Quasi-linear transformations, substitutions, numeration systems and fractals

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Abstract. In this paper we will define relations between quasi-linear transformations, substitutions, numeration systems and fractals. A Quasi-Linear Transformation (QLT) is a transformation on \( \mathbb{Z}^n \) which corresponds to the composition of a linear transformation with an integer part function. We will first give some theoretical results about QLTs. We will then point out relations between QLTs, substitutions, numeration systems and fractals. These relations allow us to define substitutions, new numeration systems and fractals associated with them. With help of some properties of the QLTs we can give the fractal dimension of these fractals.

Keywords: Gaussian integers, numeration systems, discrete linear transformations, substitutions, fractals

1 Introduction.

Fractal tiles generated by numeration systems and substitutions has been widely studied, see for example [1],[2], [5]. In this paper we define discrete linear transformations called QLT which generate tilings of \( \mathbb{Z}^n \). The tilings studied here are self-similar, they allow us to generate \( n \)-dimensional fractals. In dimension two we define substitutions that generate the border of the tiles, and we point out relations between QLTs and numeration systems which allow us to define new numeration systems. These discrete linear transformations are also in relation with discrete lines [17],[1].

Let \( g \) be a linear transformation from \( \mathbb{Z}^n \) to \( \mathbb{Q}^n \), defined by a matrix \( \frac{1}{w}A \) where \( A \) is an integer matrix and \( w \) a positive integer. If we compose this transformation with an integer part function we obtain a Quasi-Linear Transformation (QLT) from \( \mathbb{Z}^n \) to \( \mathbb{Z}^n \). We will only consider the integer part function with a positive remainder. We will denote it \( \lfloor \cdot \rfloor \). If \( x \) and \( y \) are two integers, \( \lfloor \frac{x}{y} \rfloor \) denotes the quotient of the Euclidean division of \( x \) by \( y \). We denote \( G \) as the QLT defined by \( g \). Main research results on such transformations are listed in the first section of this paper.

A substitution on an alphabet \( \mathcal{A} \) is an application that associates each letter of \( \mathcal{A} \) with a word over the alphabet \( \mathcal{A} \). In the second section of this paper we consider sequences of tiles associated with QLTs of \( \mathbb{Z}^2 \). We define substitutions
that generate the border of these tiles. Each sequence of tiles converges (after renormalization) toward a tile with a fractal border whose fractal dimension is computed with help of the substitution.

Let us consider $\beta$ an algebraic number and $\mathcal{D}$ a set of elements of $\mathbb{Z}[\beta]$. $\beta$ is a valid base using the digits set $\mathcal{D}$ if every integer $c \in \mathbb{Z}[\beta]$ has a unique representation (decomposition) of the form: $c = \sum_{j=0}^{n} a_j \beta^j$, where $a_j \in \mathcal{D}$ and $n \in \mathbb{N}$. Then $(\beta, \mathcal{D})$ is also called a numeration system. The third section will point out some relations between QLTs and numeration systems when $\beta$ is a Gaussian integer ($\beta = a + ib$, with $a, b \in \mathbb{Z}$) or an algebraic integer of order 2 ($\beta^2 + b\beta + a = 0$ with $a, b \in \mathbb{Z}$). These relations and some properties of QLTs will allow us to generate new numeration systems.

In the fourth section we will see how QLTs generate fractals (some of them are in relation with substitutions and numeration systems). Their fractal dimension can be determined by the substitution rules associated with the QLT or directly using some properties of the QLT.

2 Quasi-linear transformations

In this section we recall definitions and results that can be found in [7], [10], [3], [4] and that are useful in the rest of the paper.

Definition 1 Let $g$ be a linear transformation from $\mathbb{Z}^n$ to $\mathbb{Q}^n$, defined by a matrix $\frac{1}{w}A$ where $A = (a_{i,j})_{1 \leq i,j \leq n}$ is an integer matrix and $w$ a positive integer. The Quasi-Linear Transformation or QLT associated with $g$ is the transformation of the discrete plane $\mathbb{Z}^n$ defined by the composition of $g$ with the greatest integer part function noted $\lfloor \rfloor$. The QLT is then noted $G$.

In the following we will denote $\delta = \det(A)$, where $\det(A)$ is the determinant of $A$, and we’ll assume that $\delta > 0$. The linear transformation defined by the matrix $\frac{1}{w}A$ extends to $\mathbb{R}^n$ and for each point $Y$ of $\mathbb{R}^n$, there exists a unique point $X$ of $\mathbb{R}^n$ such that $Y = g(X)$. This is not the same for a QLT. Indeed, each point of $\mathbb{Z}^n$ can have either none, one or several antecedents: the set of antecedents of $X \in \mathbb{Z}^n$ is then called tile with index $X$.

