Quasi-linear transformations, substitutions, numeration systems and fractals

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Abstract. In this paper we will define relations between quasi-linear transformations, substitutions, numeration systems and fractals. A Quasi-Linear Transformation (QLT) is a transformation on $\mathbb{Z}^n$ which corresponds to the composition of a linear transformation with an integer part function. We will first give some theoretical results about QLTs. We will then point out relations between QLTs, substitutions, numerations systems and fractals. These relations allow us to define substitutions, new numeration systems and fractals associated with them. With help of some properties of the QLTs we can give the fractal dimension of these fractals.

Keywords: Gaussian integers, numeration systems, discrete linear transformations, substitutions, fractals

1 Introduction.

Fractal tiles generated by numeration systems and substitutions has been widely studyed, see for example [1],[2], [5]. In this paper we define discrete linear transformations called QLT which generate tilings of $\mathbb{Z}^n$. The tilings studied here are self-similar, they allow us to generate n-dimensional fractals. In dimension two we define substitutions that generate the border of the tiles, and we point out relations between QLTs and numeration systems which allow us to define new numeration systems. These discrete linear transformations are also in relation with discrete lines [17],[].

Let g be a linear transformation from $\mathbb{Z}^n$ to $\mathbb{Q}^n$, defined by a matrix $\frac{1}{w}A$ where A is an integer matrix and w a positive integer. If we compose this transformation with an integer part function we obtain a Quasi-Linear Transformation (QLT) from $\mathbb{Z}^n$ to $\mathbb{Z}^n$. We will only consider the integer part function with a positive remainder. We will denote it $\lfloor\ \rfloor$. If x and y are two integers, $\lfloor\frac{x}{y}\rfloor$ denotes the quotient of the Euclidean division of x by y. We denote G as the QLT defined by g. Main research results on such transformations are listed in the first section of this paper.

A substitution on an alphabet $\mathcal{A}$ is an application that associates each letter of $\mathcal{A}$ with a word over the alphabet $\mathcal{A}$. In the second section of this paper we consider sequences of tiles associated with QLTs of $\mathbb{Z}^2$. We define substitutions that generate the border of these tiles. Each sequence of tiles converges (after renormalization) toward a tile wih a fractal border whose fractal dimension is computed with help of the substition.

Let us consider β an algebraic number and $\mathcal{D}$ a set 672(o)-5.8887(f$\mathbb{Z}[\beta]$3.e22406(n)1.48504(e)-1.ctats is a valid base using the digits set $\mathcal{D}$ if every iateger $c \in \mathbb{Z}[\beta]$ has a unique

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poiat out somee22406(n)1.ations between QLTs and numerations system β is a Gaussian iateger ($\beta = a + \imath b$, with $a, b \in \mathbb{Z}$) or an algebraic iateger of order 2 ($\beta^2 + b\beta + a = 0$ with $a, b \in \mathbb{Z}$). Thesee22406(n)1.ations and some properties of QLTs will allow us to generate new numeration systecs.

In thefourth section wewil22-240.165(s)-3.0436(e)-1.22283(e)-218.034(h)1.48504(o)18.2077(w)-229.343(Q)-5.14863(L)82.976

In this paper we focus on a particular type of QLT, called "m-determinantal QLT".

Definition 5 *A QLT defined by a matrix $\frac{1}{w}A$ such that $w = m\det(A)$ where m is a positive integer, is called a m-determinantal QLT. A 1-determinantal QLT will be called a determinantal QLT.*

In the following G will always denote a m-determinantal QLT associated with the matrix $\frac{1}{m\delta}A$, and g the associated rational linear transformation. To simplify the notations, tiles $P_{G,X}$ and $P^p_{G,X}$ will be noted P_X and P^p_X. Moreover $(0,0,\dots,0) \in \mathbb{Z}^n$ will be simply noted O.

Proposition 1. *The tiles generated by a m-determinantal QLT G are all geometrically identical. For all $Y \in \mathbb{Z}^n$, $P_Y = \mathcal{T}_{m\widehat{A}^T Y} P_O$ where $\mathcal{T}_v$ represents the translation of vector v and $\widehat{A}^T$ is the transpose of the cofactor matrix of A.*

Proof. : If $\widehat{A}^T$ is the transpose the cofactor matrix of A, we have $A^{-1} = \dfrac{1}{\delta}\widehat{A}^T$. Let us consider $Y \in \mathbb{Z}^n$, $X \in P_O$ and $X' = m\widehat{A}^T\ Y + X$. We then have: $G(X') = \left\lfloor \frac{1}{m\delta}AX + \frac{1}{m\delta}Am\widehat{A}^T\ Y \right\rfloor = \left\lfloor \frac{1}{m\delta}AX + Y \right\rfloor = G(X) + Y$. It follows that for each point $X \in P_O$, $X' = m\widehat{A}^T\ Y + X$ is a point of P_Y. We can now conclude that $P_Y = \mathcal{T}_{m\widehat{A}^T Y} P_O$. □

Proposition 2. *The p-tiles generated by a m-determinantal QLT are all geometrically identical and can be generated recursively by translations of P_O. More precisely, if $\mathcal{T}_v$ refers to the translation of the vector v we have, for all $p \geq 1$:*

$$P^p_Y = \mathcal{T}_{\left(m\widehat{A}^T\right)^p Y} P^p_O \tag{1}$$

and

$$P^{p+1}_O = \bigcup_{X \in P_O} \mathcal{T}_{\left(m\widehat{A}^T\right)^p X} P^p_O = \bigcup_{X \in P^p_O} \mathcal{T}_{\left(m\widehat{A}^T\right) X} P_O \tag{2}$$

Proof. : If $p = 1$, the equality (1) has been proved in proposition 1.
By definition $P^2_O = \{X \in \mathbb{Z}^n | X \in P_{Y'} \text{ and } \mathrm{Y}' \in \mathrm{P_O}\}$ so

$$P^2_O = \bigcup_{Y' \in P_O} P_{Y'} = \bigcup_{Y' \in P_O} \mathcal{T}_{\widehat{A}^T\ Y'} P_O$$

which proves the equality (2) for $p = 1$.
Let us now assume the equalities (1) and (2) to be true with $p = k$ and prove that they are true for $p = k + 1$.

