal{S}$ is a connectivity class. If γ_x $(x \in \mathcal{E})$ is the system of connectivity openings associated to $\mathcal{C}$, then σ_x $(x \in \mathcal{E})$, the one corresponding to $\mathcal{S}$, is defined as follows for any $x \in \mathcal{E}$ and $A \subseteq \mathcal{E}$:*

$$\sigma_x(A) = \begin{cases} \gamma_x\alpha(A) & \text{if } x \in \alpha(A); \\ \{x\} & \text{if } x \in A \setminus \alpha(A); \\ \gamma_x\alpha(A) = \emptyset & \text{if } x \notin A. \end{cases}$$

PROOF. We show that $\mathcal{S}$ satisfies conditions (i) and (ii). It verifies (i) by definition. Given a subset $\mathcal{B}$ of $\mathcal{S}$ such that $\bigcap \mathcal{B} \neq \emptyset$, let $x \in \bigcap \mathcal{B}$; then $\{x\}$ is the only possible singleton in $\mathcal{B}$. Either $\bigcup \mathcal{B} = \{x\} \in \mathcal{S}$ or $\bigcup \mathcal{B} = \bigcup \mathcal{B}'$, where $\mathcal{B}'$ is the set of elements of $\mathcal{B}$ which are not singletons. In that case, $\mathcal{B}' \subseteq \mathcal{C} \cap \mathrm{Inv}(\alpha)$; by property (ii) for $\mathcal{C}$ we have $\bigcup \mathcal{B} \in \mathcal{C}$; as we will see in Section 2, $\mathrm{Inv}(\alpha)$ is closed under union, so $\bigcup \mathcal{B}' \in \mathrm{Inv}(\alpha)$; hence $\bigcup \mathcal{B} = \bigcup \mathcal{B}' \in \mathcal{S}$ and $\mathcal{S}$ satisfies (ii).

Let $x \in \mathcal{E}$ and $A \subseteq \mathcal{E}$. If $x \notin A$, then $x \notin \alpha(A)$, and so by (v) (for both $\mathcal{C}$ and $\mathcal{S}$) we must have $\sigma_x(A) = \emptyset = \gamma_x\alpha(A)$. If $x \in A \setminus \alpha(A)$, then $\sigma_x(A) \subseteq A$ gives $\alpha\sigma_x(A) \subseteq \alpha(A)$, and so $x \notin \alpha\sigma_x(A)$; as $x \in \sigma_x(A)$, we get $\sigma_x(A) \neq \alpha\sigma_x(A)$, that is $\sigma_x(A) \notin \mathcal{C} \cap \mathrm{Inv}(\alpha)$, and as $x \in \sigma_x(A) \in \mathcal{S}$, $\sigma_x(A)$ is a singleton, in other words $\sigma_x(A) = \{x\}$. If $x \in \alpha(A)$, then $\sigma_x(A)$ is the greatest $S \in \mathcal{S}$ such that $x \in S \subseteq A$; now $x \in \gamma_x\alpha(A)$ and $\gamma_x\alpha(A) \in \mathcal{C} \cap \mathrm{Inv}(\alpha)$ (since α is connected for C); hence $x \in \gamma_x\alpha(A) \subseteq A$ with $\gamma_x\alpha(A) \in \mathcal{S}$, which implies that $\gamma_x\alpha(A) \subseteq \sigma_x(A)$; if $\sigma_x(A)$ is a singleton, we have $\sigma_x(A) = \gamma_x\alpha(A) = \{x\}$, otherwise $\sigma_x(A) \in \mathcal{C} \cap \mathrm{Inv}(\alpha)$ and $x \in \sigma_x(A)$, that is $\gamma_x\sigma_x(A) = \sigma_x(A) = \alpha\sigma_x(A)$; but then $\sigma_x(A) = \gamma_x\alpha\sigma_x(A) \subseteq \gamma_x\alpha(A)$, and as $\gamma_x\alpha(A) \subseteq \sigma_x(A)$, we get $\sigma_x(A) = \gamma_x\alpha(A)$. ∎

Note that the first part of the proof (that $\mathcal{S}$ is a connectivity class) does not use the fact that α is connected for $\mathcal{C}$. However when α is not connected, we can replace α by $\mathbf{A}(\mathcal{B})$, where $\mathcal{B} = \mathcal{C} \cap \mathrm{Inv}(\alpha)$, and by Lemma 1.3, $\mathbf{A}(\mathcal{B})$ is connected for $\mathcal{C}$. Indeed, $\mathcal{C} \cap \mathrm{Inv}(\alpha) = \mathcal{B} \subseteq \mathrm{Inv}(\mathbf{A}(\mathcal{B}))$; on the other hand, as $\mathcal{B} \subseteq \mathrm{Inv}(\alpha)$, it follows from the general theory in Section 2 that $\mathbf{A}(\mathcal{B}) \leq \alpha$ and so $\mathrm{Inv}(\mathbf{A}(\mathcal{B})) \subseteq \mathrm{Inv}(\alpha)$; from the double inequality $\mathcal{C} \cap \mathrm{Inv}(\alpha) \subseteq \mathrm{Inv}(\mathbf{A}(\mathcal{B})) \subseteq \mathrm{Inv}(\alpha)$ we deduce $\mathcal{C} \cap \mathrm{Inv}(\alpha) = \mathcal{C} \cap \mathrm{Inv}(\mathbf{A}(\mathcal{B}))$, so that α and $\mathbf{A}(\mathcal{B})$ lead to the same connectivity class $\mathcal{S}$.

Note also that not every connectivity class $\mathcal{S}$ included in $\mathcal{C}$ arises in this way. For example if $\mathcal{C}$ is the set of 8-connected subsets of the digital plane $\mathcal{E} = \mathbb{Z}^2$, and $\mathcal{S}$ the set of 4-connected subsets of $\mathcal{E}$, there is no opening α such that $\mathcal{S}$ consists of $\emptyset$, the singletons, and the elements of $\mathcal{C} \cap \mathrm{Inv}(\alpha)$. Indeed, if A and B are two 4-connected sets of size 2 which are diagonally adjacent, then $A, B \in \mathcal{S}$, $A \cup B \in \mathcal{C}$, $A \cup B \notin \mathcal{S}$, and if we had $A, B \in \mathrm{Inv}(\alpha)$, we would have $A \cup B \in \mathrm{Inv}(\alpha)$, a contradiction.

Let us illustrate Proposition 1.4 in the two-dimensional digital case with $\mathcal{C}$ the set of 4-connected subsets of $\mathcal{E}$, and where α is the translation-invariant opening α_B by a 4-connected structuring element B. Clearly α_B takes the form described in Lemma 1.3 with $\mathcal{B}$ being the set of translates of B, and so it is connected for $\mathcal{C}$. We derive from $\mathcal{C}$ the new connectivity class $\mathcal{S}$ consisting of the empty set, the singletons, and all 4-connected sets invariant under α_B. We show in Figure 1.2 how a digital set is divided into its $\mathcal{S}$-connected components when B is the 5-pixel cross. Notice how narrow portions of 4-connected sets disconnect them in $\mathcal{S}$.

$$\begin{array}{ccc} & B & \\ B & B & B \\ & B & \end{array} \qquad\qquad \begin{array}{cccccccccc} \cdot & 1 & 1 & 1 & \cdot & \cdot & 8 & \cdot & 2 & 4 \\ 1 & 1 & 1 & 1 & 1 & 6 & 7 & 2 & 2 & 2 \\ 3 & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & 2 & 5 \end{array}$$

FIGURE 1.2. *The $\mathcal{S}$-connected components of the set are numbered 1 to 8, they consist of maximal 4-connected sets open by B, otherwise of singletons.*

Connectivity classes of the above form can be used to segment digital sets. First the set X is divided into its $\mathcal{S}$-connected components. Next, all such components which are singletons are regrouped into $\mathcal{C}$-connected components. We illustrate in Figure 1.3 what this gives for Figure 1.2.

$$\begin{array}{cccccccccc} \cdot & 1 & 1 & 1 & \cdot & \cdot & 6 & \cdot & 2 & 4 \\ 1 & 1 & 1 & 1 & 1 & 6 & 6 & 2 & 2 & 2 \\ 3 & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & 2 & 5 \end{array}$$

FIGURE 1.3. *The segmentation of Figure 1.2 obtained by regrouping singletons into 4-connected components.*

Such an operation can also be applied in the complete lattice $\mathcal{L}$ of partitions of a digital space $\mathcal{E}$. Given a partition P, first all classes are split into $\mathcal{S}$-connected components. Next, all new classes which are singletons are regrouped into $\mathcal{C}$-connected components.

Other manipulations on connectivity classes can be envisaged. For example a non-void intersection of connectivity classes is again a connectivity class. Thus the family of connectivity classes, ordered by inclusion, forms a complete lattice; the least one consists of $\emptyset$ and the singletons, while the greatest one is $\mathcal{P}(\mathcal{E})$, the set of all subsets of $\mathcal{E}$.

We have studied openings α connected for $\mathcal{C}$, that is satisfying $\alpha\gamma_x\alpha = \gamma_x\alpha$ for all $x \in \mathcal{E}$. One can also consider openings α such that $\gamma_x\alpha\gamma_x = \alpha\gamma_x$ for every $x \in \mathcal{E}$. This means in fact that for every X connected, either $\alpha(X) = X$ or $\alpha(X) = \emptyset$. As in (1.1), we have the following equivalence:

$$\gamma_x\alpha\gamma_x = \alpha\gamma_x \iff \alpha\gamma_x\alpha = \alpha\gamma_x \iff (\alpha\gamma_x)^2 = \alpha\gamma_x. \tag{1.2}$$

In particular let α be any increasing operator such that for $X \subseteq \mathcal{E}$ we have $\alpha(X) = X$ or $\alpha(X) = \emptyset$. Then it is easily seen that α is anti-extensive and idempotent, and that for any

$X \subseteq \mathcal{E}$ and $x \in \mathcal{E}$, $\gamma_x \alpha \gamma_x(X) = \alpha \gamma_x(X)$. For example we can choose $\alpha(X) = X$ if the size of X exceeds some threshold, or as in [24], p. 27, $\alpha(X) = X$ if $\alpha_B(X) \neq \emptyset$, where α_B is the translation-invariant opening by a structuring element B. Clearly such a type of opening is not necessarily connected.

An interesting study to be done is that of the wider class of openings α that *preserve connectivity*: for X connected, $\alpha(X)$ is connected. This can be expressed as follows: for every $X \subseteq \mathcal{E}$ and $x \in \mathcal{E}$, $\gamma_x \alpha \gamma_x(X) = \alpha \gamma_x(X)$ (if $x \in \alpha \gamma_x(X)$) or $\gamma_x \alpha \gamma_x(X) = \emptyset$ (if $x \notin \alpha \gamma_x(X)$). A necessary (but not sufficient) condition is that $(\alpha \gamma_x)^2 = \gamma_x \alpha \gamma_x$ for all $x \in \mathcal{E}$.

Given an anti-extensive operator ψ on $\mathcal{P}(\mathcal{E})$, let us consider the operator $\psi_{\mathcal{C}}$ resulting from applying ψ separately to each connected component of a set in a connectivity class $\mathcal{C}$. For example in [24], p. 27, ψ is the above-mentioned opening which preserves X if $\alpha_B(X) \neq \emptyset$, and removes it otherwise, and $\psi_{\mathcal{C}}$ removes all connected components too small or narrow to contain a translate of B. For $X \subseteq \mathcal{E}$ we have

$$\psi_{\mathcal{C}}(X) = \bigcup_{x \in \mathcal{E}} \psi \gamma_x(X). \tag{1.3}$$

As ψ is anti-extensive, so is $\psi_{\mathcal{C}}$. When is $\psi_{\mathcal{C}}$ increasing and idempotent, in other words an opening?

PROPOSITION 1.5. *Let $\mathcal{C}$ be a connectivity class with system of connectivity openings γ_x ($x \in \mathcal{E}$). Let ψ be an anti-extensive operator on $\mathcal{P}(\mathcal{E})$, and let $\psi_{\mathcal{C}}$ given by* (1.3). *Then for $X \subseteq \mathcal{E}$ we have*

$$\psi_{\mathcal{C}}(X) = \bigcup_{x \in \mathcal{E}} \gamma_x \psi \gamma_x(X), \tag{1.4}$$

$$\psi_{\mathcal{C}}^2(X) = \bigcup_{x \in \mathcal{E}} (\psi \gamma_x)^2(X). \tag{1.5}$$

Moreover:

(*a*) *$\psi_{\mathcal{C}}$ is increasing if and only if the restriction of ψ to $\mathcal{C}$ is increasing.*
(*b*) *$\psi_{\mathcal{C}}$ is idempotent if and only if $(\psi \gamma_x)^2 = \gamma_x \psi \gamma_x$ for every $x \in \mathcal{E}$.*
(*c*) *$\psi_{\mathcal{C}} = \psi$ if and only if $\gamma_x \psi \gamma_x = \gamma_x \psi$ for every $x \in \mathcal{E}$.*

In particular $\psi_{\mathcal{C}}$ is an opening when ψ is an opening preserving connectivity in $\mathcal{C}$, and $\psi_{\mathcal{C}} = \psi$ when ψ is an opening connected for $\mathcal{C}$.

PROOF. Note first that for $Y \subseteq X$ and $x \in \mathcal{E}$ we have $\gamma_x(Y) = \gamma_x(Y \cap \gamma_x(X))$. Indeed, as γ_x is an opening, $\gamma_x(Y) \subseteq Y$ and $\gamma_x(Y) \subseteq \gamma_x(X)$, so that $\gamma_x(Y) \subseteq Y \cap \gamma_x(X) \subseteq Y$; applying γ_x to each term of the inequality, we get

$$\gamma_x(Y) = \gamma_x \gamma_x(Y) \subseteq \gamma_x(Y \cap \gamma_x(X)) \subseteq \gamma_x(Y),$$

and the equality follows. For any $X \subseteq \mathcal{E}$ we have (for $Y = \psi_{\mathcal{C}}(X)$):

$$\begin{aligned}\gamma_x(\psi_{\mathcal{C}}(X)) &= \gamma_x\big(\psi_{\mathcal{C}}(X) \cap \gamma_x(X)\big) = \gamma_x\Big(\Big(\bigcup_{z \in \mathcal{E}} \psi \gamma_z(X)\Big) \cap \gamma_x(X)\Big) \\ &= \gamma_x\Big(\bigcup_{z \in \mathcal{E}} \big(\psi \gamma_z(X) \cap \gamma_x(X)\big)\Big) = \gamma_x\big(\psi \gamma_x(X) \cap \gamma_x(X)\big) = \gamma_x \psi \gamma_x(X),\end{aligned}$$

because for every $z \in \mathcal{E}$, either $\gamma_z(X) = \gamma_x(X)$ or $\psi\gamma_z(X) \cap \gamma_x(X) \subseteq \gamma_z(X) \cap \gamma_x(X) = \emptyset$. This gives finally

$$\psi_{\mathcal{C}}^2(X) = \bigcup_{x \in \mathcal{E}} \psi\gamma_x(\psi_{\mathcal{C}}(X)) = \bigcup_{x \in \mathcal{E}} \psi\gamma_x\psi\gamma_x(X),$$

that is (1.5).

Let $x \in \psi_{\mathcal{C}}(X)$. For any z such that $\gamma_z(X) \neq \gamma_x(X)$, we have $\gamma_z(X) \cap \gamma_x(X) = \emptyset$, hence $x \notin \gamma_z(X)$, and as ψ is anti-extensive, $x \notin \psi\gamma_z(X)$. By (1.3), we have thus $x \in \psi\gamma_x(X)$, and so $x \in \gamma_x\psi\gamma_x(X)$. Therefore $\psi_{\mathcal{C}}(X) \subseteq \bigcup_{x \in \mathcal{E}} \gamma_x\psi\gamma_x(X)$, and as each γ_x is anti-extensive, we obtain from (1.3) the equality (1.4).

For $X \in \mathcal{C}$, $\psi_{\mathcal{C}}(X) = \psi(X)$ and so for $\psi_{\mathcal{C}}$ to be increasing, the restriction of ψ to $\mathcal{C}$ must be increasing. This condition is also sufficient, because for any $X \subseteq \mathcal{E}$ and $x \in \mathcal{E}$, $\gamma_x(X) \in \mathcal{C}$, and for $X \leq Y$ we have $\gamma_x(X) \leq \gamma_x(Y)$ and so $\psi\gamma_x(X) \leq \psi\gamma_x(Y)$. Thus (a) holds.

By (1.4) and (1.5), $\psi_{\mathcal{C}}^2(X) = \psi_{\mathcal{C}}(X)$ is equivalent to

$$\bigcup_{x \in \mathcal{E}} (\psi\gamma_x)^2(X) = \bigcup_{x \in \mathcal{E}} \gamma_x\psi\gamma_x(X).$$

Now $(\psi\gamma_x)^2(X) \subseteq \gamma_x\psi\gamma_x(X)$ for each $x \in \mathcal{E}$ (since ψ is anti-extensive); moreover for $x, z \in \mathcal{E}$ $\gamma_x\psi\gamma_x(X) \cap \gamma_z\psi\gamma_z(X) \neq \emptyset$ implies that $\gamma_x(X) \cap \gamma_z(X) \neq \emptyset$, so that $\gamma_x(X) = \gamma_z(X)$ and $\gamma_x\psi\gamma_x(X) = \gamma_z\psi\gamma_z(X)$. Therefore the above equality holds if and only if $(\psi\gamma_x)^2(X) = \gamma_x\psi\gamma_x(X)$ for every $x \in \mathcal{E}$. This gives (b). As seen previously, this condition is satisfied in particular when ψ is an opening which preserves connectivity in $\mathcal{C}$.

As $\psi(X) = \bigcup_{x \in \mathcal{E}} \gamma_x\psi(X)$, we have $\psi(X) = \psi_{\mathcal{C}}(X)$ if and only if

$$\bigcup_{x \in \mathcal{E}} \gamma_x\psi(X) = \bigcup_{x \in \mathcal{E}} \gamma_x\psi\gamma_x(X).$$

As $\gamma_x\psi\gamma_x(X) \subseteq \gamma_x\psi(X)$ for each $x \in \mathcal{E}$, and for $x, z \in \mathcal{E}$ we have $\gamma_x\psi(X) \cap \gamma_z\psi(X) = \emptyset$ or $\gamma_x\psi(X) = \gamma_z\psi(X)$, the above equality holds if and only if $\gamma_x\psi\gamma_x(X) = \gamma_x\psi(X)$ for every $x \in \mathcal{E}$. Thus we get (c). From (1.1) this condition is verified in particular when ψ is an opening connected for $\mathcal{C}$. ∎

We illustrate this result in the two-dimensional digital case with 4-connectivity, and ψ defined by

$$\psi(X) = \begin{cases} X & \text{if } |X| \geq 12, \\ \alpha_B(X) & \text{otherwise}, \end{cases}$$

where α_B is the translation-invariant opening by a 4-connected structuring element B (again, the 5-pixel cross). It is easily shown that ψ is an opening. Moreover, as α_B is connected, from (1.1) we can obtain $(\alpha_B\gamma_x)^2 = \gamma_x\alpha_B\gamma_x$ for every $x \in \mathcal{E}$, from which we derive $(\psi\gamma_x)^2 = \gamma_x\psi\gamma_x$. Thus $\psi_{\mathcal{C}}$ is an opening. Note that ψ is not connected, does not preserve connectivity, and is distinct from $\psi_{\mathcal{C}}$. We show in Figure 1.4 the behaviour of $\psi_{\mathcal{C}}$ on a digital set. Large connected components are preserved, small ones are diminished, and can even disappear.