Definition 2 We call tile with index $X \in \mathbb{Z}^n$ and denote $P_{G,X}$ the set:

$$P_{G,X} = \{ Y \in \mathbb{Z}^n \mid G(Y) = X \}$$

Definition 3 We call $p$-tile or tile of order $p \in \mathbb{N}$, with index $X \in \mathbb{Z}^n$, and denote $P^p_x$ the set: $P^p_{G,X} = \{ Y \in \mathbb{Z}^n \mid G^p(Y) = X \}$ where $G^p$ denotes the transformation $G$ iterated $p$ times.

Definition 4 Two tiles are geometrically identical if one is the image of the other by a translation of an integer vector.
In this paper we focus on a particular type of QLT, called "m-determinantal QLT".

**Definition 5** A QLT defined by a matrix $\frac{1}{m}A$ such that $w = m \det(A)$ where $m$ is a positive integer, is called a m-determinantal QLT. A 1-determinantal QLT will be called a determinantal QLT.

In the following $G$ will always denote a m-determinantal QLT associated with the matrix $\frac{1}{m}A$, and $g$ the associated rational linear transformation. To simplify the notations, tiles $P_{G,X}$ and $P^p_{G,X}$ will be noted $P_X$ and $P^p_X$. Moreover $(0,0,\ldots,0) \in \mathbb{Z}^n$ will be simply noted $O$.

**Proposition 1.** The tiles generated by a m-determinantal QLT $G$ are all geometrically identical. For all $Y \in \mathbb{Z}^n$, $P_Y = T_{m\widetilde{A}^T}P_O$ where $T_v$ represents the translation of vector $v$ and $\widetilde{A}^T$ is the transpose of the cofactor matrix of $A$.

**Proof.** If $\widetilde{A}^T$ is the transpose the cofactor matrix of $A$, we have $A^{-1} = \frac{1}{\det(A)} \widetilde{A}^T$. Let us consider $Y \in \mathbb{Z}^n$, $X \in P_O$ and $X' = m\widetilde{A}^T Y + X$. We then have: $G(X') = \left[\frac{1}{m}AX + \frac{1}{m}Am\widetilde{A}^TY\right] = \left[\frac{1}{m}AX + Y\right] = G(X) + Y$. It follows that for each point $X \in P_O$, $X' = m\widetilde{A}^T Y + X$ is a point of $P_Y$. We can now conclude that $P_Y = T_{m\widetilde{A}^T}P_O$. □

**Proposition 2.** The $p$-tiles generated by a m-determinantal QLT are all geometrically identical and can be generated recursively by translations of $P_O$. More precisely, if $T_v$ refers to the translation of the vector $v$ we have, for all $p \geq 1$:

\[
\begin{align*}
P_Y^p &= T_{(m\widetilde{A}^T)^p}P_O^p \quad \text{(1)} \\
P_O^{p+1} &= \bigcup_{X \in P_O} T_{(m\widetilde{A}^T)^p}X P_O^p = \bigcup_{X \in P_O^p} T_{(m\widetilde{A}^T)^p}X P_O \quad \text{(2)}
\end{align*}
\]

**Proof.** If $p = 1$, the equality (1) has been proved in proposition 1. By definition $P_O^1 = \{X \in \mathbb{Z}^n | X \in P_G\}$ and $Y' \in P_O$ so

\[
P_O^p = \bigcup_{Y' \in P_O} P_{Y'} = \bigcup_{Y' \in P_O} T_{\widetilde{A}^T Y'} P_O
\]

which proves the equality (2) for $p = 1$.