$$
\begin{aligned}
P_Y^{k+1} &= \bigcup_{X \in P_Y} P_X^k \\
&= \bigcup_{X' \in P_O} P^k_{X'+m\widehat{A}^T Y} \\
&= \bigcup_{X' \in P_O} \mathcal{T}_{\left(m\widehat{A}^T\right)^k (X'+m\widehat{A}^T Y)} P_O^k && \text{(recurrence hypothesis)} \\
&= \mathcal{T}_{\left(m\widehat{A}^T\right)^{k+1} Y} \bigcup_{X' \in P_O} \mathcal{T}_{\left(m\widehat{A}^T\right)^k X'} P_O^k \\
&= \mathcal{T}_{\left(m\widehat{A}^T\right)^{k+1} Y} P_O^{k+1} && \text{(recurrence hypothesis)}
\end{aligned}
$$

Moreover :

$$
\begin{aligned}
P_O^{k+1} &= \bigcup_{X \in P_O} P_X^k \\
&= \bigcup_{X \in P_O} \mathcal{T}_{\left(m\widehat{A}^T\right)^k X} P_O^k && \text{(recurrence hypothesis)}
\end{aligned}
$$

And :

$$
\begin{aligned}
P_O^{k+1} &= \bigcup_{X \in P_O^k} P_X \\
&= \bigcup_{X \in P_O^k} \mathcal{T}_{\left(m\widehat{A}^T\right) X} P_O .
\end{aligned}
$$

□

For example, Figure 2 shows the tile of order 2 of the QLT defined by $\frac{1}{34}\begin{pmatrix} -3 & 5 \\ -5 & -3 \end{pmatrix}$, this tile is composed of 34 tiles of order 1. The following proposition determines the number of points of a tile of order n and will be used in the last section to detemine the fractal dimension of n-dimensional fractals.

Proposition 3. *The number of points of a p-tile generated by a m-determinantal QLT in $\mathbb{Z}^n$ is equal to $\delta^{p(n-1)} m^{np}$.*

Proof. Let first proof that the tile P_O contains exactely $\delta^{n-1} m^n$ points.

- Case of $m = 1$ and $n = 2$: it is know (see []) that in this case the number of points of P_0 equals δ.

- Case of $m = 1$ and $n > 2$. It has been proved in [] and [] that for each integer matrix A it exists an unimodular matrix U such that $AU = B$ where B is an upper triangular integer matrix and that the point of the tiles of A are in one-to-one correspondance with the points of the tiles f B. Therefore we will only do the proof for an upper triangular integer matrix $B = (b_{i,j})_{1 \le i,j \le n}$. Let denote P_0, the tile associated with B and P'_X the tiles associated with $B' = (b'_{i,j})_{1 \le i,j \le n-1}$ with $b'_{i,j} = b_{i+1,j+1}$ and $\delta' = b_{2,2} b_{3,3} \ldots b_{n,n}$. Suppose that the number of points P'_X equals δ'^{n-2}. We have:

$$\begin{array}{l} (x_1, x_2, \ldots, x_n) \in P_O \\ \Leftrightarrow \left\{ \begin{array}{lr} 0 \le b_{1,1}x_1 + b_{1,2}x_2 + \ldots + b_{1,n}x_n & < \delta \\ 0 \le \qquad b_{2,2}x_2 + \ldots + b_{2,n}x_n & < \delta \\ \vdots \qquad \vdots & \vdots \\ 0 \le \qquad b_{n,n}x_n & < \delta \end{array} \right. \\ \Leftrightarrow \left\{ \begin{array}{llll} -\frac{b_{1,2}x_2+\ldots+b1,n-1x_{n-1}+b_{1,n}x_n}{b_{1,1}} \le & x_1 < \delta' - \frac{b_{1,2}x_2+\ldots+b1,n-1x_{n-1}+b_{1,n}x_n}{b_{1,1}} & & \\ 0 \le \frac{b_{2,2}x_2+\ldots+b_{2,n}x_n}{\delta'} & < & b_{1,1} & \\ \vdots \qquad \vdots & & \vdots & \\ 0 \le \frac{b_{n,n}x_n}{\delta'} & < & b_{1,1} & \end{array} \right. \\ \Leftrightarrow \left\{ \begin{array}{l} -\frac{b_{1,2}x_2+\ldots+b1,n-1x_{n-1}+b_{1,n}x_n}{b_{1,1}} \le x_1 < \delta' - \frac{b_{1,2}x_2+\ldots+b1,n-1x_{n-1}+b_{1,n}x_n}{b_{1,1}} \\ (x_2, x_3, \ldots, x_n) \in P'_{i_2,i_3,\ldots,i_n} \text{ with } i_k = 0, 1, \ldots, b_{1,1} \end{array} \right. \end{array}$$

The number of solutions for x_1 equals δ' and each tile $P'_{i_2,i_3,\ldots,i_n}$ contains δ'^{n-2}, therefore we have $\delta'\delta'^{n-2}b_{1,1}^{n-1} = \delta^{n-1}$ points.

– If $m > 1$, we have:

$$\begin{array}{l} (x_1, x_2, \ldots, x_n) \in P_O \\ \Leftrightarrow \left\{ \begin{array}{l} 0 \le a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,n}x_n < m\delta \\ \vdots \qquad \vdots \qquad \vdots \\ 0 \le a_{n,1}x_1 + a_{n,2}x_2 + \ldots + a_{n,n}x_n < m\delta \end{array} \right. \\ \Leftrightarrow \left\{ \begin{array}{l} 0 \le \frac{a_{1,1}x_1+a_{1,2}x_2+\ldots+a_{1,n}x_n}{\delta} < m \\ 0 \le \frac{a_{2,1}x_1+a_{2,2}x_2+\ldots+a_{n,n}x_n}{\delta} < m \end{array} \right. \\ \Leftrightarrow (x_1, x_2, \ldots, x_n) \in P'_{i_1,i_2,\ldots,i_n} \text{ with } i_1, i_2, \ldots, i_n = 0, 1, \ldots, m-1 \end{array}$$

But each $P'_{i,j}$ contains δ^{n-1} points, it follows thats P_O contains $\delta^{n-1}m^n$ points.