Chapter 7 of [22] considers closings φ such that $\varphi\gamma_x\varphi = \gamma_x\varphi$ for every $x \in \mathcal{E}$, which means that for every invariant F of φ, all connected components of F for $\mathcal{C}$ are also invariants

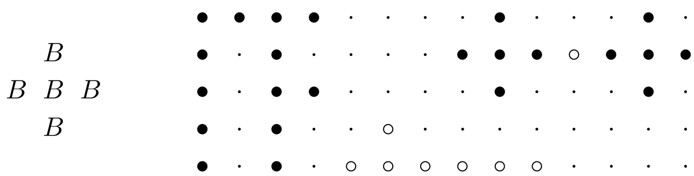

FIGURE 1.4. *The points of X which are deleted by $\psi_{\mathcal{C}}$ are written $\circ$, those which are preserved are written $\bullet$.*

of φ (this is the analogue for closings of the openings connected for $\mathcal{C}$ considered above). Now $\varphi\gamma_x\varphi = \gamma_x\varphi$ is equivalent to $\gamma_x\varphi\gamma_x = \varphi\gamma_x$, which means that φ preserves connectivity (for X connected, $\varphi(X)$ is connected). A subclass of this family consists of all closings which do not create connected components. An interesting fact proved there is that these two classes of closings form complete lattices for the ordering by inclusion. Hence for any closing φ on $\mathcal{P}(\mathcal{E})$, there is a greatest closing φ_c preserving connectivity (or not creating connected components) such that $\varphi_c(X) \subseteq \varphi(X)$ for all $X \subseteq \mathcal{E}$. For example from the convex hull (obviously a closing) we derive the "connectivity preserving convex hull" or "convex hull which does not create connected components".

The author has considered with H. Heijmans the possibility of defining connectivity classes on other object spaces than $\mathcal{P}(\mathcal{E})$, for example on the space of grey-level images on $\mathcal{E}$. In fact this is possible by translating conditions (i) to (v) into any complete lattice $\mathcal{L}$ where instead of points or singletons we have a *sup-generating family*, that is a subset ℓ of $\mathcal{L}$ such that any element of $\mathcal{L}$ is the supremum of a subset of ℓ. This applies thus to grey-level images by taking ℓ to be the set of impulse functions (see [6], Section 4.). However such notions of connectivity are generally useless, since they consist mainly of statements of the form "the umbra of the image is connected".

1.2. Annular openings

On p. 107 of [22] Serra defines a new class of translation-invariant openings on the complete lattice $\mathcal{P}(\mathcal{E})$ of parts of a digital or Euclidean space $\mathcal{E}$. He shows that for every non-empty symmetric structuring element B (that is $x \in B$ implies $-x \in B$), the operator $\gamma_B : \mathcal{P}(\mathcal{E}) \to \mathcal{P}(\mathcal{E}) : X \mapsto X \cap (X \oplus B)$ is an opening. Normally one takes B not containing the origin o, otherwise γ_B reduces to the identity $X \mapsto X$; if B is a ring centered about o, such an opening will remove isolated particles from a set (see Figure 5.2 on p. 108 of [22]). This example leads to the denomination of *annular opening* for γ_B.

Such an opening can also be defined without translation-invariance. As in Proposition 1.2, consider a map $W : \mathcal{E} \to \mathcal{P}(\mathcal{E})$, and let δ_W be the dilation given by $\delta_W(A) = \bigcup_{x\in A} W(x)$. The symmetry condition on B generalizes to W as follows: for $p, q \in \mathcal{E}$ we have $p \in W(q) \iff q \in W(p)$. Then the operator $\gamma_W : \mathcal{P}(\mathcal{E}) \to \mathcal{P}(\mathcal{E}) : X \mapsto X \cap \delta_W(X)$ is an opening. For any $X \subseteq \mathcal{E}$, $\gamma_W(X)$ is the union of all pairs $\{x, y\}$ such that $x, y \in X$ and $y \in W(x)$ (or equivalently $x \in W(y)$). Usually one takes W such that $x \notin W(x)$ for all $x \in \mathcal{E}$.

In Section 3 of [19] this construction has been generalized to a wide class of complete lattices satisfying the following *infinite supremum distributivity* law:

$$Y \wedge (\bigvee_{j \in J} X_j) = \bigvee_{j \in J} (Y \wedge X_j). \tag{ISD}$$

This excludes in particular the complete lattice of convex sets (which is not distributive).

Let for example $\mathcal{L}$ be the complete lattice of grey-level functions $\mathcal{E} \to \mathcal{G}$, where the space $\mathcal{E}$ is either $\mathbb{Z}^d$ or $\mathbb{R}^d$, and the set $\mathcal{G}$ of grey-levels is either $\overline{\mathbb{Z}} = \mathbb{Z} \cup \{+\infty, -\infty\}$ or $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$. Then the Minkowski addition $F \oplus G$ of two grey-level functions F and G is defined by setting

$$(F \oplus G)(x) = \sup_{h \in \mathcal{E}} [F(x-h) + G(h)]$$

for any $x \in \mathcal{E}$, with the further convention, in cases of ambiguous expressions of the form $+\infty - \infty$, that $F(x-h) + G(h) = -\infty$ when $F(x-h) = -\infty$ or $G(h) = -\infty$ (see [6], Section 4). For any grey-level structuring function $G : \mathcal{E} \to \mathcal{G}$, we define the support of G, $\mathrm{supp}(G)$, as the set of points $h \in \mathbb{R}^d$ for which $G(h) > -\infty$. Now assume that:

- (i) $\mathrm{supp}(G)$ is symmetric;
- (ii) $G(h) + G(-h) \geq 0$ for every $h \in \mathrm{supp}(G)$.

Then the operator $\gamma_G : \mathcal{L} \to \mathcal{L} : F \mapsto F \cap (F \oplus G)$ is an opening. In general one chooses G such that $\mathrm{supp}(G)$ does not contain the the origin o, otherwise γ_G reduces to the identity $F \mapsto F$. When G is a flat structuring function ($G(h) = 0$ for $h \in \mathrm{supp}(G)$), the behaviour of γ_G is analogous to that of the corresponding annular opening $\gamma_{\mathrm{supp}(G)}$ for sets.

In Section 5 of [5] this example has been extended to grey-level functions with a finite set of grey-levels, say $\{0, \ldots, N\}$.

1.3. *Iteration of anti-extensive operators*

We will see in Section 2 that a supremum of openings (on a given complete lattice $\mathcal{L}$) is again an opening. On the other hand an infimum of openings is generally *not* an opening: it is anti-extensive, increasing, but usually not idempotent. In fact any anti-extensive increasing operator can arise in this way:

PROPOSITION 1.6. *Every anti-extensive and increasing operator on a complete lattice $\mathcal{L}$ is an infimum of openings on $\mathcal{L}$.*

PROOF. Let ψ be anti-extensive and increasing. For any $B \in \mathcal{L}$ define the operator ψ_B as follows:

$$\psi_B(X) = \begin{cases} \psi(B) \wedge X & \text{if } X \leq B; \\ X & \text{if } X \not\leq B. \end{cases}$$

We will show that each ψ_B is an opening, and that ψ is the infimum of all ψ_B.

It is clear that ψ_B is anti-extensive. To verify that it is increasing, let $X \leq Y$; if $X \leq B$, then $\psi_B(X) = \psi(B) \wedge X \leq \psi(B) \wedge Y \leq \psi_B(Y)$; on the other hand if $X \not\leq B$, then $Y \not\leq B$ and so $\psi_B(X) = X \leq Y = \psi_B(Y)$; therefore $\psi_B(X) \leq \psi_B(Y)$ in any case. Next we show that ψ_B is idempotent; if $X \leq B$, then $\psi_B(X) = \psi(B) \wedge X \leq X \leq B$, and so

$$\psi_B^2(X) = \psi(B) \wedge \psi_B(X) = \psi(B) \wedge \big(\psi(B) \wedge X\big) = \psi(B) \wedge X = \psi_B(X);$$

on the other hand if $X \not\leq B$, then $\psi_B(X) = X$ and so $\psi_B^2(X) = X$. Thus $\psi_B^2(X) = \psi_B(X)$ in any case, and ψ_B is an opening.

Now for any $X \in \mathcal{L}$, $\psi(X) \leq \psi_B(X)$; indeed, if $X \leq B$, then $\psi(X) \leq \psi(B)$ and $\psi(X) \leq X$, so that $\psi(X) \leq \psi(B) \wedge X = \psi_B(X)$; on the other hand if $X \not\leq B$, then $\psi(X) \leq X = \psi_B(X)$. Moreover, we have $\psi_X(X) = \psi(X) \wedge X = \psi(X)$. As $\psi(X) \leq \psi_B(X)$ for each $B \in \mathcal{L}$, but $\psi_X(X) = \psi(X)$, we get $\psi(X) = \bigwedge_{B\in\mathcal{L}} \psi_B(X)$, that is $\psi = \bigwedge_{B\in\mathcal{L}} \psi_B$. ∎

Note that this result does not extend to the case where **T**-invariance is required. As counterexample, take $\mathcal{L} = \overline{\mathbb{Z}} = \mathbb{Z} \cup \{+\infty, -\infty\}$, ordered in the usual way, and $\mathbf{T} = \mathbb{Z}$ acting by translation; there are only two **T**-openings, the identity and constant $-\infty$ mappings, but there are infinitely many anti-extensive and increasing **T**-operators between them, namely all negative translations.

As an illustration of strange results that can be obtained by combining only two very simple openings, Example 5.1 in Section 5 of [19] takes in the digital plane $\mathbb{Z}^2$ a (2×2)-square A and a 5-pixel cross B, and considers the two translation-invariant morphological openings α_A and α_B by A and B on the complete lattice $\mathcal{P}(\mathbb{Z}^2)$. Then $\alpha_A\alpha_B$, $\alpha_B\alpha_A$, and $\alpha_A \cap \alpha_B$ are not openings. They share the same set of invariants, namely $\mathrm{Inv}(\alpha_A) \cap \mathrm{Inv}(\alpha_B)$, some elements of which are illustrated in Figure 5 (c) there. Choosing an operator ψ among these three, and iterating it indefinitely: $\psi, \psi^2, \psi^3, \ldots$, for $n \to \infty$ the sequence of powers ψ^n converges to an opening α, the greatest opening which is less than both α_A and α_B. Although the behaviour of α_A and α_B is easily described in terms of translates of A and B respectively, this is not the case for α: we were unable to characterize geometrically the family of structuring elements in terms of which it can be decomposed (and in fact this family is infinite).

In general terms, the problem considered in Section 5 of [19] is whether the infinite iteration $\psi, \psi^2, \psi^3, \ldots$, of an anti-extensive and increasing operator ψ on a complete lattice $\mathcal{L}$, converges to an opening, even after an infinite number of steps. This convergence is defined as follows. Clearly $\ldots \leq \psi^n \leq \psi^{n-1} \leq \ldots \leq \psi$; thus ψ^n is a decreasing sequence of **T**-operators and we define its "limit" ψ^∞ by

$$\psi^\infty = \bigwedge_{n\geq 1} \psi^n .$$

Then ψ^∞ is an increasing and anti-extensive operator. For ψ^∞ to be idempotent, it is necessary and sufficient to have $\psi \cdot \psi^\infty = \psi^\infty$. It is important to notice that this equality does not necessarily hold. Two examples where $\psi(\psi^\infty(X)) \neq \psi^\infty(X)$ for some $X \in \mathcal{L}$ can be found: on p. 113 of [22] for $\mathcal{L} = \mathcal{P}(H)$, where H is a closed half-line in $\mathbb{R}$, and in [4] for $\mathcal{L} = \mathcal{P}(\mathbb{Z})$; the latter is particularly instructive, since ψ takes the form $X \mapsto (X \oplus A) \cap X$ for some infinite subset A of $\mathbb{Z}$.

There are then two possible orientations for obtaining an opening by iteration of ψ. The first one [10] is to admit higher powers of ψ than than integers or infinity, using the concept of *ordinals* from Zermelo-Fraenkel set theory. For any ordinal ν, ψ^ν is defined by *transfinite induction* as follows: if ν is a successor, that is $\nu = \mu + 1$ of another ordinal μ, then we set $\psi^\nu = \psi \cdot \psi^\mu$; otherwise ν is a limit, and we set $\psi^\nu = \bigwedge_{0<\mu<\nu} \psi^\mu$. As $\mathcal{L}$ is smaller than the class of ordinals, it is guaranteed that for some ordinal ν we will have $\psi^\nu = \psi^\mu$ for every ordinal $\mu > \nu$, and so ψ^ν is idempotent. This approach is useless for practical applications, but has some theoretical interest.

The second orientation is to find sufficient conditions for having $\psi \cdot \psi^\infty = \psi^\infty$. This problem is studied in [4] for sets in a digital or Euclidean space, and in Section 5 of [19] for arbitrary complete lattices (see also [23]). In particular it is shown that this equality is always verified for "local" operators on sets, whose behaviour at a point p depends on the configuration of points in a finite neighbourhhod of p. This is the case for openings by a finite structuring element that we considered in the example above. Possible applications of this approach in digital geometry include the construction of an opening from a local median filter [4], or the well-known computation of the distance transform of a set by iteration of a local neighbourhood transform, which is in fact an erosion [19]. This question is further developped in Section 7 of [5] for grey-level functions with a finite set of grey-levels. A general theoretical study of this subject is in preparation [7].

1.4. Miscellany

We mentioned at the beginning the relation between openings and size distributions, following Chapter 10 of [20]: for $\lambda > 0$, the operator ψ_λ extracting from a population the subset consisting of all elements having size at least λ must clearly be an opening. This concept can be generalized to a situation where size is not quantified, but we speak of "large enough" instead of "having size at least λ". We have only to specify the class $\mathcal{B}$ of elements which are "large enough", from which we require only that "larger than large enough is large enough", that is $B \in \mathcal{B}$ and $B \leq C$ implies $C \in \mathcal{B}$.

This idea is at the basis of the recently found class of "rank-max" and related openings [15], which generalize the translation-invariant opening α_B by a structuring element B in a digital or Euclidean space: instead of taking the union of all translates of B contained in a set X, we take the union of all subsets of X which consist of "large enough" portion of a translate of B. This type of openings will be discussed in Section 3, where we will study *inf-overfilters*.

Given a complete lattice $\mathcal{L}$ with least element O and an increasing operator ψ on $\mathcal{L}$, we can define X to be "large enough" if $\psi(X) \neq O$; indeed $\psi(X) \neq O$ and $X \leq Y$ implies $\psi(Y) \neq O$. We derive then the operator $\mathrm{G}[\psi]$ which preserves all "large enough" elements of $\mathcal{L}$ and removes all others:

$$\mathrm{G}[\psi](X) = \begin{cases} X & \text{if } \psi(X) \neq O; \\ O & \text{if } \psi(X) = O. \end{cases}$$

It is easily checked that this is an opening, and that it is invariant under the same automorphisms of $\mathcal{L}$ as ψ. When $\mathcal{L} = \mathcal{P}(\mathcal{E})$, we have already met such a type of opening in Subsection 1.1 (before Proposition 1.5).

For grey-level functions $\mathcal{E} \to \mathcal{G}$ (where $\mathcal{E}$ is the space $\mathbb{Z}^d$ or $\mathbb{R}^d$, and $\mathcal{G}$ is the set of grey-levels $\overline{\mathbb{Z}} = \mathbb{Z} \cup \{+\infty, -\infty\}$ or $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$), we can also remove all grey-levels which are below a given threshold $g \in \mathcal{G}$. This gives the threshold opening α_g defined by setting for every function F and point $p \in \mathcal{E}$:

$$\alpha_g(F)(p) = \begin{cases} F(p) & \text{if } F(p) \geq g \ ; \\ -\infty & \text{if } F(p) < g \ . \end{cases}$$

Openings and closings often intervene in computational geometry, particularly when one studies convexity in relation to shape. In fact the convex hull operation, as well as most of its variants (see [16]), is a translation-invariant closing. Sometimes morphological operators are applied in this field without the authors being aware of it. For example in [2] shape is discussed in relation to "α-hull" operators on Euclidean planar subsets, which are defined as follows for any $\alpha \in \mathbb{R}$; given a set $X \subseteq \mathbb{R}^2$, its α-hull is:

— for $\alpha = 0$, its convex hull;

— for $\alpha > 0$, the intersection of all closed disks of radius $1/\alpha$ containing X;

— for $\alpha < 0$, the intersection of all sets containing X which are complements of an open disk of radius $-1/\alpha$.

The authors derive from this construction some mathematical features related to the Delaunay triangulation of a finite cluster of points. See also [9] for an extension of this analysis. In fact, the α-hull is a translation-invariant closing, and for $\alpha \neq 0$ it is an example of *structural* $\mathbf{T}$-*closing*, a concept that we will introduce in Section 2.

The reader should consult Chapter 4 of [20] (and optionally Chapters 17 and 18 of [22]) for a detailed analysis of the role of convexity in Euclidean mathematical morphology. This background can be fruitful when one studies the wide literature on convexity and shape (see in particular [16], especially Section D).

Finally let us refer to the other chapters of this book (in particular [5]) for further examples of openings.

2. Algebraic Theory of Openings

Contemporary mathematical morphology studies the algebraic properties of image transformations in a very general framework: the object space (that is, the set of all pictures on which some types of operations are defined) is any complete lattice. Only in certain circumstances do we need to make some assumptions on it (cfr. the *Basic Assumption* below) in order to obtain more precise characterizations.

In Section 1 we have met several examples of complete lattices, in particular the set $\mathcal{P}(\mathcal{E})$ of parts of a space $\mathcal{E}$, ordered by inclusion. The basic concepts related to complete lattices were introduced in Subsection 1.3 of [6]. The reader should be acquainted with it. A more detailed exposition is to be found in Chapters 1 and 5 of [1]. Let us recall briefly our notation.

We have a complete lattice $\mathcal{L}$ with the order relation $\leq$, a supremum operation written $\bigvee$ or sup, an infimum operation written $\bigwedge$ or inf, both defined on any non-void subset of $\mathcal{L}$, and two universal bounds, the least element O and the greatest element I, defined by $I = \sup \mathcal{L}$ and $O = \inf \mathcal{L}$. Note that the supremum and infimum operations are also defined for the empty set:

$$O = \bigvee \emptyset \qquad \text{and} \qquad I = \bigwedge \emptyset. \tag{2.1}$$

(Cfr. the convention setting an empty sum equal to zero and an empty product equal to one). Other elements of $\mathcal{L}$ are written by capital letters X, Y, Z, etc. We consider $\mathcal{O} = \mathcal{L}^{\mathcal{L}}$, the set of all maps $\psi : \mathcal{L} \to \mathcal{L}$, and elements of $\mathcal{O}$ are called *operators*; $\mathcal{O}$ is naturally ordered by setting $\psi \leq \xi$ if and only if $\psi(X) \leq \xi(X)$ for all $X \in \mathcal{L}$. Then $\mathcal{O}$ inherits the complete lattice structure of $\mathcal{L}$, with $\mathbf{O} : X \mapsto O$ and $\mathbf{I} : X \mapsto I$ as least and greatest elements respectively, and the supremum and infimum operations given by setting for any $X \in \mathcal{L}$ and any family ψ_j $(j \in J)$ of operators:

$$\big(\bigvee_{j\in J} \psi_j\big)(X) = \bigvee_{j\in J} \big(\psi_j(X)\big) \qquad \text{and} \qquad \big(\bigwedge_{j\in J} \psi_j\big)(X) = \bigwedge_{j\in J} \big(\psi_j(X)\big).$$

The *identity* operator $X \mapsto X$ is written $\mathbf{id}$. Other operators are written by lowercase greek letters β, γ, etc., with the letters α, δ, ε, φ, τ being reserved to openings, dilations, erosions, closings, and "translations" (that is, automorphisms of $\mathcal{L}$). The *composition* $\psi\theta$ of the operator θ by the operator ψ is defined by $\psi\theta(X) = \psi\big(\theta(X)\big)$ for $X \in \mathcal{L}$. In particular, we write ψ^2 for $\psi\psi$, and more generally ψ^n for the composition of ψ repeated n times $(n > 0)$.