Let us now assume the equalities (1) and (2) to be true with $p = k$ and prove that they are true for $p = k + 1$. 

\begin{align*}
P^k_{Y+1} &= \bigcup_{X \in P_Y} P^k_X \\
&= \bigcup_{X' \in P_Y} P^k_{X' + mA^T Y} \\
&= \bigcup_{X' \in P_Y} T^{(mA^T)^k} P^k_{X' + mA^T Y} \\
&= T^{(mA^T)^k + 1} Y \bigcup_{X' \in P_Y} T^{(mA^T)^k} P^k_{X' + mA^T Y} \\
&= T^{(mA^T)^k + 1} P^k_{Y+1} \quad \text{(recurrence hypothesis)}
\end{align*}

Moreover:
\begin{align*}
P^k_{O+1} &= \bigcup_{X \in P_O} P^k_X \\
&= \bigcup_{X \in P_O} T^{(mA^T)^k} P^k_X \\
&= T^{(mA^T)^k + 1} P^k_{O+1} \quad \text{(recurrence hypothesis)}
\end{align*}
that the number of points \( P' \) equals \( \delta^{m-2} \). We have:

\[
(x_1, x_2, \ldots, x_n) \in P_D \\
0 \leq b_{1,1}x_1 + b_{1,2}x_2 + \ldots + b_{1,n}x_n < \delta \\
0 \leq b_{2,1}x_2 + \ldots + b_{2,n}x_n < \delta \\
0 \leq b_{n,n}x_n < \delta \\
0 \leq b_{1,1}x_1 < \delta - \frac{b_{2,2}x_2+\ldots+b_{2,n}x_n}{b_{1,1}} < b_{1,1} \\
0 \leq \frac{b_{n,n}x_n}{\delta} < \frac{b_{1,1}}{b_{1,1}} \\
(x_1, x_2, \ldots, x_n) \in P'_{i_2, i_3, \ldots, i_n} \text{ with } i_k = 0, 1, \ldots, b_{1,1}
\]

The number of solutions for \( x_1 \) equals \( \delta' \) and each tile \( P'_{i_2, i_3, \ldots, i_n} \) contains \( \delta'^{n-2} \), therefore we have \( \delta' \delta^{m-2} b_{1,1} = \delta^{n-1} \) points.

- If \( m > 1 \), we have:

\[
(x_1, x_2, \ldots, x_n) \in P_D \\
0 \leq a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,n}x_n < m\delta \\
0 \leq a_{2,1}x_2 + \ldots + a_{2,n}x_n < m\delta \\
0 \leq a_{n,1}x_1 + a_{n,2}x_2 + \ldots + a_{n,n}x_n < m \\
0 \leq a_{1,1}x_1 + a_{2,2}x_2 + \ldots + a_{n,n}x_n < m \\
(x_1, x_2, \ldots, x_n) \in P'_{i_1, i_2, \ldots, i_n} \text{ with } i_1, i_2, \ldots, i_n = 0, 1, \ldots, m - 1
\]

But each \( P'_{i_1, j} \) contains \( \delta^{n-1} \) points, it follows that \( P_D \) contains \( \delta^{n-1} m^n \) points.

We conclude that the number of tiles of \( P_D \) equals \( \delta^{n-1} m^n \).

Let now suppose that the number of points of \( P''_D \) equals \( \delta^{(n-1)}m^{np} \), we have \( P''_{D,p+1} = \bigcup_{X \in P_D} T_{(m,\lambda^p)}X P''_D \) (see proposition 2), it follows that the number of points of \( P''_{D,p+1} \) equals \( \delta^{(n-1)}m^{np}\delta^{n-1}m^n = \delta^{(p+1)(n-1)}m^{n(p+1)} \).

It is well known that if \( g \) is a contracting linear transformation of \( \mathbb{R}^n \) then \( g \) has the origin as unique fixed point and for each \( X \in \mathbb{R}^n \) the sequence \( g^n(X) \) tends toward this fixed point. We will say that a QLT \( G \) is contracting if \( g \) is contracting. But such a QLT has not necessarily a unique fixed point. The behavior under iteration of 2D contracting QLTs has been studied in [7], [16], [14] and [15]. The following definition and theorem will be used to define new numeration system.

**Definition 6** A consistent Quasi-Linear Transformation is a QLT which has the origin as unique fixed point such that for each discrete point \( Y \) the sequence \( (G^n(Y))_{n \geq 0} \) tends toward this unique fixed point.
Consider a 2D-QLT defined by \( \frac{1}{2} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \), the infinite norm of \( g \) is then

\[
\|g\|_\infty = \frac{1}{\omega} \max(|a| + |b|, |c| + |d|).
\]

The two following theorems, proved in [7], states conditions such that a 2D-QLT is a consistent QLT.