We conclude that the number of tiles of P_O equals $\delta^{n-1}m^n$.
Let now supose that the number of points of P_O^p equals $\delta^{p(n-1)}m^{np}$, we have $P_O^{p+1} = \bigcup_{X \in P_O} \mathcal{T}_{(m\widehat{A}^T)^p X} P_O^p$ (see proposition 2), it follows that the number of points of P_O^{p+1} equals $\delta^{p(n-1)}m^{np}\delta^{n-1}m^n = \delta^{(p+1)(n-1)}m^{n(p+1)}$. □

It is well known that if g is a contracting linear transformation of $\mathbb{R}^n$ then g has the origin as unique fixed point and for each $X \in \mathbb{R}^n$ the sequence $g^n(X)$ tends toward this fixed point. We will say that a QLT G is contracting if g is contracting. But such a QLT has not necessarily a unique fixed point. The behavior under iteration of 2D contracting QLTs has been studied in [7], [16], [14] and [15]. The following definition and theorem will be used to define new numeration system.

Definition 6 *A consistent Quasi-Linear Transformation is a QLT which has the origin as unique fixed point such that for each discrete point Y the sequence $(G^n(Y))_{n\ge0}$ tends toward this unique fixed point.*

Consider a 2D-QLT defined by $\frac{1}{\omega}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the infinite norm of g is then

$$||g||_\infty = \frac{1}{\omega} max(\mid a \mid + \mid b \mid, \mid c \mid + \mid d \mid).$$

The two following theorems, proved in [7], states conditions such that a 2D-QLT is a consistent QLT.

Theorem 1. *Let G be a QLT such that $||g||_\infty < 1$, G is a consistent QLT if and only if one of the three following conditions is verified:*

$$\begin{array}{llll} (1) & b \leq 0, a+b \leq 0, & c > 0 \text{ and } & d \leq 0 \\ (2) & a \leq 0, & b > 0, c \leq 0 \text{ and } & c+d \leq 0 \\ (3) & a \leq 0, & b \leq 0, c \leq 0 \text{ and } & d \leq 0 \end{array}$$

Theorem 2. *Let G be a QLT such that $||g||_\infty = 1$, G is a consistent QLT if and only if one of the following conditions is verified:*

1. *If $|a| + |b| = \omega$ and $|c| + |d| < \omega$*
   - $a = 0$, $b < 0$, $c > 0$ *and* $c - d \geq 0$
   - $a > 0$, $a + b \leq 0$, $c > 0$ *and* $d \leq 0$
   - $a \leq 0$, $b > 0$, $c \leq 0$ *and* $c + d \leq 0$
   - $a < 0$, $b < 0$, $c \geq 0$ *and* $d \leq 0$
2. *If $|a| + |b| < \omega$ and $|c| + |d| = \omega$*
   - $d = 0$, $c < 0$, $b > 0$ *and* $b - a \geq 0$
   - $d > 0$, $c + d \leq 0$, $b > 0$ *and* $a \leq 0$
   - $d \leq 0$, $c > 0$, $b \leq 0$ *and* $a + b \leq 0$
   - $d < 0$, $c < 0$, $b \geq 0$ *and* $a \leq 0$
3. *If $|a| + |b| = \omega$ and $|c| + |d| = \omega$*
   - $\omega = -a + b = -c \pm d$, $a + b \leq 0$, $c + d \leq 0$ *and* $abc \neq 0$
   - $\omega = -b = c + d$, $c - g \geq 0$, *and* $bcd \neq 0$
   - $\omega = a - b = c - d$, $a + b \leq 0$, $c + d > 0$ *and* $bcd \neq 0$
   - $\omega = b = -c \pm d$, $c + d \leq 0$ *and* $bcd \neq 0$
   - $\omega = -a - b = c - d$, $a - b < 0$, $c + d > 0$ *and* $abcd \neq 0$
   - $\omega = -a \pm b = c - d$, $a + b \leq 0$, $c + d \leq 0$ *and* $bcd \neq 0$
   - $\omega = a + b = -c$, $a - b \leq 0$ *and* $abc \neq 0$
   - $\omega = a - b = -c + d$, $a + b > 0$, $c + d \leq 0$ *and* $abc \neq 0$
   - $\omega = \pm a - b = c$, $a + b \leq 0$ *and* $abc \neq 0$
   - $\omega = -a + b = -c - d$, $a + b > 0$, $c - d > 0$ *and* $abcd \neq 0$

3 Quasi-linear transformations and substitutions

In this section we only consider 2D m-determinantal QLTs.

A substitution on an alphabet $\mathcal{A}$ is an application that associates each letter of $\mathcal{A}$ with a word over the alphabet $\mathcal{A}$. In this section we will define a substitution over an alphabet $\mathcal{A}$ of 16 letters. The words generated by these substitutions corresponds to the Freeman code of the border of the tiles. The substitutions will be used to determine the fractal dimension of the border.

Let us remember that a discrete line is the set of points

$$D(a,b,\omega,i) = \left\{(x,y) \in \mathbb{Z}^2 | \left\lfloor \frac{ax+by}{\omega} \right\rfloor = i\right\}$$

where $a,b,i,\omega \in \mathbb{Z}$ and $\omega > 0$ (see [17]).

A discrete line such that $\omega = max(| a |, | b |)$ is called a discrete naive line and will be noted $\mathcal{N}(a,b,i)$. The geometrical structure of a naive line is periodic, we will use this periodicity to determine the border of a tile.

The tile with index (i,j) is the intersection of the two discrete lines $D(a,b,m\delta,i)$ and $D(c,d,m\delta,j)$. These discrete lines have been studied (see [17]) for coprime coefficients, so we note $t = \gcd(a,b), t' = \gcd(c,d)$, $a' = a/t$, $b' = b/t$, $c' = c/t'$ and $d' = d/t'$. The study of discrete lines [17] and QLTs [15],[16], [7], [9] induce the following properties.