The *range* of an operator ψ is the set $\mathrm{Ran}(\psi)$ of all $\psi(X)$ for $X \in \mathcal{L}$; an *invariant* of ψ is some $X \in \mathcal{L}$ such that $\psi(X) = X$; the *domain of invariance* of ψ is the set $\mathrm{Inv}(\psi)$ of all invariants of ψ. Clearly $\mathrm{Inv}(\psi) \subseteq \mathrm{Ran}(\psi)$; moreover we have the following characterization of the idempotence of ψ:

$$\psi^2 = \psi \quad \iff \quad \mathrm{Ran}(\psi) \subseteq \mathrm{Inv}(\psi) \quad \iff \quad \mathrm{Ran}(\psi) = \mathrm{Inv}(\psi). \tag{2.2}$$

The operators in which we are interested are generally supposed to be invariant under a certain group of automorphisms of the complete lattice $\mathcal{L}$ (for example the group of translations when $\mathcal{L}$ is the set of parts of a Euclidean space). We take thus any *group* $\mathbf{T}$ of automorphisms of $\mathcal{L}$. Given $\tau \in \mathbf{T}$ and an operator $\psi \in \mathcal{O}$, we will say that ψ *commutes with* τ, or that ψ *is* τ*-invariant*, if $\psi\tau = \tau\psi$. Moreover, we will say that ψ *is* $\mathbf{T}$*-invariant* if ψ commutes with every $\tau \in \mathbf{T}$. We will use the prefix "$\mathbf{T}$-" for "$\mathbf{T}$-invariant". We will speak thus of $\mathbf{T}$-operators, $\mathbf{T}$-dilations, $\mathbf{T}$-erosions, $\mathbf{T}$-openings, $\mathbf{T}$-closings, etc. (see Section 3 of [6]). Note that when the operator ψ is not $\mathbf{T}$-invariant, the least $\mathbf{T}$-operator $\geq \psi$ and the greatest $\mathbf{T}$-operator $\leq \psi$ are

$$\bigvee_{\tau\in\mathbf{T}} \tau\psi\tau^{-1} \qquad \text{and} \qquad \bigwedge_{\tau\in\mathbf{T}} \tau\psi\tau^{-1}$$

respectively. We recall Proposition 3.1 of [6]: the set of $\mathbf{T}$-operators is closed under the operations of composition, supremum, infimum, and it contains $\mathbf{id}$, $\mathbf{O}$, $\mathbf{I}$. When $\mathbf{T}$-invariance is not necessary, we can take $\mathbf{T} = \{\mathbf{id}\}$, and then a result concerning $\mathbf{T}$-operators for an arbitrary $\mathbf{T}$ can be particularized into a similar one for operators without translation-invariance.

Given a subset $\mathcal{B}$ of $\mathcal{L}$ and $\tau \in \mathbf{T}$, let $\tau(\mathcal{B}) = \{\tau(X) \mid X \in \mathcal{B}\}$. We will say that $\mathcal{B}$ is $\mathbf{T}$*-invariant* if for every $\tau \in \mathbf{T}$, $\mathcal{B} = \tau(\mathcal{B})$. As $\mathbf{T}$ is a group, it is sufficient to show that $\tau(\mathcal{B}) \subseteq \mathcal{B}$ for any $\tau \in \mathbf{T}$, because we have then $\tau^{-1}(\mathcal{B}) \subseteq \mathcal{B}$ and so $\mathcal{B} = \tau\big(\tau^{-1}(\mathcal{B})\big) \subseteq \tau(\mathcal{B})$. If $\mathcal{B}$ is not $\mathbf{T}$-invariant, then the $\mathbf{T}$-invariant set generated by $\mathcal{B}$ is $\mathcal{B}^{\mathbf{T}} = \bigcup_{\tau\in\mathbf{T}} \tau(\mathcal{B})$. We say that a subset $\mathcal{B}$ of $\mathcal{L}$ is *sup-closed* if $\sup \mathcal{C} \in \mathcal{B}$ for any $\mathcal{C} \subseteq \mathcal{B}$ (in particular $O \in \mathcal{B}$ by (2.1)); an *inf-closed* subset of $\mathcal{L}$ is defined similarly. By Proposition 1.1 of [6], a sup-closed subset $\mathcal{B}$ of $\mathcal{L}$ is itself a complete lattice, with universal bounds O and $\sup \mathcal{B}$, the same supremum operation as in $\mathcal{L}$, but an infimum operation $\inf_{\mathcal{B}}$ not necessarily equal to the one in $\mathcal{L}$: for $\mathcal{K} \subseteq \mathcal{L}$, $\inf_{\mathcal{B}}(\mathcal{K})$ is equal to the greatest element of $\mathcal{B}$ which is a lower bound of $\mathcal{K}$. We write $\mathcal{B}_{\sup}$ and $\mathcal{B}_{\inf}$ for the sup-closed and inf-closed subsets generated by $\mathcal{B}$. They consist of all suprema $\bigvee \mathcal{H}$ and infima $\bigwedge \mathcal{H}$ respectively of subsets $\mathcal{H}$ of $\mathcal{B}$. Note that the sup-closed subset of $\mathcal{L}$ generated by a $\mathbf{T}$-invariant set is itself a $\mathbf{T}$-invariant set; when $\mathcal{B}$ is not a $\mathbf{T}$-invariant set, the sup-closed $\mathbf{T}$-invariant set generated by $\mathcal{B}$ is $\mathcal{B}^{\mathbf{T}}_{\sup} = (\mathcal{B}^{\mathbf{T}})_{\sup}$. The same is true for inf-closed sets.

2.1. Adjunctions, dilations, and erosions

One of the main contributions of [6] is the thorough study of *adjunctions* as a general principle for pairing dilations and erosions.

We recall from [22], Chapter 1, and [6], Subsection 2.1, that a *dilation* and an *erosion* are operators commuting with the supremum and infimum operations respectively, in other words δ is a dilation if

$$\delta\Big(\bigvee_{j \in J} X_j\Big) = \bigvee_{j \in J} \delta(X_j)$$

and ε is an erosion if

$$\varepsilon\Big(\bigwedge_{j \in J} X_j\Big) = \bigwedge_{j \in J} \varepsilon(X_j)$$

for any subset $\{X_j \mid j \in J\}$ of $\mathcal{L}$. In particular by (2.1) we have $\delta(O) = O$ and $\varepsilon(I) = I$. This definition includes as special case the well-known translation-invariant dilation $X \mapsto X \oplus B$ and erosion $X \mapsto X \ominus B$ of a set X by a structuring element B in a Euclidean or digital space.

Note that any dilation or erosion is an *increasing* operator, and that any automorphism of $\mathcal{L}$ is both a dilation and an erosion.

Although idempotence, extensivity, and anti-extensivity have a meaning only for operators mapping an object space $\mathcal{L}$ into itself, this is not the case for increasing operators, dilations, and erosions. They can be defined also as operators $\mathcal{L}_1 \to \mathcal{L}_2$, where $\mathcal{L}_1$ and $\mathcal{L}_2$ are two complete lattices. This is not a futile abstraction. Operators between distinct object spaces have been considered in the following situations:

- Distance transforms: given a digital space $\mathcal{E}$, the distance transform is a map $\mathcal{P}(\mathcal{E}) \to \mathrm{Fun}(\mathcal{E})$, where $\mathrm{Fun}(\mathcal{E})$ is the set of grey-level functions $\mathcal{E} \to \overline{\mathbb{Z}}$; it involves a dilation or an erosion, according to the convention used to express the transform. See [25] and Subsection 5.3 of [19].
- Decomposition of morphological operations on power lattices: see Subsection 2.4 of [6], where several practical examples are given.
- Transformation of dilations and erosions for grey-level functions $\mathcal{E} \to \mathcal{G}$, where $\mathcal{G} = \overline{\mathbb{Z}}$ or $\overline{\mathbb{R}}$, into similar ones for grey-level functions $\mathcal{E} \to \widehat{\mathcal{G}}$, where $\widehat{\mathcal{G}}$ is a bounded closed subset of $\mathcal{G}$: see [18], Section 4.

— Sampling and reconstruction: given the set $\mathcal{G} = \overline{\mathbb{Z}}$ or $\overline{\mathbb{R}}$ of grey-levels, a space $\mathcal{E}$ and a subspace $\mathcal{D} \subseteq \mathcal{E}$, the two object spaces $\mathrm{Fun}(\mathcal{E})$ and $\mathrm{Fun}(\mathcal{D})$ of grey-level functions $\mathcal{E} \to \mathcal{G}$ and $\mathcal{D} \to \mathcal{G}$ respectively can be linked by two maps, namely a *sampling* $\sigma : \mathrm{Fun}(\mathcal{E}) \to \mathrm{Fun}(\mathcal{D})$ and a *reconstruction* $\rho : \mathrm{Fun}(\mathcal{D}) \to \mathrm{Fun}(\mathcal{E})$. H. Heijmans and A. Toet [8] considered such operators in the case where $\mathcal{E}$ is a digital space and $\mathcal{D}$ a subspace with coarser resolution, and proposed to take a dilation for σ, and the adjoint erosion for ρ. This principle is also valid when $\mathcal{E} = \mathbb{R}^d$ and $\mathcal{D} = \mathbb{Z}^d$, in the context of digitization. Moreover it can be applied to the sampling and reconstruction of sets, with $\sigma : \mathcal{P}(\mathcal{E}) \to \mathcal{P}(\mathcal{D})$ and $\rho : \mathcal{P}(\mathcal{D}) \to \mathcal{P}(\mathcal{E})$ (cfr. the well-known *square box quantization* of sets).

Now given two operators $\eta : \mathcal{L}_2 \to \mathcal{L}_1$ and $\zeta : \mathcal{L}_1 \to \mathcal{L}_2$, we call the pair (η, ζ) an *adjunction between $\mathcal{L}_2$ and $\mathcal{L}_1$* if and only if for any $X \in \mathcal{L}_1$ and $Y \in \mathcal{L}_2$ we have

$$\zeta(X) \leq Y \iff X \leq \eta(Y). \tag{2.3}$$

When ζ and η are both $\mathcal{L} \to \mathcal{L}$, we say that (η, ζ) is an adjunction *on $\mathcal{L}$*. Adjunctions on $\mathcal{L}$ were considered by Serra in Chapter 1 of [22] under the name of "morphological duality". However this concept is much older than mathematical morphology; it is linked to the classical mathematical notion of *Galois connection* (see [6], Subsection 2.3), and to category theory [3].

Examples of adjunctions include: the erosion and dilation by a structuring element B for Euclidean or digital sets, the reconstruction and sampling mappings of [8] that we mentioned above.

Adjunctions between *distinct* complete lattices are very interesting from a theoretical point of view. For example they are used by Roerdink [14] in order to define Minkowski operations $\oplus$ and $\ominus$ in the case of a space with a non-abelian group of symmetries in place of translations. In [5] an adjunction between grey-level functions and sets is defined from thresholding (see the equations (4.2) and (4.5) there), and it is used to extend set operators to "flat" operators on grey-level images. A further application of this general framework will be given in the next subsection for the structural characterization of openings. We will thus consider two or even three complete lattices, which may or may not be distinct, and adjunctions between them.

A detailed study of adjunctions is made in Subsections 2.3 and 3.1 of [6], and Subsection 2.2 of [19]. We will summarize these results here. First, adjunctions are restricted to dilations and erosions with the same invariant automorphisms:

LEMMA 2.1. *Given an adjunction (η, ζ) between two complete lattices $\mathcal{L}_2$ and $\mathcal{L}_1$ and a group $\mathbf{T}$ of automorphisms of both $\mathcal{L}_1$ and $\mathcal{L}_2$,*

(i) ζ is a dilation and η is an erosion;

(ii) ζ is $\mathbf{T}$-invariant if and only if η is $\mathbf{T}$-invariant.

(Note that when $\mathbf{T}$ acts as a group of automorphisms of both $\mathcal{L}_1$ and $\mathcal{L}_2$ it is meaningful to speak of $\mathbf{T}$-invariance for operators $\mathcal{L}_1 \to \mathcal{L}_2$ and $\mathcal{L}_2 \to \mathcal{L}_1$.) For a proof, see Propositions 2.5 and 3.2 of [6]. An adjunction (η, ζ) where ζ and η are $\mathbf{T}$-invariant is called a *$\mathbf{T}$-adjunction*.

Now let δ be a $\mathbf{T}$-dilation $\mathcal{L}_1 \to \mathcal{L}_2$. If (ε, δ) is a $\mathbf{T}$-adjunction for some $\mathbf{T}$-erosion $\varepsilon : \mathcal{L}_2 \to \mathcal{L}_1$, then for any $Y \in \mathcal{L}_2$, $\varepsilon(Y)$ is by definition the greatest $Z \in \mathcal{L}_1$ such that $Z \leq \varepsilon(Y)$. Now by (2.3) $Z \leq \varepsilon(Y) \iff \delta(Z) \leq Y$, so that:

$$\text{For every } Y \in \mathcal{L}_2,\ \varepsilon(Y) \text{ is the greatest } Z \in \mathcal{L}_1 \text{ such that } \delta(Z) \leq Y. \tag{2.4}$$

But the greatest element of a set is its supremum, and hence (2.4) implies:

$$\text{For every} \quad Y \in \mathcal{L}_2, \qquad \varepsilon(Y) = \bigvee \{Z \in \mathcal{L}_1 \mid \delta(Z) \leq Y\}. \tag{2.5}$$

Finally if (2.5) holds, it is easy to show (see in [6] the proofs of Proposition 2.6 and point (i) of Theorem 2.7), using the fact that δ commutes with supremum, that (ε, δ) is an adjunction, and so ε is a $\mathbf{T}$-erosion. Therefore the three statements: (ε, δ) is a $\mathbf{T}$-adjunction, (2.4), and (2.5), are equivalent. Similarly, given a $\mathbf{T}$-erosion $\varepsilon : \mathcal{L}_2 \to \mathcal{L}_1$, the fact that (ε, δ) is a $\mathbf{T}$-adjunction is equivalent to each of the following two statements:

$$\text{For every } X \in \mathcal{L}_1,\ \delta(X) \text{ is the least } Z \in \mathcal{L}_2 \text{ such that } X \leq \varepsilon(Z). \tag{2.6}$$

$$\text{For every} \quad X \in \mathcal{L}_1, \qquad \delta(X) = \bigwedge \{Z \in \mathcal{L}_2 \mid X \leq \varepsilon(Z)\}. \tag{2.7}$$

We conclude:

Proposition 2.2. *Given a group $\mathbf{T}$ of automorphisms of two complete lattices $\mathcal{L}_1$ and $\mathcal{L}_2$, the set of $\mathbf{T}$-adjunctions between $\mathcal{L}_2$ and $\mathcal{L}_1$ constitutes a bijection between the set of $\mathbf{T}$-dilations $\mathcal{L}_1 \to \mathcal{L}_2$ and the set of $\mathbf{T}$-erosions $\mathcal{L}_2 \to \mathcal{L}_1$; in other words:*

- (i) *Given a $\mathbf{T}$-dilation $\delta : \mathcal{L}_1 \to \mathcal{L}_2$, there is a unique $\mathbf{T}$-erosion $\varepsilon : \mathcal{L}_2 \to \mathcal{L}_1$ such that (ε, δ) is an adjunction; ε is defined by* (2.4) *or equivalently* (2.5).
- (ii) *Given a $\mathbf{T}$-erosion $\varepsilon : \mathcal{L}_2 \to \mathcal{L}_1$, there is a unique $\mathbf{T}$-dilation $\delta : \mathcal{L}_1 \to \mathcal{L}_2$ such that (ε, δ) is an adjunction; δ is defined by* (2.6) *or equivalently* (2.7).

In an adjunction (ε, δ), we say that ε is the *upper adjoint* of δ, while δ is the *lower adjoint* of ε; accordingly we write $\varepsilon = \dot{\delta}$ and $\delta = \underset{\cdot}{\varepsilon}$ in order to mean (2.5) and (2.7). The following result is easily proved from (2.3):

Lemma 2.3. *Given a group $\mathbf{T}$ of automorphisms of three complete lattices $\mathcal{L}_1$, $\mathcal{L}_2$, and $\mathcal{L}_3$:*

- (i) *If $(\varepsilon_j, \delta_j)$ is a $\mathbf{T}$-adjunction between $\mathcal{L}_2$ and $\mathcal{L}_1$ for every $j \in J$, then $(\bigwedge_{j\in J} \varepsilon_j, \bigvee_{j\in J} \delta_j)$ is a $\mathbf{T}$-adjunction between $\mathcal{L}_2$ and $\mathcal{L}_1$.*
- (ii) *Given two $\mathbf{T}$-adjunctions (ε, δ) between $\mathcal{L}_2$ and $\mathcal{L}_1$ and (ε', δ') between $\mathcal{L}_3$ and $\mathcal{L}_2$, then $(\varepsilon\varepsilon', \delta'\delta)$ is a $\mathbf{T}$-adjunction between $\mathcal{L}_3$ and $\mathcal{L}_1$.*

Following [1], one calls a *dual isomorphism* between two lattices a bijection θ which reverses the partial order relation: $X \leq Y \iff \theta(X) \geq \theta(Y)$. A bijection which transforms suprema into infima is a dual isomorphism, because we have then:

$$X \leq Y \ \Leftrightarrow\ Y = X \vee Y \ \Leftrightarrow\ \theta(Y) = \theta(X \vee Y) \ \Leftrightarrow\ \theta(Y) = \theta(X) \wedge \theta(Y) \ \Leftrightarrow\ \theta(X) \geq \theta(Y).$$

From Proposition 2.2 and Lemma 2.3 we derive then the following immediate consequence:

Corollary 2.4. *Given a group* $\mathbf{T}$ *of automorphisms of three complete lattices* $\mathcal{L}_1$, $\mathcal{L}_2$, *and* $\mathcal{L}_3$*:*

(i) *The set of* $\mathbf{T}$*-dilations* $\mathcal{L}_1 \to \mathcal{L}_2$ *is sup-closed, the set of* $\mathbf{T}$*-erosions* $\mathcal{L}_2 \to \mathcal{L}_1$ *is inf-closed; both are thus complete lattices. The set of* $\mathbf{T}$*-adjunctions between* $\mathcal{L}_2$ *and* $\mathcal{L}_1$ *is a dual isomorphism between these two complete lattices.*

(ii) *The composition of two* $\mathbf{T}$*-dilations* $\mathcal{L}_1 \to \mathcal{L}_2$ *and* $\mathcal{L}_2 \to \mathcal{L}_3$ *is a* $\mathbf{T}$*-dilation* $\mathcal{L}_1 \to \mathcal{L}_3$*, and the same holds for* $\mathbf{T}$*-erosions.* $\mathbf{T}$*-adjunctions induce an anti-automorphism for the law of composition, in other words for three dilations* $\delta, \delta_1, \delta_2$ *we have* $\delta = \delta_1\delta_2$ *if and only if* $\dot{\delta} = \dot{\delta}_2\dot{\delta}_1$.