**Theorem 1.** Let \( G \) be a QLT such that \( \|g\|_\infty < 1 \), \( G \) is a consistent QLT if and only if one of the three following conditions is verified:

1. If \( |a| + |b| = \omega \) and \( |c| + |d| < \omega \)
   - \( a = 0, b < 0, c > 0 \) and \( c - d \geq 0 \)
   - \( a > 0, a + b \leq 0, c > 0 \) and \( d \leq 0 \)
   - \( a \leq 0, b > 0, c \leq 0 \) and \( c + d \leq 0 \)
   - \( a < 0, b < 0, c \geq 0 \) and \( d \leq 0 \)
2. If \( |a| + |b| < \omega \) and \( |c| + |d| = \omega \)
   - \( d = 0, c < 0, b > 0 \) and \( b - a \geq 0 \)
   - \( d > 0, c + d \leq 0, b > 0 \) and \( a \leq 0 \)
   - \( d \leq 0, c > 0, b \leq 0 \) and \( a + b \leq 0 \)
   - \( d < 0, c < 0, b \geq 0 \) and \( a \leq 0 \)
3. If \( |a| + |b| = \omega \) and \( |c| + |d| = \omega \)
   - \( \omega = -a + b = -c + d, a + b \leq 0, c + d \leq 0 \) and \( abc \neq 0 \)
   - \( \omega = -b = c + d, c - d \geq 0, \) and \( bcd \neq 0 \)
   - \( \omega = a - b = c - d, a + b \leq 0, c + d > 0 \) and \( bcd \neq 0 \)
   - \( \omega = b = -c + d, c + d \leq 0 \) and \( bcd \neq 0 \)
   - \( \omega = a - b = c - d, a - b < 0, c + d > 0 \) and \( abc \neq 0 \)
   - \( \omega = -a \pm b = c - d, a + b \leq 0, c + d \leq 0 \) and \( bcd \neq 0 \)
   - \( \omega = a + b = -c, a - b \leq 0 \) and \( abc \neq 0 \)
   - \( \omega = a - b = -c + d, a + b > 0, c + d \leq 0 \) and \( abc \neq 0 \)
   - \( \omega = \pm a - b = c, a + b \leq 0 \) and \( abc \neq 0 \)
   - \( \omega = -a + b = -c - d, a + b > 0, c - d \geq 0 \) and \( abc \neq 0 \)

3 Quasi-linear transformations and substitutions

In this section we only consider 2D m-determinantal QLTs.

A substitution on an alphabet \( \mathcal{A} \) is an application that associates each letter of \( \mathcal{A} \) with a word over the alphabet \( \mathcal{A} \). In this section we will define a substitution over an alphabet \( \mathcal{A} \) of 16 letters. The words generated by these substitutions
corresponds to the Freeman code of the border of the tiles. The substitutions will be used to determine the fractal dimension of the border.

Let us remember that a discrete line is the set of points

\[ D(a, b, \omega, i) = \left\{ (x, y) \in \mathbb{Z}^2 \mid \frac{ax + by}{\omega} = i \right\} \]

where \( a, b, i, \omega \in \mathbb{Z} \) and \( \omega > 0 \) (see [17]).

A discrete line such that \( \omega = \max(\{a, b\}) \) is called a discrete naive line and will be noted \( N(a, b, i) \). The geometrical structure of a naive line is periodic, we will use this periodicity to determine the border of a tile.

The tile with index \((i, j)\) is the intersection of the two discrete lines \( D(a, b, m\delta, i) \) and \( D(c, d, m\delta, j) \). These discrete lines have been studied (see [17]) for coprime coefficients, so we note \( t = \gcd(a, b), t' = \gcd(c, d), a' = a/t, b' = b/t, c' = c/t' \) and \( d' = d/t' \). The study of discrete lines \([17]\) and QLTs \([15],[16],[7],[9]\) induce the following properties.

**Property 1.**

1. \( P_{O} \) is geometrically identical to \( P_{i,j}, \forall i, j \in \mathbb{Z} \);
2. \( P_{O} \) contains \( m^2 \delta \) points;
3. \( P_{O} \) contains exactly \( ml \) translates of the period of the naive line \( N(a', b', 0) \);
4. \( P_{O} \) contains exactly \( ml' \) translates of the period of the naive line \( N(c', d', 0) \);
5. \( P_{O} \) has exactly 6 neighbours (or 4 neighbours if \( a = d = 0 \) or \( b = c = 0 \))
among the sets \( P_{i,j} \) with \( i, j \in \{0, \pm 1\} \).