Property 1. 1. P_O is geometrically identical to $P_{i,j}, \forall i,j \in \mathbb{Z}$;
2. P_O contains $m^2\delta$ points;
3. P_O contains exactly $m.t$ translates of the period of the naive line $\mathcal{N}(a',b',0)$;
4. P_O contains exactly $m.t'$ translates of the period of the naive line $\mathcal{N}(c',d',0)$;
5. P_O has exactly 6 neighbours (or 4 neighbours if $a = d = 0$ or $b = c = 0$) among the sets $P_{i,j}$ with $i,j \in \{0, \pm 1\}$.

Figure 1 corresponds to the tile P_O associated to the QLT of matrix $\frac{1}{34}\begin{pmatrix} -3 & 5 \\ -5 & -3 \end{pmatrix}$, we see also the period of the naive lines defined by $\mathcal{N}(-3,5,0)$ and $\mathcal{N}(-5,-3,0)$. The border of P_O can be split into six parts which are the frontiers between P_O and its six neighbours. These six parts are noted Γ_i with $i = 0,1,...,5$ as shown in figure 1. The arrows indicate the direction. With each border is associated a word whose letters correspond to the direction of the successive segments (U = up, D = down, L = left, R = right). In our example: Γ_0 =RURRURR, Γ_1 =U, Γ_2 =LULUULU, Γ_3 =LLDLLDL, Γ_4 =D and Γ_5 =DRDDRD.

We now explain how to obtain Γ_i for $i = 0,1,...,5$. We limit the study to the case $a' < 0$, $b' > 0$, $| b' | \geq | a' |$, $c' < 0$, $d' < 0$ and $| c' | \leq | d' |$; the other cases are similar and can be obtained by symmetry.

- Γ_0 is the bottom of mt times the period of the naive line $\left\lfloor \frac{a'x+b'y}{b'} \right\rfloor = 0$. Réveilles [17] proved that the period of a naive line contains long and short steps. We will note them l and s. We will also note $q = \left\lfloor \frac{b'}{-a'} \right\rfloor$. The steps s and l have the respective length q and $q+1$. Let $[0, q_1, q_2, \ldots, q_n]$ be the continued fraction development of $\frac{-a'}{b'}$. The period of the naive line $\left\lfloor \frac{a'x+b'y}{b'} \right\rfloor = 0$ can be obtained by the following recurrence (Réveilles [17]):
  $\mu_0 = l$

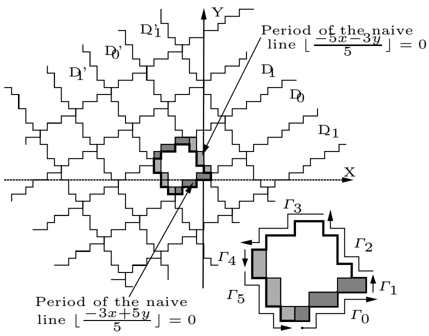

Fig. 1. Border of P_O.

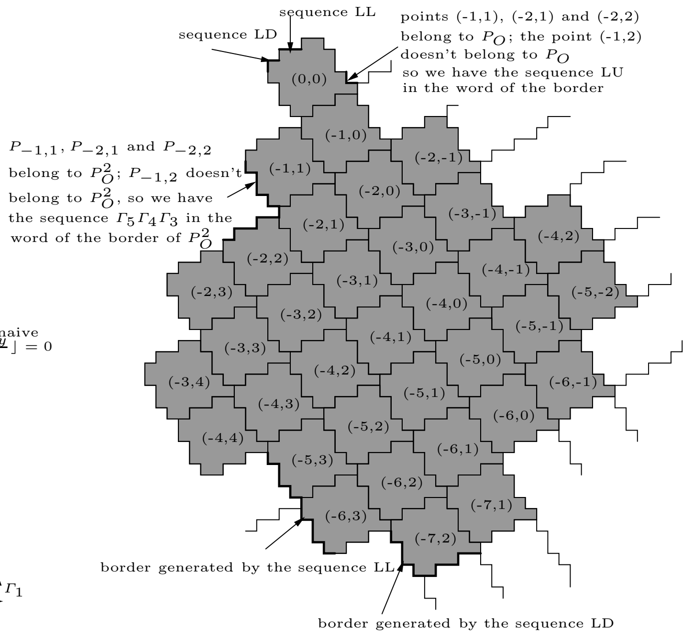

Fig. 2. Border of P_O^2 containing 34 tiles of order 1.

$\mu_1 = s$
$\mu_2 = ls^{q_2-1}$
if $i+1$ is even, $\mu_{i+1} = \mu_{i-1}\mu_i^{q_i+1}$
if $i+1$ is odd, $\mu_{i+1} = \mu_i^{q_i+1}\mu_{i-1}$
μ_n is the period.

- As Γ_0 is the bottom of mt times this period, the word corresponding to a short (resp. long) step will be R^q (resp. R^{q+1}). These successive steps give rise to a vertical segment which corresponds to the letter U. The following recurrence determines then Γ_0:
  $\nu_1 = R^p$
  $\nu_2 = R^{p+1}(UR^p)^{q_2-1}$
  if $i+1$ is even, $\nu_{i+1} = \nu_{i-1}\nu_i^{q_{i+1}}$
  if $i+1$ is odd, $\nu_{i+1} = \nu_i^{q_{i+1}}\nu_{i-1}$
  Finally: $\Gamma_0 = (\nu_n)^{(mt)}$.

- Γ_5 is the bottom of mt' times the period of the digital straight line $\left\lfloor \dfrac{c'x+d'y}{-c'} \right\rfloor$ without the first point of the period which is a neighbour of $P_{1,1}$ and corresponds to the border Γ_4. Again, the recurrence given by Réveilles [17] to determine this period induces the recurrence that permits to obtain Γ_5.

– To obtain Γ_3 (resp. Γ_2) we remark that they are the "inverse" of Γ_0 (resp. Γ_5): If we read Γ_0 (resp. Γ_5) from right to left and replace U by D and R by L (and vice versa) we obtain Γ_3 (resp. Γ_2). Moreover, independently of a, b, c and d, $\Gamma_4 = D$ and $\Gamma_1 = U$. Finally $\Gamma_0\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5$ corresponds to the border of P_O.