In particular when we restrict ourselves to one complete lattice $\mathcal{L}$, the set of $\mathbf{T}$-dilations (or $\mathbf{T}$-erosions) is a *monoid*, in the sense that it is closed under composition and contains the identity **id**. We end with the following result coming from the Propositions 2.8 of [6] and [19], except the third point, whose proof is left to the reader:

Proposition 2.5. *Given a group* $\mathbf{T}$ *of automorphisms of two complete lattices* $\mathcal{L}_1$ *and* $\mathcal{L}_2$*, and a* $\mathbf{T}$*-adjunction* (ε, δ) *between* $\mathcal{L}_2$ *and* $\mathcal{L}_1$*,*

(i) $\delta\varepsilon\delta = \delta$, $\varepsilon\delta\varepsilon = \varepsilon$, $\delta\varepsilon$ *is a* $\mathbf{T}$*-opening on* $\mathcal{L}_2$*, and* $\varepsilon\delta$ *is a* $\mathbf{T}$*-closing on* $\mathcal{L}_1$.

(ii) $\mathrm{Inv}(\delta\varepsilon) = \mathrm{Ran}(\delta\varepsilon) = \mathrm{Ran}(\delta)$ *and* $\mathrm{Inv}(\varepsilon\delta) = \mathrm{Ran}(\varepsilon\delta) = \mathrm{Ran}(\varepsilon)$.

(iii) $\mathrm{Ran}(\delta)$ *is sup-closed,* $\mathrm{Ran}(\varepsilon)$ *is inf-closed, and both are* $\mathbf{T}$*-invariant; they are isomorphic complete lattices, where* $D \in \mathrm{Ran}(\delta)$ *and* $E \in \mathrm{Ran}(\varepsilon)$ *correspond under this isomorphism by the equivalent relations* $E = \varepsilon(D)$ *and* $D = \delta(E)$.

We can use (2.5) and (2.7) in order to describe the behaviour of $\delta\varepsilon$ and $\varepsilon\delta$. Applying δ to both sides of (2.5), the fact that δ commutes with the supremum operation gives for any $Y \in \mathcal{L}_2$:

$$\delta\varepsilon(Y) = \bigvee\{\delta(Z) \mid Z \in \mathcal{L}_1,\ \delta(Z) \leq Y\}. \tag{2.8}$$

On the other hand (2.7) with $X = \varepsilon(Y)$ gives:

$$\delta\varepsilon(Y) = \bigwedge\{Z \in \mathcal{L}_2 \mid \varepsilon(Y) \leq \varepsilon(Z)\}. \tag{2.9}$$

Similarly for any $X \in \mathcal{L}_1$ we obtain:

$$\varepsilon\delta(X) = \bigwedge\{\varepsilon(Z) \mid Z \in \mathcal{L}_2,\ X \leq \varepsilon(Z)\}; \tag{2.10}$$

$$= \bigvee\{Z \in \mathcal{L}_1 \mid \delta(Z) \leq \delta(X)\}. \tag{2.11}$$

If we use (2.4) and (2.6) instead of (2.5) and (2.7), then in the above four equations we may replace $\bigvee$ by "the greatest element of" and $\bigwedge$ by "the least element of".

2.2. Structural characterization of openings

Now we stay within a single complete lattice $\mathcal{L}$ with a group $\mathbf{T}$ of automorphisms. Let us call a *contraction* an increasing and anti-extensive operator. Thus an opening is an idempotent contraction. Write $\mathcal{C}^{\mathbf{T}}$ for the set of $\mathbf{T}$-contractions. We will exhibit an adjunction between $\mathcal{C}^{\mathbf{T}}$ and $\mathcal{P}(\mathcal{L})$ which induces an isomorphism between the set of $\mathbf{T}$-openings and the set of sup-closed $\mathbf{T}$-invariant subsets of $\mathcal{L}$.

We first exhibit the structure of $\mathcal{C}^{\mathbf{T}}$ as complete lattice and monoid. The following result is easily proved (part of it comes from Propositions 2.2 and 3.1 of [6]):

Lemma 2.6. *$\mathcal{C}^{\mathbf{T}}$ is closed under composition, arbitrary suprema, and non-empty infima, and its has $\mathbf{O}$ and $\mathbf{id}$ as least and greatest element.*

Thus $\mathcal{C}^{\mathbf{T}}$ is a complete lattice, with the same supremum and infimum operations as in the set $\mathcal{O}$ of operators, except that the empty infimum is $\mathbf{id}$ in $\mathcal{C}^{\mathbf{T}}$ and $\mathbf{I}$ in $\mathcal{O}$ (cfr. (2.1)).

Now let us see how the partial order on $\mathcal{C}^{\mathbf{T}}$ is translated into the domain of invariance of its elements. Given $\psi, \psi' \in \mathcal{C}^{\mathbf{T}}$, $\psi \leq \psi'$ implies $\mathrm{Inv}(\psi) \subseteq \mathrm{Inv}(\psi')$; indeed, for $X \in \mathrm{Inv}(\psi)$ we have $X = \psi(X) \leq \psi'(X) \leq X$ and so $X \in \mathrm{Inv}(\psi')$. For $\psi_1, \ldots, \psi_n \in \mathcal{C}^{\mathbf{T}}$ $(n > 1)$ we have

$$\mathrm{Inv}(\psi_1 \cdots \psi_n) = \mathrm{Inv}(\psi_1) \cap \cdots \cap \mathrm{Inv}(\psi_n). \tag{2.12}$$

Indeed, as $\psi_1 \cdots \psi_n \leq \psi_j$ for $j = 1, \ldots, n$, we get $\mathrm{Inv}(\psi_1 \cdots \psi_n) \subseteq \mathrm{Inv}(\psi_j)$; on the other hand it is trivial that for $X \in \mathrm{Inv}(\psi_1) \cap \cdots \cap \mathrm{Inv}(\psi_n)$ we have $\psi_1 \cdots \psi_n(X) = X$. A similar argument shows that for a *non-empty* family ψ_j $(j \in J)$ of $\mathbf{T}$-contractions,

$$\mathrm{Inv}\Big(\bigwedge_{j \in J} \psi_j\Big) = \bigcap_{j \in J} \mathrm{Inv}(\psi_j) \qquad\qquad (J \neq \emptyset). \tag{2.13}$$

Now let $\mathcal{P}(\mathcal{L})$ be the complete lattice of parts of $\mathcal{L}$, ordered by inclusion. We obtain the following:

Proposition 2.7. *The map $\mathcal{C}^{\mathbf{T}} \to \mathcal{P}(\mathcal{L}) : \psi \mapsto \mathrm{Inv}(\psi)$ is an erosion. It has a lower adjoint dilation $\mathcal{P}(\mathcal{L}) \to \mathcal{C}^{\mathbf{T}} : \mathcal{B} \mapsto \mathbf{A}^{\mathbf{T}}(\mathcal{B})$ defined as follows: for any $\mathcal{B} \subseteq \mathcal{L}$, $\mathbf{A}^{\mathbf{T}}(\mathcal{B})$ is the least $\mathbf{T}$-contraction ξ such that $\mathcal{B} \subseteq \mathrm{Inv}(\xi)$. Conversely, for any $\mathbf{T}$-contraction ψ, $\mathrm{Inv}(\psi)$ is the greatest $\mathcal{X} \subseteq \mathcal{L}$ such that $\mathbf{A}^{\mathbf{T}}(\mathcal{X}) \leq \psi$.*

Proof. By (2.13) the map $\psi \mapsto \mathrm{Inv}(\psi)$ commutes with non-empty infima. As the empty infimum is $\mathbf{id}$ in $\mathcal{C}^{\mathbf{T}}$ and $\mathcal{L}$ in $\mathcal{P}(\mathcal{L})$, since we have $\mathrm{Inv}(\mathbf{id}) = \mathcal{L}$, that map commutes also with the empty infimum, and it is an erosion. Given the lower adjoint dilation $\mathcal{B} \mapsto \mathbf{A}^{\mathbf{T}}(\mathcal{B})$, applying (2.6) and (2.4) to them gives the expression of $\mathbf{A}^{\mathbf{T}}(\mathcal{B})$ and $\mathrm{Inv}(\psi)$ respectively. ∎

Following [22], in Subsection 2.2 of [19] we called $\mathbf{A}^{\mathbf{T}}(\mathcal{B})$ the *least $\mathbf{T}$-extension of the identity on $\mathcal{B}$*, which means that it is the least increasing $\mathbf{T}$-operator having $\mathcal{B}$ in its domain of invariance. As the map $\mathcal{B} \mapsto \mathbf{A}^{\mathbf{T}}(\mathcal{B})$ is a dilation $\mathcal{P}(\mathcal{L}) \to \mathcal{C}^{\mathbf{T}}$, for any family $\mathcal{B}_j$ $(j \in J)$ of subsets of $\mathcal{L}$ (even an empty one) we have

$$\mathbf{A}^{\mathbf{T}}\Big(\bigcup_{j \in J} \mathcal{B}_j\Big) = \bigvee_{j \in J} \mathbf{A}^{\mathbf{T}}(\mathcal{B}_j). \tag{2.14}$$

The following result describes $\mathbf{A}^{\mathbf{T}}(\mathcal{B})$ in more detail:

Proposition 2.8. *Given $\mathcal{B} \subseteq \mathcal{L}$,*

(i) $\mathbf{A}^{\mathbf{T}}(\mathcal{B})$ *is a $\mathbf{T}$-opening.*

(ii) *For any $X \in \mathcal{L}$, $\mathbf{A}^{\mathbf{T}}(\mathcal{B})(X) = \bigvee\{B \in \mathcal{B}^{\mathbf{T}} \mid B \leq X\}$, and $\mathbf{A}^{\mathbf{T}}(\mathcal{B})(X)$ is the greatest $Y \in \mathcal{B}^{\mathbf{T}}_{\sup}$ such that $Y \leq X$.*

(iii) $\mathrm{Inv}(\mathbf{A}^{\mathbf{T}}(\mathcal{B})) = \mathcal{B}^{\mathbf{T}}_{\sup}$.

Proof. By definition, $\mathbf{A^T}(\mathcal{B})$ is the least $\xi \in \mathcal{C}^{\mathbf{T}}$ such that $\mathcal{B} \subseteq \mathrm{Inv}(\xi)$; but then $\mathcal{B} \subseteq \mathrm{Inv}(\xi^2)$ and $\xi^2 \leq \xi$, from which we deduce $\xi^2 = \xi$, that is $\mathbf{A^T}(\mathcal{B})$ is idempotent, and (i) holds.

Let ψ be defined by $\psi(X) = \bigvee\{B \in \mathcal{B}^{\mathbf{T}} \mid B \leq X\}$. Obviously $\psi(X) \in \mathcal{B}^{\mathbf{T}}_{\mathrm{sup}}$ and $\psi(X) \leq X$. Given $A \in \mathcal{B}^{\mathbf{T}}_{\mathrm{sup}}$ with $A \leq X$, then either $A = O \leq \psi(X)$, or $A = \sup \mathcal{X}$ for some non-empty $\mathcal{X} \subseteq \mathcal{B}^{\mathbf{T}}$, and for each $B \in \mathcal{X}$ we have $B \leq A \leq X$, so that $\mathcal{X}$ enters in the decomposition of $\psi(X)$ and $A \leq \psi(X)$. Thus $\psi(X)$ is the greatest $Y \in \mathcal{B}^{\mathbf{T}}_{\mathrm{sup}}$ such that $Y \leq X$. It is then easy to show that ψ is a $\mathbf{T}$-contraction and $\mathrm{Inv}(\psi) = \mathcal{B}^{\mathbf{T}}_{\mathrm{sup}}$.

Let ξ be a $\mathbf{T}$-contraction such that $\mathcal{B} \subseteq \mathrm{Inv}(\xi)$. The $\mathbf{T}$-invariance of ξ implies that $\mathcal{B}^{\mathbf{T}} \subseteq \mathrm{Inv}(\xi)$. For every $B \in \mathcal{B}^{\mathbf{T}}$ such that $B \leq X$, we have $B = \xi(B) \leq \xi(X)$, and so $\psi(X) \leq \xi(X)$ by definition of ψ. Thus ψ is the least $\mathbf{T}$-contraction having $\mathcal{B}$ in its domain of invariance, and so $\psi = \mathbf{A^T}(\mathcal{B})$ and $\mathrm{Inv}(\mathbf{A^T}(\mathcal{B})) = \mathrm{Inv}(\psi) = \mathcal{B}^{\mathbf{T}}_{\mathrm{sup}}$. Hence (ii) and (iii) hold. ∎

The next result is Proposition 2.3 of [19]:

Proposition 2.9. *Let α be an opening and ψ a contraction. Then the following four statements are equivalent:*

(i) $\alpha \leq \psi$.

(ii) $\alpha\psi = \alpha$.

(iii) $\psi\alpha = \alpha$.

(iv) $\mathrm{Inv}(\alpha) \subseteq \mathrm{Inv}(\psi)$.

Consider for example size distributions: for each $\lambda > 0$, there is an opening α_λ extracting from a population the subset consisting of all elements of size at least λ; then for $\lambda > \mu > 0$ we have obviously $\alpha_\lambda \leq \alpha_\mu$ and $\alpha_\lambda\alpha_\mu = \alpha_\mu\alpha_\lambda = \alpha_\lambda$.

Now the above results allow us to formulate the structural characterization of $\mathbf{T}$-openings in terms of sup-closed $\mathbf{T}$-invariant subsets of $\mathcal{L}$:

Theorem 2.10.

(i) *For any $\mathbf{T}$-contraction ψ, ψ is a $\mathbf{T}$-opening if and only if $\psi = \mathbf{A^T}(\mathcal{B})$ for some subset $\mathcal{B}$ of $\mathcal{L}$.*

(ii) *The set of $\mathbf{T}$-openings is sup-closed, with universal bounds $\mathbf{O}$ and $\mathbf{id}$.*

(iii) *For any $\mathbf{T}$-contraction ψ, $\mathbf{A^T}(\mathrm{Inv}(\psi))$ is the greatest $\mathbf{T}$-opening $\leq \psi$.*

(iv) *For any subset $\mathcal{B}$ of $\mathcal{L}$, $\mathcal{B}$ is sup-closed and $\mathbf{T}$-invariant if and only $\mathcal{B} = \mathrm{Inv}(\psi)$ for some $\mathbf{T}$-contraction ψ.*

(v) *The set of sup-closed $\mathbf{T}$-invariant subsets of $\mathcal{L}$ is closed under intersection, with universal bounds $\emptyset$ and $\mathcal{L}$.*

(vi) *For any subset $\mathcal{B}$ of $\mathcal{L}$, $\mathrm{Inv}(\mathbf{A^T}(\mathcal{B})) = \mathcal{B}^{\mathbf{T}}_{\mathrm{sup}}$, the least sup-closed $\mathbf{T}$-invariant subset of $\mathcal{L}$ containing $\mathcal{B}$.*

(vii) *The set of $\mathbf{T}$-openings, ordered by $\leq$, and the set of sup-closed $\mathbf{T}$-invariant subsets of $\mathcal{L}$, ordered by inclusion, are isomorphic complete lattices. A $\mathbf{T}$-opening α and a sup-closed $\mathbf{T}$-invariant set $\mathcal{B}$ correspond under this isomorphism by the equivalent relations $\mathcal{B} = \mathrm{Inv}(\alpha)$ and $\alpha = \mathbf{A^T}(\mathcal{B})$.*

Proof. We will use freely Proposition 2.5 with $\delta : \mathcal{P}(\mathcal{L}) \to \mathcal{C}^{\mathbf{T}} : \mathcal{B} \mapsto \mathbf{A^T}(\mathcal{B})$ and $\varepsilon : \mathcal{C}^{\mathbf{T}} \to \mathcal{P}(\mathcal{L}) : \psi \mapsto \mathrm{Inv}(\psi)$.

(i): By Proposition 2.8 (i), $\mathbf{A}^{\mathbf{T}}(\mathcal{B})$ is a $\mathbf{T}$-opening. Conversely, given a $\mathbf{T}$-opening α, $\mathbf{A}^{\mathbf{T}}(\mathrm{Inv}(\alpha))$ is by definition the least $\mathbf{T}$-contraction ξ such that $\mathrm{Inv}(\alpha) \subseteq \mathrm{Inv}(\xi)$ (see Proposition 2.7). Now by Proposition 2.9, $\mathrm{Inv}(\alpha) \subseteq \mathrm{Inv}(\xi)$ is equivalent to $\alpha \leq \xi$, and so this least ξ must be α, that is $\alpha = \mathbf{A}^{\mathbf{T}}(\mathrm{Inv}(\alpha))$.
(ii): By (i) the range of δ is the set of $\mathbf{T}$-openings, and $\mathrm{Ran}(\delta)$ is sup-closed. See also (2.14).
(iii): $\mathbf{A}^{\mathbf{T}}(\mathrm{Inv}(\psi))$ is a $\mathbf{T}$-opening (see (i)), and as $\delta\varepsilon$ is anti-extensive, $\mathbf{A}^{\mathbf{T}}(\mathrm{Inv}(\psi)) \leq \psi$. Given any other $\mathbf{T}$-opening $\alpha \leq \psi$, by Proposition 2.9 we have $\mathrm{Inv}(\alpha) \subseteq \mathrm{Inv}(\psi)$; now as $\varepsilon\delta\varepsilon = \varepsilon$, we have $\mathrm{Inv}(\mathbf{A}^{\mathbf{T}}(\mathrm{Inv}(\psi))) = \mathrm{Inv}(\psi)$, so that $\mathrm{Inv}(\alpha) \subseteq \mathrm{Inv}(\mathbf{A}^{\mathbf{T}}(\mathrm{Inv}(\psi)))$; by Proposition 2.9 this gives $\alpha \leq \mathbf{A}^{\mathbf{T}}(\mathrm{Inv}(\psi))$. Thus $\mathbf{A}^{\mathbf{T}}(\mathrm{Inv}(\psi))$ is the greatest such α.
(iv): Given a $\mathbf{T}$-contraction ψ and $\mathcal{B} = \mathrm{Inv}(\psi)$, by Proposition 2.8 ($iii$) and the fact that $\varepsilon\delta\varepsilon = \varepsilon$ we obtain $\mathcal{B}^{\mathbf{T}}_{\mathrm{sup}} = \mathrm{Inv}(\mathbf{A}^{\mathbf{T}}(\mathcal{B})) = \mathrm{Inv}(\mathbf{A}^{\mathbf{T}}(\mathrm{Inv}(\psi))) = \mathrm{Inv}(\psi) = \mathcal{B}$, that is $\mathcal{B}$ is sup-closed and $\mathbf{T}$-invariant. Conversely if $\mathcal{B}$ is sup-closed and $\mathbf{T}$-invariant, Proposition 2.8 (iii) again gives $\mathcal{B} = \mathcal{B}^{\mathbf{T}}_{\mathrm{sup}} = \mathrm{Inv}(\mathbf{A}^{\mathbf{T}}(\mathcal{B}))$.
(v): By (iv) the range of ε is the set of sup-closed $\mathbf{T}$-invariant subsets of $\mathcal{L}$, and $\mathrm{Ran}(\varepsilon)$ is inf-closed. See also (2.13).
(vi) follows from Proposition 2.8 (iii), given the obvious fact that $\mathcal{B}^{\mathbf{T}}_{\mathrm{sup}}$ is the least sup-closed $\mathbf{T}$-invariant subset of $\mathcal{L}$ containing $\mathcal{B}$.
(vii): We have shown that $\mathrm{Ran}(\delta)$ is the set of $\mathbf{T}$-openings, while $\mathrm{Ran}(\varepsilon)$ is the set of sup-closed $\mathbf{T}$-invariant subsets of $\mathcal{L}$. Both complete lattice are isomorphic, the isomorphism being given by the restriction of δ to $\mathrm{Ran}(\varepsilon)$ or conversely the restriction of ε to $\mathrm{Ran}(\delta)$. ∎

Given $B \in \mathcal{L}$, let us write $\mathbf{A}^{\mathbf{T}}_{B}$ for $\mathbf{A}^{\mathbf{T}}(\{B\})$; we call it the *structural* $\mathbf{T}$*-opening by* B, and for each $X \in \mathcal{L}$ we have

$$\mathbf{A}^{\mathbf{T}}_{B}(X) = \bigvee\{\tau(B) \mid \tau \in \mathbf{T},\ \tau(B) \leq X\}.$$

Clearly $\mathbf{A}^{\mathbf{T}}_{B} = \mathbf{A}^{\mathbf{T}}_{\tau(B)}$ for every $\tau \in \mathbf{T}$. For example in the case where $\mathcal{L} = \mathcal{P}(\mathcal{E})$ for a digital or Euclidean space $\mathcal{E}$ and $\mathbf{T}$ is the group of translations of $\mathcal{E}$, for $B \subseteq \mathcal{E}$ the structural $\mathbf{T}$-opening $\mathbf{A}^{\mathbf{T}}_{B}$ by B is equal to $\delta_B\varepsilon_B$, the composition of the erosion and dilation by B.