Figure 1 corresponds to the tile \( P_{O} \) associated to the QLT of matrix \( \frac{1}{34} \begin{pmatrix} -3 & 5 \\ -5 & -3 \end{pmatrix} \),
we see also the period of the naive lines defined by \( N(-3, 5, 0) \) and \( N(-5, -3, 0) \).
The border of \( P_{O} \) can be split into six parts which are the frontiers between \( P_{O} \)
and its six neighbours. These six parts are noted \( I_{i} \) with \( i = 0, 1, \ldots, 5 \) as shown in figure 1. The arrows indicate the direction. With each border is associated a word whose letters correspond to the direction of the successive segments (\( U = \text{up}, D = \text{down}, L = \text{left}, R = \text{right} \)). In our example: \( I_{0} = \text{RURURRR}, I_{1} = \text{U}, I_{2} = \text{LULULUL}, I_{3} = \text{LLDDLDDL}, I_{4} = \text{D} \) and \( I_{5} = \text{DRDRD} \).

We now explain how to obtain \( I_{i} \) for \( i = 0, 1, \ldots, 5 \). We limit the study to the case \( a' < 0, b' > 0, |b'| \geq |a'|, c' < 0, d' < 0 \) and \( |c'| \leq |d'| \); the other cases are similar and can be obtained by symmetry.

- \( I_{0} \) is the bottom of \( ml \) times the period of the naive line \( \frac{a'x + b'y}{b'} = 0 \).

Réveilles [17] proved that the period of a naive line contains long and short steps. We will note them \( l \) and \( s \). We will also note \( q = \left\lfloor \frac{b'}{a'} \right\rfloor \). The steps \( s \) and \( l \) have the respective length \( q \) and \( q + 1 \). Let \([0, q_1, q_2, \ldots, q_n]\) be the continued fraction development of \( \frac{a'}{b'} \). The period of the naive line \( \frac{a'x + b'y}{b'} = 0 \) can be obtained by the following recurrence (Réveilles [17]):
Fig. 1. Border of $P_0$.  

\[
\mu_1 = s \\
\mu_2 = ls^{q_2 - 1}
\]
if \(i + 1\) is even, \(\mu_{i+1} = \mu_i s^{q_i+1}\)
if \(i + 1\) is odd, \(\mu_{i+1} = \mu_i s^{q_i+1} \mu_i^{-1}\)

\(\mu_0\) is the period.

- As \(\Gamma_0\) is the bottom of \(mt\) times this period, the word corresponding to a short (resp. long) step will be \(R^t\) (resp. \(R^{t+1}\)). These successive steps give rise to a vertical segment which corresponds to the letter \(U\). The following recurrence determines then \(\Gamma_0\):
   \[
   \nu_1 = R^p \\
   \nu_2 = R^{p+1}(U R^p)s^{q_2-1}
   \]
if \(i + 1\) is even, \(\nu_{i+1} = \nu_i - 1 \nu_i^{q_i+1}\)
if \(i + 1\) is odd, \(\nu_{i+1} = \nu_i^{q_i+1} \nu_i^{-1}\)
Finally, \(\Gamma_0 = (\nu_0)^{mt}\).

- \(\Gamma_5\) is the bottom of \(mt'\) times the period of the digital straight line \(\left\lfloor \frac{c'x + d'y}{c'} \right\rfloor\) without the first point of the period which is a neighbour of \(P_{1,1}\), and corresponds to the border \(\Gamma_3\). Again, the recurrence given by Réveilles [17] to determine this period induces the recurrence that permits to obtain \(\Gamma_5\).
To obtain $\Gamma_3$ (resp. $\Gamma_2$) we remark that they are the "inverse" of $\Gamma_0$ (resp. $\Gamma_5$): If we read $\Gamma_0$ (resp. $\Gamma_5$) from right to left and replace $U$ by $D$ and $R$ by $L$ (and vice versa) we obtain $\Gamma_3$ (resp. $\Gamma_2$). Moreover, independently of $a$, $b$, $c$ and $d$, $\Gamma_4 = D$ and $\Gamma_1 = U$. Finally $\Gamma_0 \Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4 \Gamma_5$ corresponds to the border of $P_O$.