We now consider p-tiles, we have (see proposition 2) $P_O^p = \bigcup_{X \in P_O^{p-1}} \mathcal{T}_{(m\widehat{A}^T)X} P_O$: we can obtain P_O^p from P_O^{p-1} by replacing each point $X \in P_O^{p-1}$ by P_X which is the image of P_O by a given translation (see figure 2). Suppose that the border of P_O^{p-1} contains the letter U, that means that P_O^{p-1} contains a point of coordinates (i, j) such that the point $(i+1, j)$ doesn't belong to P_O^{p-1} (see figure 2). The letter U in the border of P_O^{p-1} generates the sequence Γ_3 (border between $P_{i,j}$ and $P_{i+1,j}$) in P_O^p. In the same manner, the letters L, D and R respectively generate the sequences Γ_5, Γ_0 and Γ_2. But this is not sufficient to generate the border of P_O^p. Indeed, suppose that the border of P_O^n contains the sequence LU, this sequence corresponds to the border between three points (i, j), $(i-1, j)$ and $(i-1, j+1)$ which belong to P_O^{p-1} and the point $(i, j+1)$ which does not belong to P_O^{p-1} (see figure 2), so the sequence LU generates the sequence $\Gamma_5\Gamma_4\Gamma_3$ in the border of P_O^p. We can see in figure 2 that the sequence LL (resp. LU and LD) generates the sequence $\Gamma_5\Gamma_4\Gamma_5$ (resp. $\Gamma_5\Gamma_4\Gamma_3$ and $\Gamma_5\Gamma_0$). So we have to add 12 letters (one letter for each of the 12 sequences of two letters) and to examine which word has to be associated with these letters. Call L_1, L_2, …, L_{12} these letters which will be respectively between LL, DL, UL, RR, DR, UR, UU, LU, RU, DD, LD and RD. First we have to modify Γ_i by adding these 12 letters, we denote Γ_i' the new words. In the case studied here ($a' < 0$, $b' > 0$, $\mid b' \mid \geq \mid a' \mid$, $c' < 0$, $d' < 0$ and $\mid c' \mid \leq \mid d' \mid$) Γ_5' always begins and ends with the letter R and $\Gamma_4' = D$, so the sequence $\Gamma_5\Gamma_4\Gamma_5$ becomes $\Gamma_5' L_{12} \Gamma_4' L_5 \Gamma_5'$. Now determine the word generated by L_1: LL_1L generates $\Gamma_5' L_{12} \Gamma_4 L_5 \Gamma_5'$, L generates Γ_5' so L_1 will generate $L_{12}\Gamma_4 L_5$. In the same way we examine all other letters and we obtain a substitution over an alphabet of 16 letters.

In figures 3 and 4 we can see two examples of borders of P_O^n. Their corresponding matrix and substitutions are:

Figure 4		Figure 3	
matrix	$\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$	matrix	$\begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$
D	$\rightarrow L$	D	$\rightarrow R$
U	$\rightarrow R$	U	$\rightarrow L$
L	$\rightarrow UL_6RL_9U$	L	$\rightarrow R$
R	$\rightarrow DL_2LL_{11}D$	R	$\rightarrow L$
L_1	$\rightarrow L_6RL_9$	L_1	$\rightarrow L_{12}DL_5$
L_2	$\rightarrow L_8$	L_2	$\rightarrow L_4$
L_3	$\rightarrow L_4RL_9$	L_3	$\rightarrow L_{11}DL_5$
L_4	$\rightarrow L_2LL_{11}$	L_4	$\rightarrow L_8UL_3$
L_5	$\rightarrow L_1LL_{11}$	L_5	$\rightarrow L_9UL_3$
L_6	$\rightarrow L_{12}$	L_6	$\rightarrow L_1$
L_7	$\rightarrow L_4RL_4$	L_7	$\rightarrow L_{11}DL_2$
L_8	$\rightarrow L_6RL_4$	L_8	$\rightarrow L_{12}DL_2$
L_9	$\rightarrow L_5$	L_9	$\rightarrow L_1$
L_{10}	$\rightarrow L_1LL_1$	L_{10}	$\rightarrow L_9UL_6$
L_{11}	$\rightarrow L_3$	L_{11}	$\rightarrow L_4$
L_{12}	$\rightarrow L_2LL_1$	L_{12}	$\rightarrow L_8UL_6$

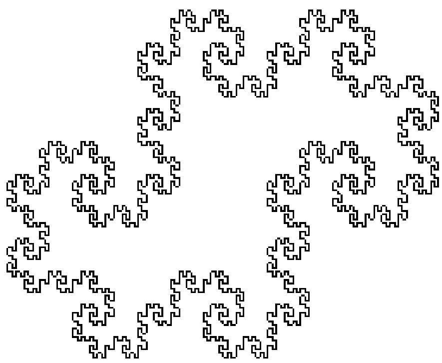

Fig. 3. Border of P_O^{12}.

4 UL_6 Quasi-linel

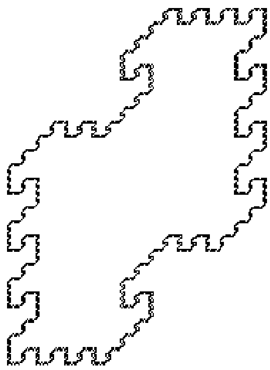

Fig. 4. Border of P_O^7.

form

$$c = c_0 + c_1\beta + c_2\beta^2 \ldots + c_n\beta^n$$

with $c_i \in \mathcal{D}$ and $n \in \mathbb{N}$; the length of the decomposition is then $n+1$. We also say that $(\beta, \mathcal{D})$ is a numeration system of $\mathbb{Z}[\beta]$ and $\mathcal{D}$ is the set of digits of this numeration system.

4.1 Case of Gaussian integers

The results below can be found in [8]: they allow us to determine numeration systems of $\mathbb{Z}[i]$ by considering $\beta = a + ib$ a Gaussian integer and $\mathcal{D}$ a set of Gaussian integers.