Given a $\mathbf{T}$-opening α and any $\mathcal{B} \subseteq \mathcal{L}$ such that $\mathrm{Inv}(\alpha) = \mathcal{B}^{\mathbf{T}}_{\mathrm{sup}}$, we have $\alpha = \mathbf{A}^{\mathbf{T}}(\mathcal{B})$ by the above theorem, and (2.14) gives $\mathbf{A}^{\mathbf{T}}(\mathcal{B}) = \sup_{B\in\mathcal{B}} \mathbf{A}^{\mathbf{T}}_{B}$. Thus *every* $\mathbf{T}$*-opening is a supremum of structural* $\mathbf{T}$*-openings*, something which is well-known in the case of $\mathcal{P}(\mathcal{E})$ with translations.

There are two remarks to be made now. First, the results given here can be obtained in a more "down-to-earth" way, without recourse to the adjunction between $\mathcal{C}^{\mathbf{T}}$ and $\mathcal{P}(\mathcal{L})$; this is indeed the approach followed in [19], Section 2; however the abstract proofs given here are much shorter.

Second, the above theory can be translated to closings by duality, interverting dual notions such as supremum and infimum, dilation and erosion, etc. Call an *expansion* an increasing and extensive operator. The map $\psi \mapsto \mathrm{Inv}(\psi)$ is a dilation from the complete lattice of $\mathbf{T}$-expansions to $(\mathcal{P}(\mathcal{L}), \supseteq)$, the dual of $(\mathcal{P}(\mathcal{L}), \subseteq)$; alternately, it is an erosion from the dual of the complete lattice of $\mathbf{T}$-expansions to $(\mathcal{P}(\mathcal{L}), \subseteq)$. Its adjoint $\mathcal{B} \mapsto \mathbf{F}^{\mathbf{T}}(\mathcal{B})$ associates to each $\mathcal{B} \subseteq \mathcal{L}$ the *greatest* $\mathbf{T}$*-extension of identity on* $\mathcal{B}$, that is the greates $\mathbf{T}$-expansion having $\mathcal{B}$ in its domain of invariance. Under this adjunction, $\mathbf{T}$-closings correspond by dual isomorphism to inf-closed $\mathbf{T}$-invariant subsets of $\mathcal{L}$. We define also $\mathbf{F}^{\mathbf{T}}_B = \mathbf{F}^{\mathbf{T}}(\{B\})$, the *structural* $\mathbf{T}$*-closing by* B, and then every $\mathbf{T}$-closing is an infimum of structural $\mathbf{T}$-closings.

2.3. Basic Assumption and duality under inversion

In the case of sets or grey-level functions on a Euclidean or digital space with translation-invariance, a structural $\mathbf{T}$-opening or $\mathbf{T}$-closing can be obtained by the composition of a $\mathbf{T}$-dilation and its adjoint $\mathbf{T}$-erosion. We will give here the conditions which guarantee this property; they are of course satisfied in the above two cases.

Given $\ell \subseteq \mathcal{L}$, we say that ℓ is *sup-generating* if every $X \in \mathcal{L}$ is the supremum of a subset of ℓ. Elements in ℓ will be written as lower-case letters x, y, z, etc. For $X \in \mathcal{L}$ we define $\ell(X) = \{x \in \ell \mid x \leq X\}$. The fact that ℓ is sup-generating means that $X = \bigvee \ell(X)$. In Subsection 3.2 of [6] we introduced the following:

Basic Assumption. *$\mathbf{T}$ is commutative and $\mathcal{L}$ has a sup-generating subset ℓ such that:*

(i) ℓ is $\mathbf{T}$-invariant, in other words for every $\tau \in \mathbf{T}$ and $x \in \ell$, $\tau(x) \in \ell$;

(ii) $\mathbf{T}$ is transitive on ℓ, in other words for every $x, y \in \ell$, there exists $\tau \in \mathbf{T}$ such that $\tau(x) = y$.

As explained in Subsection 3.2 of [6], the Basic Assumption implies that $\mathbf{T}$ acts *regularly* on ℓ, in other words that for every $x, y \in \ell$, there is a *unique* $\tau \in \mathbf{T}$ such that $\tau(x) = y$. Let o be some fixed element of ℓ which we call the *origin*. For every $x \in \ell$ there is a unique $\tau_x \in \mathbf{T}$ such that $\tau_x(o) = x$. This bijection between ℓ and $\mathbf{T}$ allows us to endow ℓ with the commutative group structure of $\mathbf{T}$. For $x, y \in \ell$ we define $x + y = \tau_y(x) = \tau_x(y)$, $-x = \tau_x^{-1}(o)$, and $x - y = x + (-y)$. This makes ℓ an additive group isomorphic to $\mathbf{T}$. For $X \in \mathcal{L}$ and $h \in \ell$, we define $X_h = \tau_h(X)$. Then in Subsection 3.2 of [6] we proved that every $\mathbf{T}$-adjunction on $\mathcal{L}$ is of the form $(\varepsilon_A, \delta_A)$ for some $A \in \mathcal{L}$, where

$$\delta_A(X) = X \oplus A = \bigvee_{a \in \ell(A)} X_a,$$
$$\varepsilon_A(X) = X \ominus A = \bigwedge_{a \in \ell(A)} X_{-a}.$$

Note that for $a \in \ell$, $\delta_a = \tau_a$ and $\varepsilon_a = \tau_a^{-1}$. Moreover, the map $A \mapsto \delta_A$ is an isomorphism between $\mathcal{L}$ and the complete lattice of $\mathbf{T}$-dilations, while the map $A \mapsto \varepsilon_A$ is a dual isomorphism between $\mathcal{L}$ and the complete lattice of $\mathbf{T}$-erosions, that is

$$B \leq C \quad \Longleftrightarrow \quad \delta_B \leq \delta_C \quad \Longleftrightarrow \quad \varepsilon_B \geq \varepsilon_C$$

for any $B, C \in \mathcal{L}$. This implies in particular that for $A_j \in \mathcal{L}$ $(j \in J)$,

$$\delta_{\sup_{j \in J} A_j} = \bigvee_{j \in J} \delta_{A_j} \qquad \text{and} \qquad \varepsilon_{\inf_{j \in J} A_j} = \bigwedge_{j \in J} \varepsilon_{A_j}.$$

More properties of $\mathbf{T}$-adjunctions under the Basic Assumption are given in [6], Subsection 3.2. In particular we obtained *Matheron's theorem*, which states that an increasing

$\mathbf{T}$-operator is a supremum of $\mathbf{T}$-erosions. Then in Subsection 2.3 of [19] we proved that under the Basic Assumption:

(i) *Given a* $\mathbf{T}$*-opening* α *and a* $\mathbf{T}$*-dilation* δ, $\mathrm{Inv}(\alpha)$ *is invariant under* δ, *in other words* $\alpha\delta\alpha = \delta\alpha$.

(ii) *For any* $B \in \mathcal{L}$, $\mathbf{A}_B^{\mathbf{T}} = \delta_B \varepsilon_B$.

The Basic Assumption is obviously satisfied in the case of sets or grey-level functions (see [6], Section 4). This shows that the opening by a structuring element (a structural $\mathbf{T}$-opening) can be obtained as the composition of the erosion and dilation by that element.

But what about the closing by that structuring element? The Basic Assumption is not sufficient for this purpose. We will see two examples of a structural $\mathbf{T}$-closing which cannot be obtained from a $\mathbf{T}$-adjunction.

Let $\mathcal{L}$ be a complete lattice and φ a $\mathbf{T}$-closing for which there is some $X \in \mathrm{Inv}(\varphi)$ with $X \neq I$, such that for every $A \in \mathcal{L}$, $\delta_A \geq \varphi$ implies $\delta_A(X) = I$. Then φ is not an infimum (either empty or non-empty) of $\mathbf{T}$-dilations, because this would give $\varphi(X) = I$. Thus the dual version of Matheron's theorem does not hold. It follows moreover that φ is not an infimum of $\mathbf{T}$-closings of the form $\varepsilon_A\delta_A$, because each $\varepsilon_A\delta_A$ is an infimum of $\mathbf{T}$-dilations:

$$\varepsilon_A\delta_A = \bigwedge_{a \in \ell(A)} \tau_{-a}\delta_A = \bigwedge_{a \in \ell(A)} \delta_{A_{-a}}.$$

In fact this situation arises in the following two cases:

(a) $\mathcal{L}$ is the complete lattice of all topologically closed subsets of $\mathbb{R}^d$, and $\varphi = \mathbf{F}_B^{\mathbf{T}}$, where B is a closed set whose complement is bounded but non-empty.

(b) $\mathcal{L}$ is the complete lattice of all convex subsets of $\mathbb{R}^d$, and $\varphi = \mathbf{F}_B^{\mathbf{T}}$, where B is a convex set such that there exists a bounded convex set C with $B \ominus C = \emptyset$ (B contains no translate of C); for example B can be a segment or a half-line (it contains no translate of a disk).

Thus structural $\mathbf{T}$-closings do generally not coincide with closings arising from $\mathbf{T}$-adjunctions on $\mathcal{L}$. For this we need the dual of the Basic Assumption. In practice, it will be easier to obtain it from an operation similar to the complementation for sets, or the grey-level inversion for grey-level function, which turns the complete lattice upside down.

Let us call an *inversion* an operator θ on $\mathcal{L}$ such that: $\theta^2 = \mathbf{id}$ and θ is decreasing, that is $X \leq Y \implies \theta(X) \geq \theta(Y)$ for all $X, Y \in \mathcal{L}$. As $\theta^2 = \mathbf{id}$, θ is a bijection and we have in fact $X \leq Y \iff \theta(X) \geq \theta(Y)$, in other words *an inversion is a dual automorphism* of $\mathcal{L}$.

An inversion θ transforms an operator ψ into $\theta\psi\theta$, the *dual by* θ *of* ψ. For example in the case of sets, the complementation $\theta : X \mapsto X^c$ gives $\theta\psi\theta : X \mapsto \psi(X^c)^c$, the *dual by complementation of* ψ. Now for an inversion θ on $\mathcal{L}$, the map $\psi \mapsto \theta\psi\theta$ is itself an inversion of the complete lattice $\mathcal{O}$ of operators, which preserves the law of composition, interverts openings and closings, dilations and erosions, and reverses adjunctions in the sense that for an adjunction (ε, δ) on $\mathcal{L}$, $(\theta\delta\theta, \theta\varepsilon\theta)$ is again an adjunction. Moreover if we put $\theta(\mathcal{B}) = \{\theta(B) \mid B \in B\}$ for $\mathcal{B} \subseteq \mathcal{L}$, we get for any $\psi \in \mathcal{O}$: $\mathrm{Inv}(\theta\psi\theta) = \theta(\mathrm{Inv}(\psi))$ and $\mathrm{Ran}(\theta\psi\theta) = \theta(\mathrm{Ran}(\psi))$.

If we set $\theta\mathbf{T}\theta = \{\theta\tau\theta \mid \tau \in \mathbf{T}\}$, then the fact that $\theta^2 = \mathbf{id}$ implies the equivalence between $\mathbf{T} = \theta\mathbf{T}\theta$, $\mathbf{T} \subseteq \theta\mathbf{T}\theta$, and $\mathbf{T} \supseteq \theta\mathbf{T}\theta$. We say then that θ *preserves* $\mathbf{T}$. When the inversion θ preserves $\mathbf{T}$, then an operator ψ is $\mathbf{T}$-invariant if and only if $\theta\psi\theta$ is $\mathbf{T}$-invariant; moreover for any $\mathcal{B} \subseteq \mathcal{L}$, we will have $\theta\mathbf{A}^{\mathbf{T}}(\mathcal{B})\theta = \mathbf{F}^{\mathbf{T}}(\theta(\mathcal{B}))$.

Let us see what happen when $\mathcal{L}$ satisfies the Basic Assumption and has an inversion θ which preserves $\mathbf{T}$. Given $B \in \mathcal{L}$, we have first $\theta\mathbf{A}^{\mathbf{T}}_B\theta = \mathbf{F}^{\mathbf{T}}_{\theta(B)}$. Second, as $(\varepsilon_B, \delta_B)$ is a $\mathbf{T}$-adjunction, $(\theta\delta_B\theta, \theta\varepsilon_B\theta)$ is also a $\mathbf{T}$-adjunction, and there is some $\widetilde{B} \in \mathcal{L}$ such that $\theta\delta_B\theta = \varepsilon_{\widetilde{B}}$ and $\theta\varepsilon_B\theta = \delta_{\widetilde{B}}$. Third, we have $\mathbf{A}^{\mathbf{T}}_B = \delta_B\varepsilon_B$. Combining these three facts, we get

$$\mathbf{F}^{\mathbf{T}}_{\theta(B)} = \theta\mathbf{A}^{\mathbf{T}}_B\theta = \theta\delta_B\varepsilon_B\theta = \varepsilon_{\widetilde{B}}\delta_{\widetilde{B}}. \tag{2.15}$$

Thus every structural $\mathbf{T}$-closing arises from a $\mathbf{T}$-adjunction on $\mathcal{L}$.

Let us examine in more detail the map $B \mapsto \widetilde{B}$. As $\theta^2 = \mathbf{id}$, we have $\widetilde{\widetilde{B}} = B$, and so this map is a bijection; now for $B, C \in \mathcal{L}$ we have

$$B \leq C \iff \delta_B \leq \delta_C \iff \theta\delta_B\theta \geq \theta\delta_C\theta \iff \varepsilon_{\widetilde{B}} \geq \varepsilon_{\widetilde{C}} \iff \widetilde{B} \leq \widetilde{C},$$

and it is thus an automorphism of the complete lattice $\mathcal{L}$. In particular $\widetilde{B} = \sup_{b\in\ell(B)} \widetilde{b}$. Moreover, for any $b \in \ell$ we have

$$\theta\tau_b\theta = \theta\delta_b\theta = \varepsilon_{\widetilde{b}} = \tau^{-1}_{\widetilde{b}}. \tag{2.16}$$

For example in the case of subsets of a digital or Euclidean space $\mathcal{E}$, the complementation is an inversion which commutes with any automorphism of $\mathcal{P}(\mathcal{E})$, in particular with translations. Here (2.16) gives $\tau_b = \tau^{-1}_{\widetilde{b}}$ for any point b, so that $\widetilde{b} = -b$ and we get thus $\widetilde{B} = \{-b \mid b \in B\}$. Note that since $B \mapsto \widetilde{B}$ is an automorphism and complementation commutes with automorphisms, we have $(\widetilde{B})^c = \widetilde{B^c}$ and we write it $\widetilde{B}^c$. Now (2.15) gives $\mathbf{F}^{\mathbf{T}}_B = \varepsilon_{\widetilde{B}^c}\delta_{\widetilde{B}^c}$. In the literature $\widetilde{B} = \{-b \mid b \in B\}$ is usually written $\check{B}$. See also [6], Subsection 4.1, and [19], Subsection 2.4 for more details on this particular case.

In the case of grey-level functions on $\mathcal{E}$, ℓ consists of all impulse functions $f_{h,v}$ having value v at point h and $-\infty$ elsewhere, while θ is given by $\theta(F)(x) = -F(x)$ for any grey-level function F (grey-level inversion). It is easily seen that $\theta\tau_{h,v}\theta = \tau_{h,-v}$ and so (2.16) gives $\widetilde{f}_{h,v} = f_{-h,v}$; hence $\widetilde{F}$ is given by $\widetilde{F}(x) = F(-x)$ (inversion in the spatial domain). Here we have also $\theta(\widetilde{F}) = \widetilde{\theta(F)}$. For more details on this complete lattice, see also [6], Subsection 4.4.

2.4. *The case without translation-invariance, and some generalizations*

We said at the beginning of this section that all our general results concerning $\mathbf{T}$-operators can be applied to the case where $\mathbf{T}$-invariance is not required, by simply taking $\mathbf{T} = \{\mathbf{id}\}$; indeed, any operator is invariant under $\mathbf{id}$. This is essentially the point of view adopted by Serra and his followers in [22] and other works.

For $\mathbf{T} = \{\mathbf{id}\}$ the operators $\mathbf{A}^{\mathbf{T}}(\mathcal{B})$, $\mathbf{F}^{\mathbf{T}}(\mathcal{B})$, $\mathbf{A}^{\mathbf{T}}_B$, and $\mathbf{F}^{\mathbf{T}}_B$ are written $\mathbf{A}(\mathcal{B})$, $\mathbf{F}(\mathcal{B})$, $\mathbf{A}_B$, and $\mathbf{F}_B$. It is easily seen from Proposition 2.8 that $\mathbf{A}^{\mathbf{T}}(\mathcal{B}) = \mathbf{A}(\mathcal{B}^{\mathbf{T}})$ and similarly $\mathbf{F}^{\mathbf{T}}(\mathcal{B}) = \mathbf{F}(\mathcal{B}^{\mathbf{T}})$. In particular if $\mathcal{B}$ is $\mathbf{T}$-invariant (which is for example the case when $\mathcal{B} = \mathrm{Inv}(\psi)$ or $\mathcal{B} = \mathrm{Ran}(\psi)$ for a $\mathbf{T}$-operator ψ), then $\mathbf{A}^{\mathbf{T}}(\mathcal{B}) = \mathbf{A}(\mathcal{B})$ and $\mathbf{F}^{\mathbf{T}}(\mathcal{B}) = \mathbf{F}(\mathcal{B})$.

There are some additional results which hold in this special case. We have already seen Proposition 1.6, which states that any contraction is an infimum of openings, and we gave a counterexample to this result when $\mathbf{T}$-invariance is needed by taking $\mathcal{L} = \overline{\mathbb{Z}} = \mathbb{Z} \cup \{+\infty, -\infty\}$ and $\mathbf{T} = \mathbb{Z}$; note that here $\mathcal{L}$ satisfies the Basic Assumption and has an inversion preserving $\mathbf{T}$. There is also *Matheron's theorem*, whose expression by Serra on complete lattices is as follows (see [6], Theorem 2.4, and [22], Theorem 1.2):

> *An operator ψ is a non-empty supremum of erosions if and only if ψ is increasing and $\psi(I) = I$. Dually, ψ is a non-empty infimum of dilations if and only if ψ is increasing and $\psi(O) = O$.*

The corresponding statements for $\mathbf{T}$-operators require respectively the Basic Assumption and its dual (cfr. the two counterexamples we mentioned with the family of closed sets and the one of convex sets).