Definition 7 *Let $c = x+iy$ and $\beta = a+ib$ be two Gaussian integers. The integer division of c by β, noted $\left\lfloor \frac{c}{\beta} \right\rfloor$, is defined by:* $\left\lfloor \frac{c}{\beta} \right\rfloor = \left\lfloor \frac{ax+by}{a^2+b^2} \right\rfloor + i \left\lfloor \frac{-bx+ay}{a^2+b^2} \right\rfloor$.

This division corresponds to the usual division of complex numbers composed with the integer part function: so we have the following relation with QLTs.

Proposition 4. *Let $\beta = a+ib$ and $c = x+iy$ be two Gaussian integers and let $c' = x' + iy' = \left\lfloor \frac{c}{\beta} \right\rfloor$. The point (x', y') is then the image of the point (x, y) by the QLT $\lfloor g_\beta \rfloor$ defined on $\mathbb{Z}^2$ by:*

$$\begin{aligned} \lfloor g_\beta \rfloor : \ \mathbb{Z}^2 &\longrightarrow \mathbb{Z}^2 \\ (x,y) &\longmapsto \begin{cases} x' = \left\lfloor \dfrac{ax+by}{a^2+b^2} \right\rfloor \\ y' = \left\lfloor \dfrac{-bx+ay}{a^2+b^2} \right\rfloor \end{cases} \end{aligned}$$

Theorem 3. *Let $\beta = a + ib$ be a Gaussian integer and $\mathcal{D}$ the set of Gaussian integers c such that* $\left\lfloor \dfrac{c}{\beta} \right\rfloor = 0$*, the three following properties are then equivalent:*

1. *$(\beta, \mathcal{D})$ is a numeration system,*
2. *The QLT $\lfloor g_\beta \rfloor$ is a consistent Quasi-Linear Transformation,*
3. *$a \leq 0$ and $\mid a \mid + \mid b \mid > 1$.*

Remark 1. consider the QLT $\lfloor g_\beta \rfloor$ and note P_O and P_O^p the tiles defined by this QLT. The set of digits is given by $\mathcal{D} = \{c = x + iy \mid \lfloor \frac{c}{\beta} \rfloor = 0\} = \{c = x + iy \mid (x, y) \in P_O\}$. In [7] and [9] we can find an algorithm that allows to determine P_O.

4.2 Case of algebraic integers of order 2

Now we consider β an algebraic integer such that $\beta^2 + b\beta + a = 0$ with $a, b \in \mathbb{Z}$. We only consider the case where β is a complex number, that is to say $b^2 - 4a < 0$. We will define numeration systems of $\mathbb{Z}[\beta]$ where the digits are elements of $\mathbb{Z}[\beta]$. First we will define an integer division by β, this integer division corresponds to a QLT that will define the digits of the numeration system. As in preceeding section, the QLT has to be a consistent QLT. We will define the division by using another base of $\mathbb{Z}[\beta]$, the QLT associated to the division will depend on these base, a good choice of the base will raise to a consistent QLT and so defined a numeration system.

Lemma 1. *Let $\beta_1 = n + \beta$ with $n \in \mathbb{Z}$, we have $\mathbb{Z}[\beta] = \mathbb{Z}[\beta_1]$.*

Proof. Let $x = x_1 + x_2\beta \in \mathbb{Z}[\beta]$, we have:

$$\begin{aligned}
&\Leftrightarrow \exists x_1, x_2 \in \mathbb{Z} \mid x = x_1 + x_2\beta \\
&\Leftrightarrow \exists x_1, x_2 \in \mathbb{Z} \mid x = x_1 - x_2 n + x_2\beta_1 \\
&\Leftrightarrow \exists x_1', x_2' \in \mathbb{Z} \mid x = x_1' + x_2'\beta_1 \\
&\Leftrightarrow x \in \mathbb{Z}[\beta_1]
\end{aligned}$$

□

Let now define the integer division using this base. We have $\beta^2 + b\beta + a = 0$, so $\frac{1}{\beta} = -\frac{\beta + b}{a} = \frac{-\beta_1 + n - b}{a}$ and $\beta_1^2 = \beta_1(2n - b) - n^2 + nb - a$. Let $x = x_1 + x_2\beta_1$, we have:

$$\begin{aligned}
\frac{x}{\beta} &= \frac{(x_1 + x_2\beta_1)(-\beta_1 + n - b)}{a} \\
&= \frac{x_2(\beta_1(b - 2n) + n^2 - nb + a) + \beta_1(x_2(n - b) - x_1) + x_1(n - b)}{a} \\
&= \frac{x_1(n - b) + x_2(n^2 - nb + a) + (-x_1 - x_2 n)\beta_1}{a}.
\end{aligned}$$

We define the integer division of x by β by the composition of the division above with the integer part function.

Definition 1. *Let $x = x_1 + x_2\beta_1$ and G_β the QLT defined by the matrix* $\frac{1}{a}[\begin{pmatrix} n-b & a-nb+n^2 \\ -1 & -n \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. *The quotient of the integer division of x by β, noted $\lfloor \frac{x}{\beta}$ is defined by $x'_1 + x'_2\beta_1$, where $(x'_1, x'_2) = G_\beta(x_1, x_2)$.*

The set of digits is

$$\mathcal{D} = \{x = x_1 + x_2\beta_1 \mid \lfloor \frac{x}{\beta} \rfloor = 0\} = \{x = x_1 + x_2\beta_1 \mid G(x_1, x_2) \in P_O\}$$

where P_O is the tile of order one associated to the QLT G.