There is a third property peculiar to the case $\mathbf{T} = \{\mathbf{id}\}$ (see [19], Proposition 2.9). Given an adjunction (ε, δ) *on* $\mathcal{L}$, we call the opening $\delta\varepsilon$ a *morphological opening*, and the closing $\varepsilon\delta$ a *morphological closing*. Then:

> *A structural opening $\mathbf{A}_B$ is a morphological opening; every opening is a supremum of morphological openings. Dually, a structural closing $\mathbf{F}_B$ is a morphological closing; every closing is an infimum of morphological closings.*

Again the corresponding statements for $\mathbf{T}$-operators require respectively the Basic Assumption and its dual (cfr. the same two counterexamples). This contradicts the suggestion made in [24], Theorem 2.4 p. 24, that they are generally valid under translation-invariance.

This distinction explains our choice of *structural* $\mathbf{T}$-openings (resp. $\mathbf{T}$-closings), rather than *morphological* ones, as the basic blocks for the decomposition of $\mathbf{T}$-openings (resp. $\mathbf{T}$-closings). Note however that if we do not restrict adjunctions to $\mathcal{L}$, but take them between two complete lattices, then every $\mathbf{T}$-opening becomes trivially “morphological”:

Proposition 2.11. *Given a $\mathbf{T}$-opening α, there is a $\mathbf{T}$-adjunction (ε, δ) between $\mathcal{L}$ and $\mathrm{Inv}(\alpha)$ such that $\alpha = \delta\varepsilon$.*

(This result is proved in Proposition 3.10 of Chapter 0 of [3] in the case without $\mathbf{T}$-invariance). Indeed, $\mathrm{Inv}(\alpha)$ is a $\mathbf{T}$-invariant complete lattice, and we have only to take $\delta : \mathrm{Inv}(\alpha) \to \mathcal{L} : X \mapsto X$ and $\varepsilon : \mathcal{L} \to \mathrm{Inv}(\alpha) : Y \mapsto \alpha(Y)$; both δ and ε are $\mathbf{T}$-invariant and for $X \in \mathrm{Inv}(\alpha)$ and $Y \in \mathcal{L}$ we have $\delta(X) = X \leq Y \iff X \leq \alpha(Y) = \varepsilon(Y)$.

3. Inf-overfilters

Despite their forbidding name, *inf-overfilters* are very useful in practice, because they allow the construction of new types of openings. Indeed, although any opening can be decomposed as a supremum of structural openings, this is not always the most economical way to define a new opening. For example in Subsection 1.2 we defined from a symmetric structuring element B in a space $\mathcal{E}$ with translation group $\mathbf{T}$, the *annular* $\mathbf{T}$-opening $\mathbf{id} \wedge \delta_B$; its minimal decomposition in terms of structural $\mathbf{T}$-openings would be $\bigvee_{b \in B} \mathbf{A}^{\mathbf{T}}_{\{o,b\}}$, which is clearly more complicated than $\mathbf{id} \wedge \delta_B$. Here we will again consider openings of the form $\mathbf{id} \wedge \eta$ for an increasing operator η; now η will not be a dilation, but an inf-overfilter: the terminology stems from Matheron (see [22], Chapter 6). It will generally have a decomposition as an infimum of suprema of terms of the form $\delta\varepsilon$, where δ is a $\mathbf{T}$-dilation, ε a $\mathbf{T}$-erosion, and

$\delta \geq \underset{\cdot}{\varepsilon}$ (or equivalently $\varepsilon \geq \dot{\delta}$). Again, this decomposition will generally be easier than the one as a supremum of structural $\mathbf{T}$-openings.

In Section 6.3 of [22] Matheron made the following definition: An *inf-overfilter* is an increasing operator η such that $\eta(\mathbf{id} \wedge \eta) = \eta$. Dually, a *sup-underfilter* is an increasing operator ζ such that $\zeta(\mathbf{id} \vee \zeta) = \zeta$. By duality, we can restrict our analysis to inf-overfilters and openings, the corresponding results for sup-underfilters and closings following immediately. Note that for an increasing operator η we always have $\eta(\mathbf{id} \wedge \eta) \leq \eta\, \mathbf{id} = \eta$; hence η will be an inf-overfilter if $\eta(\mathbf{id} \wedge \eta) \geq \eta$. We call a $\mathbf{T}$-*inf-overfilter* a $\mathbf{T}$-invariant inf-overfilter. When $\mathbf{T}$-invariance is not necessary, one can set $\mathbf{T} = \{\mathbf{id}\}$ and drop the prefix "$\mathbf{T}$-".

The following elementary result (see [19], Proposition 4.1) highlights the meaning of the concept of an inf-overfilter:

Proposition 3.1. *Given an inf-overfilter η, then $\eta \leq \eta^2$ and $\mathbf{id} \wedge \eta$ is an opening.*

An operator ψ such that $\psi^2 \geq \psi$ is called by Matheron an *overfilter*, and this explains the origin of the term "inf-overfilter". Any opening is an inf-overfilter as it corresponds to the particular case where $\eta = \mathbf{id} \wedge \eta$. An inf-overfilter can be interpreted as an increasing operator η applying to $X \in \mathcal{L}$ an opening $\mathbf{id} \wedge \eta$, but adding to it something more (the difference between $\eta(X)$ and $(\mathbf{id} \wedge \eta)(X)$), which does not depend on X, but only on the result $(\mathbf{id} \wedge \eta)(X)$ of that opening.

Inf-overfilters were studied from a purely algebraic point of view by Matheron in Chapter 6 of [22], especially in Sections 6.3 and 6.4. Independently of this work, we introduced the so-called "rank-max" openings on sets or grey-level functions (see [15]) as a generalization of the opening by a structuring element A. Given a rank filter ρ_A^k associated to a rank k and a structuring element A, and the dilation δ_A by A (in other words the max filter associated to the reflected structuring element $\check{A}$), the operator $\mathbf{id} \wedge \delta_A \rho_A^k$ is an opening; more generally, given a family C^j $(j \in J)$ of subsets of A, the operator

$$\alpha = \mathbf{id} \wedge \delta_A\Big(\bigvee_{j \in J} \varepsilon_{C^j}\Big) \tag{3.1}$$

mapping a set X onto

$$\alpha(X) = X \cap \Big(\big(\bigcup_{j \in J} (X \ominus C^j)\big) \oplus A\Big), \tag{3.2}$$

is an opening. As explained in Subsection 4.2 of [19], this opening can be interpreted as follows: it transforms a binary image X into the supremum of all portions of it which consist of a "sufficiently large" subset of a translate of A; the subsets C^j of A are precisely the minimal ones which can be considered as "sufficiently large". When the family of C^j reduces to A, the opening α reduces to the usual $\mathbf{T}$-opening $\delta_A \varepsilon_A$ by A.

Serra pointed out that the operator $\delta_A(\bigvee_{j \in J} \varepsilon_{C^j}) = \bigvee_{j \in J} \delta_A \varepsilon_{C^j}$ is an inf-overfilter, linking thus this abstract concept to practical considerations. This remark was expanded in Section 9.9 of [22], where a characterization of inf-overfilters was given. Further results were obtained in Section 4 of [19], with the assumption of $\mathbf{T}$-invariance.

The purpose of this section is to deepen this study. In Subsection 3.1 we recall some of elementary results from [22] and [19]. Subsection 3.2 studies the complete lattice of inf-overfilters associated to any given opening. Finally in Subsection 3.3 we give decomposition formulas for such inf-overfilters, generalizing some similar characterizations from [19].

But beforehand we will briefly illustrate the interest of this family of operators. Given $\mathcal{L} = \mathcal{P}(\mathcal{E})$, the set of parts of the digital plane $\mathcal{E} = \mathbb{Z}^2$, let B be the 5-pixel cross (as in Figure 1.2), and let A be the (3×3)-square, both centered about the origin. Then $\delta_A \varepsilon_B$ is a $\mathbf{T}$-inf-overfilter and $\mathbf{id} \wedge \delta_A \varepsilon_B$ is a $\mathbf{T}$-opening which can be decomposed as the supremum of the structural $\mathbf{T}$-openings by B, C, D, E, F, where C, D, E, F are the four subsets obtained by adding to B one of the four corners of A (see Figure 3.1). This opening will preserve in a set X all 5-pixel crosses and all pixels 4-adjacent to two pixels of such a cross. Now if in the segmentation of Figures 1.2 and 1.3 we had taken $\mathbf{id} \wedge \delta_A \varepsilon_B$ instead of α_B, the components labelled 3, 4, and 5 would have been merged with the neighbouring components labelled 1 and 2 respectively, a better result.

$$\begin{array}{ccc} A & A & A \\ A & A & A \\ A & A & A \end{array} \qquad \begin{array}{ccc} & B & \\ B & B & B \\ & B & \end{array} \qquad \begin{array}{ccc} C & C & \\ C & C & C \\ & C & \end{array} \qquad \begin{array}{ccc} & D & D \\ D & D & D \\ & D & \end{array} \qquad \begin{array}{ccc} & E & \\ E & E & E \\ & E & E \end{array} \qquad \begin{array}{ccc} & F & \\ F & F & F \\ F & F & \end{array}$$

FIGURE 3.1. *The domain of invariance of the* $\mathbf{T}$*-opening* $\mathbf{id} \wedge \delta_A \varepsilon_B$ *is sup-generated by the translates of* B, C, D, E, F.

3.1. Basic properties of inf-overfilters

The results stated below are proved in Subsection 4.1 of [19], and generalize some earlier findings of Matheron and Serra (without $\mathbf{T}$-invariance).

The converse of Proposition 3.1 is not true: if ψ is increasing and $\mathbf{id} \wedge \psi$ is an opening, then ψ is not necessarily an inf-overfilter, even when $\psi^2 \geq \psi$. Consider for example $\mathcal{L} = \mathcal{P}(\mathbb{Z}^2)$, the complete lattice of parts of the digital plane $\mathbb{Z}^2$, take B to be the set of 8 neighbours of the origin o, and let $\psi = \delta_B$. As B is symmetric ($B = \check{B}$), $\mathbf{id} \wedge \delta_B$ is an annular opening; as $B \subseteq B \oplus B$, we get $\delta_B^2 = \delta_{B \oplus B} \geq \delta_B$; however we have $\delta_B(\{o\}) = B$, so that $(\mathbf{id} \wedge \delta_B)(\{o\}) = \{o\} \cap B = \emptyset$, and hence $\delta_B(\mathbf{id} \wedge \delta_B)(\{o\}) = \delta_B(\emptyset) = \emptyset$, that is $\delta_B(\{o\}) \neq \delta_B(\mathbf{id} \wedge \delta_B)(\{o\})$.

Nevertheless, an opening of the form $\mathbf{id} \wedge \psi$, where ψ is increasing, arises in fact from an inf-overfilter:

PROPOSITION 3.2. *Let* ψ *be an increasing operator such that* $\mathbf{id} \wedge \psi$ *is an opening. Let* $\eta = \psi(\mathbf{id} \wedge \psi)$*. Then* η *is an inf-overfilter and* $\mathbf{id} \wedge \eta = \mathbf{id} \wedge \psi$.

The following result is essentially due to Matheron, and it should be compared to Theorem 2.10 (*ii*):

PROPOSITION 3.3. *The set of* $\mathbf{T}$*-inf-overfilters is sup-closed, with universal bounds* $\mathbf{O}$ *and* $\mathbf{I}$.

Our next result is a generalization of properties found by Serra and Matheron:

PROPOSITION 3.4. *Let* η *be a* $\mathbf{T}$*-inf-overfilter,* α *a* $\mathbf{T}$*-opening,* (ε, δ) *a* $\mathbf{T}$*-adjunction, and* ψ *an increasing* $\mathbf{T}$*-operator. Then the following operators are* $\mathbf{T}$*-inf-overfilters:*

(*i*) $\psi\eta$, *if* $\psi \geq \mathbf{id} \wedge \eta$.

(*ii*) η^2.

(*iii*) $\psi\alpha$, *if* $\psi \geq \alpha$.

(*iv*) $\psi\varepsilon$, *if* $\psi \geq \delta$.

An example of the form (*iv*) was given in Figure 3.1 with $\eta = \delta_A\varepsilon_B$, where $A \supseteq B$. Let us now illustrate (*i*) with that same $\eta = \delta_A\varepsilon_B$ and with $\psi = \varepsilon_B\delta_B$ (ψ is a closing, so that $\psi \geq \mathbf{id} \geq \mathbf{id} \wedge \eta$). We show in Figure 3.2 how the two openings $\mathbf{id} \wedge \eta$ and $\mathbf{id} \wedge \psi\eta$ behave differently.

$$\begin{array}{ccc} A & A & A \\ A & A & A \\ A & A & A \end{array} \qquad \begin{array}{ccc} & B & \\ B & B & B \\ & B & \end{array} \qquad \qquad \begin{array}{ccccccc} * & * & \cdot & * & \cdot & * & \cdot \\ * & * & * & * & * & * & * \\ \cdot & * & \cdot & \cdot & \cdot & * & \cdot \end{array}$$

$$\begin{array}{ccccccc} * & * & * & \cdot & * & * & * \\ * & * & * & \cdot & * & * & * \\ * & * & * & \cdot & * & * & * \end{array} \qquad \qquad \begin{array}{ccccccc} * & * & * & \cdot & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & \cdot & * & * & * \end{array}$$

$$\begin{array}{ccccccc} * & * & \cdot & \cdot & \cdot & * & \cdot \\ * & * & * & \cdot & * & * & * \\ \cdot & * & \cdot & \cdot & \cdot & * & \cdot \end{array} \qquad \qquad \begin{array}{ccccccc} * & * & \cdot & \cdot & \cdot & * & \cdot \\ * & * & * & * & * & * & * \\ \cdot & * & \cdot & \cdot & \cdot & * & \cdot \end{array}$$

FIGURE 3.2. *On top we have the structuring elements A and B (both centered about the origin), and the original set X. In the middle are shown $\delta_A\varepsilon_B(X)$ and $\varepsilon_B\delta_B\delta_A\varepsilon_B(X)$. At bottom we get $(\mathbf{id} \wedge \delta_A\varepsilon_B)(X)$ and $(\mathbf{id} \wedge \varepsilon_B\delta_B\delta_A\varepsilon_B)(X)$, two distinct results.*

Note that any constant operator $\gamma_A : X \mapsto A$ is an inf-overfilter. It is $\mathbf{T}$-invariant if A is fixed by $\mathbf{T}$.

Given a $\mathbf{T}$-inf-overfilter η, then $\mathbf{id} \wedge \eta$ is the greatest $\mathbf{T}$-opening $\leq \eta$. From Proposition 3.4 (*iii*) we deduce the following characterization due to Matheron:

COROLLARY 3.5. *Given an increasing $\mathbf{T}$-operator η, the following three statements are equivalent:*

(*i*) *η is a $\mathbf{T}$-inf-overfilter.*

(*ii*) *If α_η is the greatest $\mathbf{T}$-opening $\leq \eta$, then $\eta\alpha_\eta = \eta$.*

(*iii*) *There is a $\mathbf{T}$-opening α and an increasing $\mathbf{T}$-operator θ such that $\theta \geq \alpha$ and $\eta = \theta\alpha$.*

3.2. *The complete lattice of $\mathbf{T}$-inf-overfilters associated to a $\mathbf{T}$-opening*

From Corollary 3.5 we know that $\mathbf{T}$-inf-overfilters can be characterized as operators of the form $\theta\alpha$ for a $\mathbf{T}$-opening α and an increasing $\mathbf{T}$-operator $\geq \alpha$. We make thus the following definition. Given a $\mathbf{T}$-opening α, write $\mathcal{H}_\mathbf{T}(\alpha)$ for the set of all $\mathbf{T}$-inf-overfilters $\theta\alpha$, where θ is an increasing $\mathbf{T}$-operator $\geq \alpha$, and $\mathcal{A}_\mathbf{T}(\alpha)$ for the set of all $\mathbf{T}$-openings of the form $\mathbf{id} \wedge \eta$, where $\eta \in \mathcal{H}_\mathbf{T}(\alpha)$. For any $\mathbf{T}$-inf-overfilter η we have $\eta \in \mathcal{H}_\mathbf{T}(\mathbf{id} \wedge \eta)$, and for any $\mathbf{T}$-opening α we have $\alpha \in \mathcal{H}_\mathbf{T}(\alpha)$ and $\alpha \in \mathcal{A}_\mathbf{T}(\alpha)$. Let now α be a fixed $\mathbf{T}$-opening.

LEMMA 3.6. *Given a* $\mathbf{T}$*-operator* η, $\eta \in \mathcal{H}_{\mathbf{T}}(\alpha)$ *if and only if* η *is increasing,* $\eta \geq \alpha$, *and* $\eta\alpha = \eta$.

PROOF. If $\eta \in \mathcal{H}_{\mathbf{T}}(\alpha)$, that is $\eta = \theta\alpha$ for an increasing $\theta \geq \alpha$, then η is increasing, $\eta = \theta\alpha \geq \alpha\alpha = \alpha$, and $\eta\alpha = \theta\alpha\alpha = \theta\alpha = \eta$. If η is increasing, $\eta \geq \alpha$, and $\eta\alpha = \eta$, then we take $\theta = \eta$. ∎

COROLLARY 3.7. α *is the unique* $\mathbf{T}$*-opening in* $\mathcal{H}_{\mathbf{T}}(\alpha)$.

PROOF. For a $\mathbf{T}$-opening $\alpha' \in \mathcal{H}_{\mathbf{T}}(\alpha)$, by Lemma 3.6 we have $\alpha' \geq \alpha$ and $\alpha'\alpha = \alpha'$; now the latter equality implies $\alpha' \leq \alpha$ (by Proposition 2.9), so that $\alpha' = \alpha$. ∎

THEOREM 3.8. $\mathcal{H}_{\mathbf{T}}(\alpha)$ *is inf-closed and closed under non-empty suprema.*

PROOF. Consider a non-empty family of elements η_j of $\mathcal{H}_{\mathbf{T}}(\alpha)$ ($j \in J \neq \emptyset$). As $\eta_j \geq \alpha$ and $\eta_j\alpha = \alpha$ for each $j \in J$ (by Lemma 3.6), we get $\bigwedge_{j\in J} \eta_j \geq \alpha$ and $(\bigwedge_{j\in J} \eta_j)\alpha = \bigwedge_{j\in J}(\eta_j\alpha) = \bigwedge_{j\in J} \eta_j$; now $\bigwedge_{j\in J} \eta_j$ is $\mathbf{T}$-invariant, hence it belongs to $\mathcal{H}_{\mathbf{T}}(\alpha)$. Similarly $\bigvee_{j\in J} \eta_j \in \mathcal{H}_{\mathbf{T}}(\alpha)$. Thus $\mathcal{H}_{\mathbf{T}}(\alpha)$ is closed under non-empty suprema and infima. Now $\mathbf{I} = \bigwedge \emptyset \in \mathcal{H}_{\mathbf{T}}(\alpha)$, and so $\mathcal{H}_{\mathbf{T}}(\alpha)$ is inf-closed. ∎

It follows that $\mathcal{H}_{\mathbf{T}}(\alpha) \cup \{\mathbf{O}\}$ is a complete sublattice of $\mathcal{O}$. Note that $\mathcal{H}_{\mathbf{T}}(\alpha)$ is itself a complete lattice, with the same supremum and infimum operations as in $\mathcal{O}$, except that $\sup \emptyset = \alpha$ instead of $\mathbf{O}$ (see (2.1)).