Theorem 4. *Let β be an algebraic integer such that $\beta^2 + b\beta + a = 0$ and $\mathcal{D}$ the set of Gaussian integers c such that $\left\lfloor \frac{c}{\beta} \right\rfloor = 0$, the three following properties are then equivalent:*

1. *It exists n such that $(\beta, \mathcal{D})$ is a numeration system,*
2. *It exists n such that the QLT $\lfloor g_\beta \rfloor$ (defining the integer division) is a consistent QLT,*
3. *$(b > 2$) or ($b = 2$ and*

Proof. $(\beta, \mathcal{D})$ is then a numeration if and only if for all $c \in \mathbb{Z}[\beta]$, there exist $c_0, c_1, \ldots, c_p \in \mathcal{D}$ such that

$$c = c_0 + c_1\beta + \ldots + c_p\beta^p$$

Moreover, this decomposition has to be unique. It is easy to see that if this decomposition exists then $c_i = x_i + y_i\beta_1$ with $\begin{pmatrix} x_i \\ y_i \end{pmatrix} = G^i \begin{pmatrix} x \\ y \end{pmatrix}$ where $x + y\beta_1 = c$. We conclude that the decomposition exists and is unique if there exists $p \in \mathbb{N}$ such that $G^p \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Finally $(\mathcal{D}, \beta)$ is a numeration system if and only if G is a consistent QLT. Theorem 1 gives the necessary and sufficient conditions such that G is a consistent QLT but only for QLTs such that $||g||_\infty < 1$. The first and the third case of this theorem can not be satisfied. Indeed, in these two cases we have the condition $n^2 - nb + a \geq 0$ and we have assumed that $b^2 - 4a < 0$ so that $n^2 - nb + a$ is always positive. The second case corresponds to the conditions

$$n - b \leq 0, \quad n^2 - nb + a > 0, \quad -1 \leq 0 \text{ and } -1 - n \leq 0$$

which is equivalent to $-1 \leq n \leq b$. If we choose n such that $||g||_\infty < 1$, we have

$$||G||_\infty = \tfrac{1}{a} max(b - n + n^2 - nb + a; 1 + |n|) < 1$$

$$\Leftrightarrow \qquad \begin{cases} n^2 - (b+1)n + b < 0 \ (1) \\ 1 + |n| < a \ (2) \end{cases}$$

If $b \leq 2$, the conditions $-1 \leq n \leq b$ and $n^2 - (b+1)n + b < 0$ are conflicting. But if $b > 2$, the condition (1) is satisfied for $1 < n < b$ and the condition (3) $1 + n < a$ can always be satisfied (because $b^2 < 4a$).

Remark 2. when $b = 2$ the QLT associated with the integer division is of norm 1. In [7] the conditions for such a QLT to be a consistent QLT are also given, we can choose $n = 1$ or $n = 2$ to obtain a consistent QLT and so to define a numeration system. In the same way, if $n = b$ or $1 + n = a$ the QLT is of norm 1; the conditions given in [7] will determine the conditions such that $(\mathcal{D}, \beta)$ is a numeration system.

Examples: let $\beta^2 + 3\beta + 3 = 0$, and $\beta_1 = 1 + \beta$, the QLT corresponding to the integer division by β is defined by the matrix $\frac{1}{3}\begin{pmatrix} -2 & 1 \\ -1 & -1 \end{pmatrix}$ and the set of digits is $\mathcal{D} = \{0, -1, -2 - \beta\}$, if we choose $\beta_1 = 2 + \beta$, the QLT corresponding to the integer division by β is defined by the matrix $\frac{1}{3}\begin{pmatrix} -1 & 1 \\ -1 & -2 \end{pmatrix}$ and the set of digits is $\mathcal{D} = \{0, -1, -2\}$.

5 Quasi-linear transformations and fractals

5.1 Border of tiles in 2-dimension

Gilbert [6] proved that if $(\beta, \mathcal{D})$ is a valid base for $\mathbb{Z}[\beta]$, then every complex number $c \in \mathbb{C}$ has an infinite representation (not necessarily unique) in the form:

$$c = \sum_{j=-\infty}^{p} c_j \beta^j, \; c_j \in \mathcal{D}.$$

Let us consider the set of complex numbers with zero integer part in this numeration system (also called "fundamental domain" in the literature [12]), that is to say the set $\mathcal{K}$ of complex numbers z such that:

$$c = \sum_{j=-\infty}^{-1} c_j \beta^j, \; c_j \in \mathcal{D}.$$

In [5] Gilbert determined the fractal dimension of the border of this set considering the base $\beta = -n + i$ with $n \in \mathbb{N}$ and $\mathcal{D} = 0, 1, \ldots, n^2$. Note $K^p = \left\{ c \in \mathbb{C} | c = \sum_{j=1}^{p} \frac{c_j}{\beta^j} \right\}$, the set $\mathcal{K}$ is the limit of K_p when p tends toward infinity.

Denote ρ and θ the module and argument of β: $\beta = \rho e^{i\theta}$. We have $\rho = \sqrt{\delta}$ where δ is the determinant of the matrix associated to the division, and so:

$$\begin{aligned} K^p &= \left\{ c \in \mathbb{C} | c = \sum_{j=1}^{p} \frac{c_j}{\beta^j} \text{ with } c_j \in \mathcal{D} \right\} \\ &= \left\{ c \in \mathbb{C} | c = \frac{1}{\beta^p} \sum_{j=1}^{p} c_j \beta^{p-j} \text{ with } c_j \in \mathcal{D} \right\} \\ &= \frac{1}{\beta^p} \left\{ c \in \mathbb{C} | c = \sum_{j=0}^{p-1} c_j \beta^j \text{ with } c_j \in \mathcal{D} \right\} \end{aligned}$$

If β is a Gaussian integer $\mathcal{D} = P_O$, if β is an algebraic integer

$$\mathcal{D} = \{x = x_1 + x_2\beta_1 | (x_1, x_2) \in P_O\}\,.$$

In [8] we studied particular determinantal QLTs associated with matrices of the form $\frac{1}{a^2+b^2}\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ and showed how the border of the p-tiles of these QLTs can be generated by a substitution. This study has been generalized to every 2D determinantal QLT, so we define substitutions that generate the border of the p-tiles P_O^p. We don't give the method to generate the substitutions in his paper, it can be found in []. The susbtitution associated with the QLT (associated with the numeration system), determines the border of P_O. If at each step of the substitution we divide the length of these segments $\sqrt{\delta}$, we obtain the border of a fractal set noted $\mathcal{F}$ (which corresponds exactely to the fundamental domain in case of Gaussian integers). Let design by N_p the number of segments of P_O^p, the susbtitution allows to determine N_p. Using the counting box dimension we can determine the fractal dimension of the border which is given by $d = \lim_{p\to+\infty} \frac{\log N_p}{\log \delta^{\frac{p}{2}}}$. For the examples in section 3 we find a fractal dimension $d = 1,303$ for figure 4 and $d = 1,5236$ for figure 3 (which conforms to the result found in [5]).