PROPOSITION 3.9. *Given an increasing* $\mathbf{T}$*-operator* $\theta \geq \alpha$ *and* $\eta \in \mathcal{H}_{\mathbf{T}}(\alpha)$, *we have* $\theta\eta \in \mathcal{H}_{\mathbf{T}}(\alpha)$. *In particular* $\mathcal{H}_{\mathbf{T}}(\alpha)$ *is closed under composition.*

PROOF. By Lemma 3.6 we have $\eta\alpha = \eta$ and $\eta \geq \alpha$, so that $\theta\eta\alpha = \theta\eta$ and $\theta\eta \geq \alpha\alpha = \alpha$, that is $\theta\eta \in \mathcal{H}_{\mathbf{T}}(\alpha)$. For any $\eta' \in \mathcal{H}_{\mathbf{T}}(\alpha)$, $\eta' \geq \alpha$, so we can take $\theta = \eta'$, and $\eta'\eta \in \mathcal{H}_{\mathbf{T}}(\alpha)$. ∎

LEMMA 3.10. *Let* α' *be a* $\mathbf{T}$*-opening such that* $\alpha' \geq \alpha$. *Then* $\eta\alpha' = \eta$ *for every* $\eta \in \mathcal{H}_{\mathbf{T}}(\alpha)$; *in particular when* $\eta \geq \alpha'$ *we have* $\eta \in \mathcal{H}_{\mathbf{T}}(\alpha')$.

PROOF. As $\mathbf{id} \geq \alpha' \geq \alpha$ and $\eta\alpha = \eta$, we get $\eta \geq \eta\alpha' \geq \eta\alpha = \eta$, that is $\eta\alpha' = \eta$. If $\eta \geq \alpha'$, then $\eta \in \mathcal{H}_{\mathbf{T}}(\alpha')$ by Lemma 3.6. ∎

COROLLARY 3.11. *For every* $\eta, \eta' \in \mathcal{H}_{\mathbf{T}}(\alpha)$, $\eta(\mathbf{id} \wedge \eta') = \eta$ *and* $(\mathbf{id} \wedge \eta)(\mathbf{id} \wedge \eta') = \mathbf{id} \wedge \eta \wedge \eta'$.

PROOF. As $\mathbf{id} \wedge \eta'$ is a $\mathbf{T}$-opening $\geq \alpha$, $\eta(\mathbf{id} \wedge \eta') = \eta$ by Lemma 3.10. Thus

$$(\mathbf{id} \wedge \eta)(\mathbf{id} \wedge \eta') = \mathbf{id}(\mathbf{id} \wedge \eta') \wedge \eta(\mathbf{id} \wedge \eta') = (\mathbf{id} \wedge \eta') \wedge \eta = \mathbf{id} \wedge \eta \wedge \eta'. \blacksquare$$

PROPOSITION 3.12. $\mathcal{A}_{\mathbf{T}}(\alpha)$ *is closed under non-empty infima, and for any* $\alpha_1, \ldots, \alpha_n \in \mathcal{A}_{\mathbf{T}}(\alpha)$ $(n \geq 2)$,

$$\alpha_1 \cdots \alpha_n = \alpha_1 \wedge \ldots \wedge \alpha_n.$$

Moreover, if $\mathcal{L}$ *satisfies the infinite supremum distributy condition*

$$X \wedge (\bigvee_{j\in J} Y_j) = \bigvee_{j\in J} (X \wedge Y_j), \tag{ISD}$$

then $\mathcal{A}_{\mathbf{T}}(\alpha)$ *is closed under non-empty suprema.*

Proof. For $\alpha_1 = \mathbf{id} \wedge \eta_1$ and $\alpha_2 = \mathbf{id} \wedge \eta_2$, where $\eta_1, \eta_2 \in \mathcal{H}_{\mathbf{T}}(\alpha)$, Corollary 3.11 says that

$$\alpha_1\alpha_2 = (\mathbf{id} \wedge \eta_1)(\mathbf{id} \wedge \eta_2) = \mathbf{id} \wedge \eta_1 \wedge \eta_2 = \alpha_1 \wedge \alpha_2.$$

For $\alpha_1, \ldots, \alpha_n \in \mathcal{A}_{\mathbf{T}}(\alpha)$, where $n > 2$, the equality $\alpha_1 \cdots \alpha_n = \alpha_1 \wedge \ldots \wedge \alpha_n$ follows by induction:

$$\begin{aligned} &\alpha_1 \cdots \alpha_n = (\alpha_1 \cdots \alpha_{n-1})\alpha_n = (\alpha_1 \wedge \ldots \wedge \alpha_{n-1})\alpha_n \\ &= \alpha_1\alpha_n \wedge \ldots \wedge \alpha_{n-1}\alpha_n = (\alpha_1 \wedge \alpha_n) \wedge \ldots \wedge (\alpha_{n-1} \wedge \alpha_n) = \alpha_1 \wedge \ldots \wedge \alpha_n. \end{aligned}$$

Given a non-empty family of elements η_j of $\mathcal{H}_{\mathbf{T}}(\alpha)$ ($j \in J \neq \emptyset$), by Theorem 3.8 we have $\bigwedge_{j\in J} \eta_j \in \mathcal{H}_{\mathbf{T}}(\alpha)$. Hence $\bigwedge_{j\in J}(\mathbf{id}\wedge\eta_j) = \mathbf{id}\wedge\bigwedge_{j\in J} \eta_j \in \mathcal{A}_{\mathbf{T}}(\alpha)$. Thus $\mathcal{A}_{\mathbf{T}}(\alpha)$ is closed under non-empty infima.

By Theorem 3.8 again, $\bigvee_{j\in J} \eta_j \in \mathcal{H}_{\mathbf{T}}(\alpha)$; now if $\mathcal{L}$ satisfies the condition (ISD), then $\mathcal{O}$ satisfies it also and we get $\bigvee_{j\in J}(\mathbf{id}\wedge\eta_j) = \mathbf{id}\wedge(\bigvee_{j\in J} \eta_j) \in \mathcal{A}_{\mathbf{T}}(\alpha)$. Thus $\mathcal{A}_{\mathbf{T}}(\alpha)$ is closed under non-empty suprema. ∎

When (ISD) holds, $\mathcal{A}_{\mathbf{T}}(\alpha) \cup \{\mathbf{O}, \mathbf{I}\}$ is a complete sublattice of $\mathcal{O}$; in fact $\mathcal{A}_{\mathbf{T}}(\alpha)$ is then itself a complete lattice, with the same supremum and infimum operations as in $\mathcal{O}$, except that $\sup\emptyset = \alpha$ instead of $\mathbf{O}$ and $\inf\emptyset = \mathbf{id}$ instead of $\mathbf{I}$ (see (2.1)). Note that (ISD) is satisfied for sets or grey-level functions.

We saw in Subsection 1.3 that an infimum or a composition of openings is usually not an opening. Hence Proposition 3.12 is very interesting, since it gives a class of openings which can be algebraically combined in various ways. This will be illustrated in the next subsection, where we will express elements of $\mathcal{H}_{\mathbf{T}}(\alpha)$ in terms of dilations and erosions intervening in the decomposition of α.

3.3. Decomposition formulas

Inspired by our "rank-max" openings, Serra gave in Section 9.9 of [22] a characterization of inf-overfilters on a complete lattice satisfying the infinite distributivity conditions ((ISD) and its dual). This characterization was generalized in [19], Subsection 4.1 to any complete lattice, and extended to the case of **T**-invariance. Here we will express this result in the framework of $\mathcal{H}_{\mathbf{T}}(\alpha)$. Indeed, recall that any **T**-inf-overfilter η is in $\mathcal{H}_{\mathbf{T}}(\alpha)$ for some **T**-opening α.

If we look back at Proposition 3.4, point (*iii*) expresses a **T**-inf-overfilter in $\mathcal{H}_{\mathbf{T}}(\alpha)$, while point (*iv*) considers a **T**-inf-overfilter of the form $\psi\varepsilon$ for a **T**-erosion ε and ψ an increasing **T**-operator such that $\psi \geq \varepsilon$. In fact this point (*iv*) was derived from point (*iii*) (see [19], Proposition 4.4), and we prove similarly the following:

Proposition 3.13. *For any* **T***-adjunction* (ε, δ)*, the lattice* $\mathcal{H}_{\mathbf{T}}(\delta\varepsilon)$ *is the set of all* $\psi\varepsilon$*, where* ψ *is an increasing* **T***-operator* $\geq \delta$*.*

Proof. If $\eta \in \mathcal{H}_{\mathbf{T}}(\delta\varepsilon)$, then $\eta = \eta\delta\varepsilon$ and $\eta \geq \delta\varepsilon$. Setting $\psi = \eta\delta$, ψ is an increasing **T**-operator, $\eta = \eta\delta\varepsilon = \psi\varepsilon$, and $\psi = \eta\delta \geq \delta\varepsilon\delta = \delta$. Conversely, if $\eta = \psi\varepsilon$ for an increasing **T**-operator $\psi \geq \delta$, then $\eta = \psi\varepsilon \geq \delta\varepsilon$ and $\eta\delta\varepsilon = \psi\varepsilon\delta\varepsilon = \psi\varepsilon = \eta$, that is $\eta \in \mathcal{H}_{\mathbf{T}}(\delta\varepsilon)$. ∎

For example consider again the digital plane $\mathbb{Z}^2$ and the 5-pixel cross B shown in Figure 3.1. If H and V are respectively the horizontal and vertical (3×5)-rectangles, centered about the origin, then $(\delta_H \wedge \delta_V)\varepsilon_B \in \mathcal{H}_{\mathbf{T}}(\delta_B\varepsilon_B)$. The opening $\mathbf{id} \wedge (\delta_H \wedge \delta_V)\varepsilon_B$ will preserve in a digital set any pixel p which lies in a (3×3)-square containing a translate of B (cfr. the sets C, D, E, F in Figure 3.1), or such that there exist two (3×3)-squares, each containing a translate of B, which are neighbouring p in the horizontal and vertical directions respectively. See Figure 3.3.

$$
\begin{array}{ccccc}
 & V & V & V & \\
H & {}^H{}_V & {}^H{}_V & {}^H{}_V & H \\
H & {}^H{}_V & {}^H{}_V & {}^H{}_V & H \\
H & {}^H{}_V & {}^H{}_V & {}^H{}_V & H \\
 & V & V & V &
\end{array}
\qquad
\begin{array}{ccccccc}
 & & & & & & \\
\cdot & * & \cdot & & \cdot & * & * \\
* & * & * & x & * & * & * \\
* & * & \cdot & & \cdot & * & \cdot \\
 & & & & & &
\end{array}
\qquad
\begin{array}{cccccc}
 & & & \cdot & * & \cdot \\
 & & y & * & * & * \\
\cdot & * & \cdot & \cdot & * & * \\
* & * & * & & & \\
* & * & * & & &
\end{array}
$$

FIGURE 3.3. *The opening* $\mathbf{id} \wedge (\delta_H \wedge \delta_V)\varepsilon_B$ *removes pixel* x*, but preserves pixel* y*, as well as pixels marked* $*$.

Now we can start the characterization of $\mathcal{H}_{\mathbf{T}}(\alpha)$ for a $\mathbf{T}$-opening decomposable as a supremum of morphological $\mathbf{T}$-openings:

LEMMA 3.14. *Given* $\mathbf{T}$*-openings* α_j *and* $\eta_j \in \mathcal{H}_{\mathbf{T}}(\alpha_j)$ $(j \in J)$*, we have* $\bigvee_{j\in J} \eta_j \in \mathcal{H}_{\mathbf{T}}(\bigvee_{j\in J} \alpha_j)$.

PROOF. For $J = \emptyset$, this reduces to $\mathbf{O} \in \mathcal{H}_{\mathbf{T}}(\mathbf{O})$. Assume thus that $J \neq \emptyset$. Clearly $\eta = \bigvee_{j\in J} \eta_j$ is increasing and $\mathbf{T}$-invariant. Now $\alpha = \bigvee_{j\in J} \alpha_j$ is a $\mathbf{T}$-opening, and for each $j \in J$ we have $\eta \geq \eta_j = \eta_j\alpha_j$ and $\mathbf{id} \geq \alpha \geq \alpha_j$; hence $\eta \geq \eta\alpha \geq \eta_j\alpha_j = \eta_j$, so that $\eta \geq \eta\alpha \geq \bigvee_{j\in J} \eta_j = \eta$, that is $\eta\alpha = \eta$. As $\eta_j \geq \alpha_j$ for each $j \in J$, we get $\eta \geq \alpha$. ∎

PROPOSITION 3.15. *Given two non-empty index sets* J, K*, let* $(\varepsilon_j, \delta_j)$ *be a* $\mathbf{T}$*-adjunction for* $j \in J$*, let* ψ_{kj} *be an increasing* $\mathbf{T}$*-operator for* $j \in J$ *and* $k \in K$*, and assume that* $\psi_{kj} \geq \delta_j$ *for every* j, k*. Then the operator*

$$\eta = \bigwedge_{k\in K} \bigvee_{j\in J} \psi_{kj}\varepsilon_j \tag{3.3}$$

belongs to $\mathcal{H}_{\mathbf{T}}(\bigvee_{j\in J} \delta_j\varepsilon_j)$.

PROOF. By Proposition 3.13 we have $\psi_{kj}\varepsilon_j \in \mathcal{H}_{\mathbf{T}}(\delta_j\varepsilon_j)$ for every $j \in J$, $k \in K$. Lemma 3.14 implies that $\eta_k = \bigvee_{j\in J} \psi_{kj}\varepsilon_j \in \mathcal{H}_{\mathbf{T}}(\bigvee_{j\in J} \delta_j\varepsilon_j)$ for every $k \in K$. As $\mathcal{H}_{\mathbf{T}}(\bigvee_{j\in J} \delta_j\varepsilon_j)$ is closed under non-empty infima (by Theorem 3.8), $\eta = \bigwedge_{k\in K} \eta_k \in \mathcal{H}_{\mathbf{T}}(\bigvee_{j\in J} \delta_j\varepsilon_j)$. ∎

The invariants of the opening $\mathbf{id} \wedge \eta$ for η as in (3.3) were characterized in Theorem 4.6 and Corollary 4.7 of [19]. We recall this result here:

PROPOSITION 3.16. *Let* η *be given by* (3.3)*. Then the domain of invariance of* $\mathbf{id} \wedge \eta$ *consists of all* $B \in \mathcal{L}$ *such that for every* $j \in J$ *there is some* $C_j \in \mathcal{L}$ *with*

$$\bigvee_{j\in J} \delta_j(C_j) \leq B \leq \bigwedge_{k\in K} \bigvee_{j\in J} \psi_{kj}(C_j). \tag{3.4}$$

Moreover, if $\mathcal{L}$ satisfies (ISD), *then* (3.4) *holds if and only if for every $k \in K$ and $j \in J$ there are some $B_{kj}, C_j \in \mathcal{L}$ with*

$$\delta_j(C_j) \leq B_{kj} \leq \psi_{kj}(C_j) \qquad \text{and} \qquad B = \bigwedge_{k\in K} \bigvee_{j\in J} B_{kj}. \tag{3.5}$$

In order to give a converse of Proposition 3.15, we will assume a decomposition of any increasing $\mathbf{T}$-operator θ such that $\theta(O) = O$ as an infimum of $\mathbf{T}$-dilations.

THEOREM 3.17. *Suppose that in $\mathcal{L}$ every increasing $\mathbf{T}$-operator fixing O is a non-empty infimum of $\mathbf{T}$-dilations. Let $\alpha = \bigvee_{j\in J} \delta_j\varepsilon_j$, where $J \neq \emptyset$ and each $(\varepsilon_j, \delta_j)$ is a $\mathbf{T}$-adjunction. Then for any $\eta \in \mathcal{H}_{\mathbf{T}}(\alpha)$ there exists $\eta_0 \in \mathcal{H}_{\mathbf{T}}(\alpha)$ with $\eta_0(O) = O$, a non-empty index set K, a family of $\mathbf{T}$-dilations δ'_k ($k \in K$) such that for every j,k, $\delta'_k \geq \delta_j\varepsilon_j$ or equivalently $\delta'_k\delta_j \geq \delta_j$, and we have*

$$\eta_0 = \bigwedge_{k\in K} \bigvee_{j\in J} \delta'_k\delta_j\varepsilon_j \qquad \text{and} \qquad \eta = \eta_0 \vee \gamma = \bigwedge_{k\in K} \bigvee_{j\in J} (\delta'_k \vee \gamma)\delta_j\varepsilon_j, \tag{3.6}$$

where γ is the constant operator defined by $\gamma(X) = \eta(O)$ for $X \in \mathcal{L}$. In particular η takes the form (3.3), *and $\eta = \eta_0$ when $\eta(O) = O$.*

PROOF. Define the operator θ by $\theta(O) = O$ and $\theta(X) = \eta(X)$ for $X \neq O$, and let $\eta_0 = \theta\alpha$. Clearly θ is an increasing $\mathbf{T}$-operator, $\theta \geq \alpha$, and so $\eta_0 \in \mathcal{H}_{\mathbf{T}}(\alpha)$ by definition; moreover $\eta_0(O) = O$ and $\eta = \theta \vee \gamma = \eta_0 \vee \gamma$.

By our assumption we have the decomposition $\theta = \bigwedge_{k\in K} \delta'_k$, where δ'_k is a dilation for each $k \in K$, and $K \neq \emptyset$. Moreover, for each j,k we have $\delta'_k \geq \theta \geq \alpha \geq \delta_j\varepsilon_j$. From the properties of adjunctions (in particular Proposition 2.5) it is easily seen that $\delta'_k \geq \delta_j\varepsilon_j$ is equivalent to $\delta'_k\delta_j \geq \delta_j$. Now we have the decomposition

$$\eta_0 = \theta\alpha = \big(\bigwedge_{k\in K} \delta'_k\big)\big(\bigvee_{j\in J} \delta_j\varepsilon_j\big) = \bigwedge_{k\in K} \bigvee_{j\in J} \delta'_k\delta_j\varepsilon_j,$$

because each dilation δ'_k commutes with suprema. Thus the left half of (3.6) holds.