5.2 Fractals of $\mathbb{Z}^n$

Consider the recurrence given in proposition 2 which allows the construction of $P_O^n \in \mathbb{Z}^n$. If we apply this recurrence but starting with a subset P'_O of P_O, that is to say that we define

$$P_O'^{p+1} = \bigcup_{X\in P'_O} \mathcal{T}_{(m\widehat{A}^T)^p X} P_O'^p = \bigcup_{X\in P_O'^p} \mathcal{T}_{\left(\widehat{mA^T}\right) X} P'_O$$

Property 2. Let denote N_p the number of points of $P_O'^p$ and N the number of points removed from P_0 to obtain P'_O. We have $N_p = (m^n\delta^{n-1} - N)^p$.

Proof. In property 3 we proved that the number of points of P_O equals $m^n\delta^{n-1}$ therefore $N_1 = m^n\delta^{n-1} - N$. Suppose that $N_{p-1} = (m^n\delta^{n-1} - N)^{p-1}$, we have $P_O'^{p+1} = \bigcup_{X\in P'_O} \mathcal{T}_{(m\widehat{A}^T)^p X} P_O'^p$, it follows that $N_p = N_1 N_p = (m^n\delta^{n-1} - N)(m^n\delta^{n-1} - N)^p = (m^n\delta^{n-1} - N)^{p+1}$. □

Like above, at each step of the recurrence, we divide the size of the points by $m\delta^{\frac{n-1}{n}}$. We then obtain a fractal whose box-counting dimension is given by

$$d = \lim_{p\to\infty} \frac{\log(m^n\delta^{n-1} - N)^p}{\log((m\delta^{\frac{n-1}{n}})^p)} = \frac{\log(m^n\delta^{n-1} - N)}{\log(m\delta^{\frac{n-1}{n}})}$$

If we consider numeration systems, P'_O corresponds to a subset $\mathcal{D}'$ of the set of digits $\mathcal{D}$, so that the fractal obtained corresponds to the set of numbers with zero integer part and whose decomposition uses only the digits of $\mathcal{D}'$.

Example 1. In figures 5,6 and 7 we see P_O and the fractal generated.The black points of P_0 are removed to obtain P'_O (which corresponds to the grey points). The QLT in figure 5 is defined by $\frac{1}{9}\begin{pmatrix} 0 & -3 \\ 3 & 0 \end{pmatrix}$, the fractal dimension of the set is $\frac{\log(7)}{\log(3)} = 1,7712$, it corresponds to the numeration system $(-3i, \{u + iv|u = 0, 1, 2\ v = 0, -1, -2\})$ and represents the set of numbers with zero integer part and whose decomposition don't use the digits 0 and $2-2i$. In figure 6 the QLT is defined by $\frac{1}{13}\begin{pmatrix} -2 & 3 \\ -3 & -2 \end{pmatrix}$, the fractal dimension of the set is $2\frac{\log(8)}{\log(13)} = 1,6214$, it corresponds to the numeration system $(-2+3i, \{0, -1, -2, -3, -4, -1+i, -2+i, -3+i-4+i, -2+2i, -3+2I, -2-i, -3-i\})$ and represents the set of numbers with zero integer part and whose decomposition don't use the digits $0, -4, -1+i, -3+2i$ and $-2-2i$. In figure 7 the QLT is defined by $\frac{1}{9}\begin{pmatrix} 2 & 3 \\ -2 & 1 \end{pmatrix}$ and the fractal dimension of the set is $\frac{2\log(5)}{\log(8)} = 1,5479$.

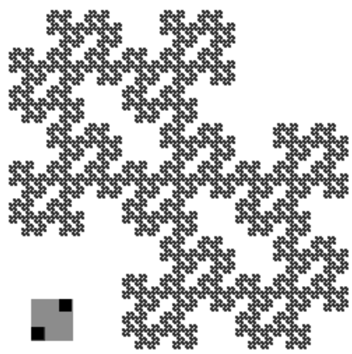

Fig. 5. Fractal associated with numeration systems

6 Conclusion

We have seen relations that exists between QLTs, substitutions, numeration systems and fractals. Quasi-linear transformations generates tilings of $\mathbb{Z}^n$ and n-dimensional fractals,we give the fractal dimension of these fractals. We defined substitutions that generate the border of the tiles in 2 dimension and allow to compute the fractal dimension of the border. In a future work we will show how these results can be extended in dimension n. We also used consistent 2D-QLTs (see definition 1) to define new numeration systems of $\mathbb{Z}[\beta]$ where β is an agebraic integer of order 2. In [2], the author studied fractals associated with shift radix

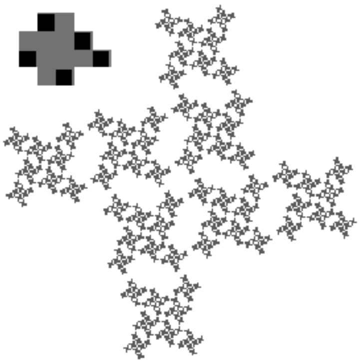

Fig. 6. Another fractal associated with numeration systems

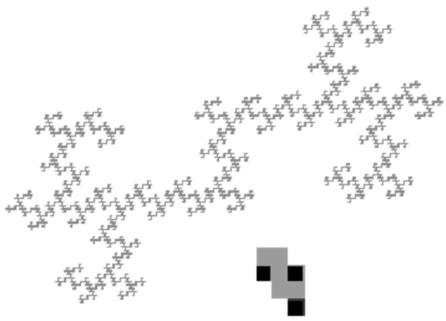

Fig. 7. Fractal associated with QLT

systems (which generalize numeration systems). Some of these fractals can be obtained usign QLTs, one of our future works is to study the relations between QLTs and these numeration systems.

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