As γ is a constant operator, it satisfies $(\gamma \vee \psi)\xi = \gamma \vee \psi\xi$ for any two operators ψ, ξ, and so for every $k \in K$ we have

$$\bigvee_{j\in J} (\delta'_k \vee \gamma)\delta_j\varepsilon_j = \bigvee_{j\in J} (\gamma \vee \delta'_k\delta_j\varepsilon_j) = \gamma \vee \big(\bigvee_{j\in J} \delta'_k\delta_j\varepsilon_j\big) = \gamma \vee \big(\delta'_k \bigvee_{j\in J} \delta_j\varepsilon_j\big) = \gamma \vee \delta'_k\alpha = (\gamma \vee \delta'_k)\alpha.$$

Hence we get

$$\bigwedge_{k\in K} \bigvee_{j\in J} (\delta'_k \vee \gamma)\delta_j\varepsilon_j = \bigwedge_{k\in K} (\gamma \vee \delta'_k)\alpha = \big(\bigwedge_{k\in K} (\gamma \vee \delta'_k)\big)\alpha.$$

As $\eta = \eta\alpha = (\gamma \vee \theta)\alpha = \big(\gamma \vee (\bigwedge_{k\in K} \delta'_k)\big)\alpha$, in order to obtain the right half of (3.6) we have only to show that

$$\bigwedge_{k\in K} (\gamma \vee \delta'_k) = \gamma \vee \big(\bigwedge_{k\in K} \delta'_k\big). \tag{3.7}$$

First, given $X \neq O$, for each $k \in K$ we have $\delta'_k(X) \geq \theta(X) = \eta(X) \geq \eta(O) = \gamma(X)$, and so $\bigwedge_{k\in K} \delta'_k(X) \geq \gamma(X)$; then

$$\bigwedge_{k\in K} (\delta'_k \vee \gamma)(X) = \bigwedge_{k\in K} \big(\delta'_k(X) \vee \gamma(X)\big) = \bigwedge_{k\in K} \delta'_k(X) = \gamma(X) \vee \big(\bigwedge_{k\in K} \delta'_k(X)\big).$$

Next, for each $k \in K$ we have $\delta'_k(O) = O$ and so

$$\bigwedge_{k\in K} (\delta'_k \vee \gamma)(O) = \bigwedge_{k\in K} \big(\delta'_k(O) \vee \gamma(O)\big) = \gamma(O) = \gamma(O) \vee \big(\bigwedge_{k\in K} \delta'_k(O)\big).$$

Thus the equality holds for any $X \in \mathcal{L}$, and (3.7) follows. ∎

Remark. (*i*) The real difficulty in the proof is in showing (3.7) and

$$\gamma \vee \bigwedge_{k\in K} \bigvee_{j\in J} \delta'_k \delta_j \varepsilon_j = \bigwedge_{k\in K} \bigvee_{j\in J} (\delta'_k \vee \gamma)\delta_j \varepsilon_j$$

without assuming (ISD). If we assume it, this is trivial.
(*ii*) For $\mathbf{T} = \{\mathbf{id}\}$, every increasing operator fixing O is an infimum of dilations, and every opening is a supremum of morphological openings (cfr. Subsection 2.4; for more details see Theorem 2.4 of [6] and Proposition 2.9 of [19]). In this case Theorem 3.17 characterizes $\mathcal{H}_{\{\mathbf{id}\}}(\alpha)$ for any opening α. In particular we obtain Serra's characterization of inf-overfilters (Theorem 9.7 of [22]).
(*iii*) If $\mathbf{T} \neq \{\mathbf{id}\}$, then we have not always such decompositions. However:

- — if $\mathcal{L}$ satisfies the Basic Assumption (cfr. Subsection 2.3), then every $\mathbf{T}$-opening is a supremum of morphological $\mathbf{T}$-openings (see also [19], Theorem 2.11);
- — if $\mathcal{L}$ satisfies the *dual* of that Basic Assumption, then every increasing $\mathbf{T}$-operator is an infimum of $\mathbf{T}$-dilations (see also [6], Theorem 3.11 and Remark 3.2 (*iv*)).

Note that if $\mathcal{L}$ satisfies the Basic Assumption or its dual, $\mathbf{I}$ is the only increasing $\mathbf{T}$-operator which does not fix O. Thus in the case of binary or grey-level images on a Euclidean or digital space, Theorem 3.17 characterizes $\mathcal{H}_{\mathbf{T}}(\alpha)$ for any $\mathbf{T}$-opening α, and $\eta = \eta_0$, except for $\eta = \mathbf{I}$.

Let us now give some particular case. Consider again the opening on sets given by (3.1) and (3.2). It arises from the inf-overfilter $\delta_A(\bigvee_{j\in J} \varepsilon_{C^j}) = \bigvee_{j\in J} \delta_A \varepsilon_{C^j}$, where $C^j \subseteq A$ for each $j \in J$. The latter clearly takes the form (3.3) with $\psi_{kj} = \delta_A$ and $\varepsilon_j = \varepsilon_{C^j}$ for each $k \in K$ and $j \in J$, and so it belongs to $\mathcal{H}_{\mathbf{T}}(\alpha)$ for $\alpha = \bigvee_{j\in J} \delta_{C^j}\varepsilon_{C^j}$.

While keeping the C^j constant ($j \in J$), we can modify δ_A (with the constraint $A \supseteq \bigcup_{j=1}^m C^j$, in other words $\delta_A \geq \bigvee_{i=1}^m \delta_{C^j}$), and we obtain thus different openings in $\mathcal{A}_{\mathbf{T}}(\alpha)$. They satisfy the property of Proposition 3.12: given a non-empty family of such openings, we can take their composition or equivalently their infimum, and also their supremum, and we still get an opening in $\mathcal{A}_{\mathbf{T}}(\alpha)$. Clearly such combined openings take the form

$$\mathbf{id} \wedge \psi(\bigvee_{j\in J} \varepsilon_{C^j}), \qquad \text{where} \quad \psi(\emptyset) = \emptyset, \quad \psi \text{ is increasing}, \quad \text{and} \quad \psi \geq \bigvee_{j\in J} \delta_{C^j}. \tag{3.8}$$

Conversely every such ψ is an infimum of a non-empty family of dilations $\delta_A \geq \bigvee_{j\in J} \varepsilon_{C^j}$, and so an opening of the form (3.8) can be written as

$$\mathbf{id} \wedge (\bigwedge_{k\in K} \delta_{A_k})(\bigvee_{j\in J} \varepsilon_{C^j}), \qquad \text{where} \quad K \neq \emptyset \quad \text{and} \quad A_k \supseteq \bigcup_{j\in J} C^j \quad \text{for} \quad k \in K. \tag{3.9}$$

Thus (3.8) and (3.9) are equivalent characterizations of openings in this subfamily of $\mathcal{A}_{\mathbf{T}}(\alpha)$, and so such openings are non-empty infima of openings of the form (3.1) with the C^j fixed but A varying.

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Appendix. Morphological filters and their domain of invariance

At the beginning of Section 1, we defined a *morphological filter* as an increasing and idempotent operator, but afterwards we restricted ourselves to extensive or anti-extensive ones, namely closings and openings. The main advantage of this limitation of scope is obtained in Section 2: the characterization of openings by their domain of invariance, which gives an isomorphism between the complete lattice of **T**-openings and the one of sup-closed **T**-invariant sets. A dual characterization holds for closings.

We will see here to what extent such a characterization can be made for morphological filters. We will show that the domain of invariance of a **T**-invariant morphological filter is a **T**-invariant complete lattice embedded in the original one, but with possibly distinct supremum and infimum operations. Now this mapping from **T**-invariant morphological filters to **T**-invariant complete lattices is not one-to-one, but only *onto*: a **T**-invariant complete lattice is the domain of invariance of a whole family of $\mathbf{T}$-invariant morphological filters, whose universal bounds are known. Moreover this mapping does not relate ordering relations between morphological filters to corresponding inclusion relations between their respective domains of invariance. Hence we cannot give a simple structural decomposition of morphological filters, as we did for openings and closings.

For the sake of brievity, let us write a *filter* for a morphological filter, and a $\mathbf{T}$-*filter* for a $\mathbf{T}$-invariant morphological filter. Matheron made a detailed study of filters in Chapter 6 of [22], without $\mathbf{T}$-invariance. We will prove here in a different way some of his results, with the additional constraint of $\mathbf{T}$-invariance. In the case where it is not taken into account, all results on $\mathbf{T}$-filters apply to filters by taking $\mathbf{T} = \{\mathbf{id}\}$.

Let us recall some notation from Section 2. We consider operators on a complete lattice $\mathcal{L}$. The *range* of an operator ψ is the set $\mathrm{Ran}(\psi)$ of all $\psi(X)$ for $X \in \mathcal{L}$; an *invariant* of ψ is some $X \in \mathcal{L}$ such that $\psi(X) = X$; the *domain of invariance* of ψ is the set $\mathrm{Inv}(\psi)$ of all invariants of ψ. Clearly $\mathrm{Inv}(\psi) \subseteq \mathrm{Ran}(\psi)$; moreover the idempotence of ψ can be expressed in three equivalent ways (see (2.2)):

$$\psi^2 = \psi \iff \mathrm{Ran}(\psi) \subseteq \mathrm{Inv}(\psi) \iff \mathrm{Ran}(\psi) = \mathrm{Inv}(\psi).$$

We will use the following fact from Matheron's Criterion 6.6, proved in Section 6.1 of [22]: If ψ and ξ are filters and $\psi \leq \xi$, then $\psi\xi$ and $\xi\psi$ are filters. This is for example the case if ψ is an opening and ξ a closing.

The following three results characterising $\mathbf{T}$-filters in terms of $\mathbf{T}$-invariant complete lattices embedded in $\mathcal{L}$ are simply extensions to $\mathbf{T}$-invariance of results from Section 6.2 of [22]:

PROPOSITION A.1. *Let ψ be a $\mathbf{T}$-filter. Then $\mathrm{Inv}(\psi)$ is a $\mathbf{T}$-invariant complete lattice for the same order relation $\leq$ as $\mathcal{L}$, with least element $\psi(O)$ and greatest element $\psi(I)$; given $\mathcal{S} \subseteq \mathrm{Inv}(\psi)$, the supremum and infimum of $\mathcal{S}$ in $\mathrm{Inv}(\psi)$ are $\psi(\bigvee \mathcal{S})$ and $\psi(\bigwedge \mathcal{S})$.*

PROOF. $\mathrm{Inv}(\psi)$ is $\mathbf{T}$-invariant because ψ is $\mathbf{T}$-invariant.

For $X \in \mathcal{L}$, $O \leq X \leq I$, and as ψ is increasing, $\psi(O) \leq \psi(X) \leq \psi(I)$, so that $\psi(O)$ and $\psi(I)$ are the least and greatest elements of $\mathrm{Ran}(\psi) = \mathrm{Inv}(\psi)$. Let $\mathcal{S} \subseteq \mathrm{Inv}(\psi)$ and suppose that $U \in \mathrm{Inv}(\psi)$ is an upper bound of $\mathcal{S}$: for all $S \in \mathcal{S}$, $U \geq S$. Thus for $S \in \mathcal{S}$ we have $S \leq \bigvee \mathcal{S} \leq U$, and as ψ is increasing,

$$S = \psi(S) \leq \psi(\bigvee \mathcal{S}) \leq \psi(U) = U;$$

this means that $\psi(\bigvee \mathcal{S})$ is the least upper bound of $\mathcal{S}$ in $\mathrm{Ran}(\psi) = \mathrm{Inv}(\psi)$. We prove in the same way that $\psi(\bigwedge \mathcal{S})$ is the greatest lower bound of $\mathcal{S}$ in $\mathrm{Inv}(\psi)$. ∎

PROPOSITION A.2. *Let $\mathcal{B}$ be a $\mathbf{T}$-invariant complete lattice included in $\mathcal{L}$, with supremum and infimum operations written $\bigvee^{\mathcal{B}}$ and $\bigwedge^{\mathcal{B}}$. The set of $\mathbf{T}$-filters having $\mathcal{B}$ as domain of invariance is not empty; its least element is $\mathbf{F}^{\mathbf{T}}(\mathcal{B})\mathbf{A}^{\mathbf{T}}(\mathcal{B})$ and its greatest element is $\mathbf{A}^{\mathbf{T}}(\mathcal{B})\mathbf{F}^{\mathbf{T}}(\mathcal{B})$. Moreover, for any $X \in \mathcal{L}$, we have*

$$\mathbf{F}^{\mathbf{T}}(\mathcal{B})\mathbf{A}^{\mathbf{T}}(\mathcal{B})(X) = \bigvee^{\mathcal{B}}\{B \in \mathcal{B} \mid B \leq X\}$$

and $$\mathbf{A}^{\mathbf{T}}(\mathcal{B})\mathbf{F}^{\mathbf{T}}(\mathcal{B})(X) = \bigwedge^{\mathcal{B}}\{B \in \mathcal{B} \mid B \geq X\}.$$

Proof. We show only the half of the statement concerning $\mathbf{F}^{\mathbf{T}}(\mathcal{B})\mathbf{A}^{\mathbf{T}}(\mathcal{B})$. The other half about $\mathbf{A}^{\mathbf{T}}(\mathcal{B})\mathbf{F}^{\mathbf{T}}(\mathcal{B})$ follows by duality. As $\mathbf{F}^{\mathbf{T}}(\mathcal{B})$ and $\mathbf{A}^{\mathbf{T}}(\mathcal{B})$ are $\mathbf{T}$-invariant, so is $\mathbf{F}^{\mathbf{T}}(\mathcal{B})\mathbf{A}^{\mathbf{T}}(\mathcal{B})$. Now $\mathbf{F}^{\mathbf{T}}(\mathcal{B})\mathbf{A}^{\mathbf{T}}(\mathcal{B})$ is a filter by Matheron's criterion. Let $X \in \mathcal{L}$ and $\mathcal{B}(X)$ the set of $B \in \mathcal{B}$ such that $B \leq X$. As $\mathcal{B}$ is $\mathbf{T}$-invariant, $\mathbf{A}^{\mathbf{T}}(\mathcal{B})(X) = \bigvee \mathcal{B}(X)$ (see Section 2). For any $C \in \mathcal{B}$, $C \geq \mathbf{A}^{\mathbf{T}}(\mathcal{B})(X) = \bigvee \mathcal{B}(X)$ if and only if $C \geq B$ for every $B \in \mathcal{B}(X)$, in other words if and only if $C \geq \bigvee^{\mathcal{B}} \mathcal{B}(X)$; hence $\bigvee^{\mathcal{B}} \mathcal{B}(X)$ is the least element of $\mathcal{B}$ which is $\geq \mathbf{A}^{\mathbf{T}}(\mathcal{B})(X)$. As $\mathcal{B}$ is $\mathbf{T}$-invariant, we have (see Section 2):

$$\mathbf{F}^{\mathbf{T}}(\mathcal{B})\mathbf{A}^{\mathbf{T}}(\mathcal{B})(X) = \bigwedge\{C \in \mathcal{B} \mid C \geq \mathbf{A}^{\mathbf{T}}(\mathcal{B})(X)\} = \bigwedge\{C \in \mathcal{B} \mid C \geq \bigvee^{\mathcal{B}} \mathcal{B}(X)\} = \bigvee^{\mathcal{B}} \mathcal{B}(X).$$

In particular $\mathbf{F}^{\mathbf{T}}(\mathcal{B})\mathbf{A}^{\mathbf{T}}(\mathcal{B})(X) \in \mathcal{B}$, while for $B \in \mathcal{B}$ we have $B = \mathbf{F}^{\mathbf{T}}(\mathcal{B})\mathbf{A}^{\mathbf{T}}(\mathcal{B})(B)$; hence $\mathrm{Ran}(\mathbf{F}^{\mathbf{T}}(\mathcal{B})\mathbf{A}^{\mathbf{T}}(\mathcal{B})) = \mathrm{Inv}(\mathbf{F}^{\mathbf{T}}(\mathcal{B})\mathbf{A}^{\mathbf{T}}(\mathcal{B})) = \mathcal{B}$. Let ψ be a $\mathbf{T}$-filter such that $\mathrm{Inv}(\psi) = \mathcal{B}$. For all $B \in \mathcal{B}(X)$ we have $B \leq X$ and so $B = \psi(B) \leq \psi(X)$, since ψ is increasing; as $\psi(X) \in \mathcal{B}$, we have thus $\mathbf{F}^{\mathbf{T}}(\mathcal{B})\mathbf{A}^{\mathbf{T}}(\mathcal{B})(X) = \bigvee^{\mathcal{B}} \mathcal{B}(X) \leq \psi(X)$. Therefore $\mathbf{F}^{\mathbf{T}}(\mathcal{B})\mathbf{A}^{\mathbf{T}}(\mathcal{B})$ is the least $\mathbf{T}$-filter having $\mathcal{B}$ as domain of invariance. ∎

Corollary A.3. *Let $\mathcal{B}$ be a $\mathbf{T}$-invariant complete lattice included in $\mathcal{L}$, with supremum and infimum operations written $\bigvee^{\mathcal{B}}$ and $\bigwedge^{\mathcal{B}}$. For any $\mathcal{S} \subseteq \mathcal{B}$,*

$$\bigvee^{\mathcal{B}} \mathcal{S} = \mathbf{F}^{\mathbf{T}}(\mathcal{B})(\bigvee \mathcal{S}) \qquad \textit{and} \qquad \bigwedge^{\mathcal{B}} \mathcal{S} = \mathbf{A}^{\mathbf{T}}(\mathcal{B})(\bigwedge \mathcal{S}).$$

Proof. By definition of $\mathbf{A}^{\mathbf{T}}(\mathcal{B})$, $\mathcal{B} \subseteq \mathrm{Inv}(\mathbf{A}^{\mathbf{T}}(\mathcal{B}))$ and $\mathrm{Inv}(\mathbf{A}^{\mathbf{T}}(\mathcal{B}))$ is sup-closed in $\mathcal{L}$ (see Section 2); so $\mathbf{A}^{\mathbf{T}}(\mathcal{B})(\bigvee \mathcal{S}) = \bigvee \mathcal{S}$. As $\mathcal{B} = \mathrm{Inv}(\mathbf{F}^{\mathbf{T}}(\mathcal{B})\mathbf{A}^{\mathbf{T}}(\mathcal{B}))$, by Proposition A.1 we have $\bigvee^{\mathcal{B}} \mathcal{S} = \mathbf{F}^{\mathbf{T}}(\mathcal{B})\mathbf{A}^{\mathbf{T}}(\mathcal{B})(\bigvee \mathcal{S})$. Hence $\bigvee^{\mathcal{B}} \mathcal{S} = \mathbf{F}^{\mathbf{T}}(\mathcal{B})\big(\mathbf{A}^{\mathbf{T}}(\mathcal{B})(\bigvee \mathcal{S})\big) = \mathbf{F}^{\mathbf{T}}(\mathcal{B})(\bigvee \mathcal{S})$. The other equality concerning $\bigwedge^{\mathcal{B}} \mathcal{S}$ is proved in the same way. ∎

In particular, for any $\mathbf{T}$-filter ψ having $\mathcal{B}$ as domain of invariance, we have $\psi(\bigvee \mathcal{S}) = \mathbf{F}^{\mathbf{T}}(\mathcal{B})\mathbf{A}^{\mathbf{T}}(\mathcal{B})(\bigvee \mathcal{S}) = \mathbf{F}^{\mathbf{T}}(\mathcal{B})(\bigvee \mathcal{S})$ and $\psi(\bigwedge \mathcal{S}) = \mathbf{A}^{\mathbf{T}}(\mathcal{B})\mathbf{F}^{\mathbf{T}}(\mathcal{B})(\bigwedge \mathcal{S}) = \mathbf{A}^{\mathbf{T}}(\mathcal{B})(\bigwedge \mathcal{S})$.

Given a $\mathbf{T}$-filter ψ, it is easy to show that the set of $\mathbf{T}$-filters ξ such that $\mathrm{Inv}(\xi) = \mathrm{Inv}(\psi)$ is the set of operators $\psi\theta$, where θ is an increasing $\mathbf{T}$-operator such that $\mathrm{Inv}(\psi) \subseteq \mathrm{Inv}(\theta)$. One can also prove that for two $\mathbf{T}$-operators ψ and ξ, they are $\mathbf{T}$-filters with the same domain of invariance if and only if $\psi\xi = \xi$ and $\xi\psi = \psi$.

Let us see to what features of filters correspond inclusion relations between their domains of invariance. This cannot be the ordering $\leq$, because $\mathrm{Inv}(\xi) \subseteq \mathrm{Inv}(\psi)$ means $\xi \leq \psi$ when ξ and ψ are openings, but $\xi \geq \psi$ when ξ and ψ are closings. However there is an analogue of points (*iii*) and (*iv*) of Proposition 2.9 is as follows: given two filters ψ and ξ,

$$\mathrm{Inv}(\xi) \subseteq \mathrm{Inv}(\psi) \quad \Longleftrightarrow \quad \psi\xi = \xi.$$

The set of $\mathbf{T}$-filters is a complete lattice (this was shown by Matheron in [22] without $\mathbf{T}$-invariance), but this structure is not reflected in the invariance domain of $\mathbf{T}$-filters.