We now consider p-tiles, we have (see proposition 2) $P_O^p = \bigcup_{X \in P_O^{p-1}} T_{(m \times r)} X P_O$: we can obtain $P_O^p$ from $P_O^{p-1}$ by replacing each point $X \in P_O^{p-1}$ by $P_X$ which is the image of $P_O$ by a given translation (see figure 2). Suppose that the border of $P_O^{p-1}$ contains the letter $U$, that means that $P_O^{p-1}$ contains a point of coordinates $(i,j)$ such that the point $(i+1,j)$ doesn’t belong to $P_O^{p-1}$ (see figure 2). The letter $U$ in the border of $P_O^{p-1}$ generates the sequence $\Gamma_3$ (border between $P_{i,j}$ and $P_{i+1,j}$) in $P_O^p$. In the same manner, the letters $L$, $D$ and $R$ respectively generate the sequences $\Gamma_5$, $\Gamma_0$ and $\Gamma_2$. But this is not sufficient to generate the border of $P_O^p$. Indeed, suppose that the border of $P_O^p$ contains the sequence $LU$, this sequence corresponds to the border between three points $(i,j)$, $(i-1,j)$ and $(i-1,j+1)$ which belong to $P_O^{p-1}$ and the point $(i,j+1)$ which does not belong to $P_O^{p-1}$ (see figure 2), so the sequence $LU$ generates the sequence $\Gamma_5 \Gamma_4 \Gamma_3$ in the border of $P_O^p$. We can see in figure 2 that the sequence $LU$ (resp. $LU'$ and $LD$) generates the sequence $\Gamma_3 \Gamma_4 \Gamma_5$ (resp. $\Gamma_5 \Gamma_4 \Gamma_3$ and $\Gamma_5 \Gamma_0$). So we have to add 12 letters (one letter for each of the 12 sequences of two letters) and to examine which word has to be associated with these letters. Call $L_1$, $L_2$, ..., $L_{12}$ these letters which will be respectively between $LL$, $DL$, $UL$, $RR$, $DR$, $UR$, $UU$, $LU$, $RU$, $DD$, $LD$ and $RD$. First we have to modify $\Gamma_3$ by adding these 12 letters, we denote $\Gamma_3'$ the new words. In the case studied here $(a' < 0$, $b' > 0$, $|b'| \geq |a'|$, $c' < 0$, $d' < 0$ and $|c'| \leq |d'|$) $\Gamma_3'$ always begins and ends with the letter $R$ and $\Gamma_4' = D$, so the sequence $\Gamma_3' \Gamma_4' \Gamma_5'$ becomes $\Gamma_3' \Gamma_4' \Gamma_5' \Gamma_5' \Gamma_5'$. Now determine the word generated by $L_1$: $LL_1L$ generates $\Gamma_3' \Gamma_4' \Gamma_5' \Gamma_5' \Gamma_5'$, $L$ generates $\Gamma_3'$ so $L_1$ will generate $L_1 \Gamma_3'$ $L_5$. In the same way we examine all other letters and we obtain a substitution over an alphabet of 16 letters.
In figures 3 and 4 we can see two examples of borders of $P^n_O$. Their corresponding matrix and substitutions are:

<table>
<thead>
<tr>
<th>Figure 4 matrix ( \begin{pmatrix} 0 &amp; -1 \ 1 &amp; -1 \end{pmatrix} )</th>
<th>Figure 3 matrix ( \begin{pmatrix} -1 &amp; 1 \ -1 &amp; -1 \end{pmatrix} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( D \rightarrow L )</td>
<td>( D \rightarrow R )</td>
</tr>
<tr>
<td>( U \rightarrow R )</td>
<td>( U \rightarrow L )</td>
</tr>
<tr>
<td>( L \rightarrow UL_6RL_9U )</td>
<td>( L \rightarrow R )</td>
</tr>
<tr>
<td>( R \rightarrow DL_2LL_{11}D )</td>
<td>( R \rightarrow L )</td>
</tr>
<tr>
<td>( L_1 \rightarrow L_6RL_9 )</td>
<td>( L_1 \rightarrow L_{12}DL_5 )</td>
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<tr>
<td>( L_2 \rightarrow L_8 )</td>
<td>( L_2 \rightarrow L_4 )</td>
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<td>( L_3 \rightarrow L_{11}DL_5 )</td>
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<td>( L_4 \rightarrow L_8UL_3 )</td>
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<td>( L_7 \rightarrow L_4RL_4 )</td>
<td>( L_7 \rightarrow L_{11}DL_2 )</td>
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<td>( L_8 \rightarrow L_{12}DL_2 )</td>
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<td>( L_9 \rightarrow L_1 )</td>
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<td>( L_{10} \rightarrow L_9UL_6 )</td>
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<tr>
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<td>( L_{11} \rightarrow L_4 )</td>
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<tr>
<td>( L_{12} \rightarrow L_2LL_1 )</td>
<td>( L_{12} \rightarrow L_8UL_6 )</td>
</tr>
</tbody>
</table>

Fig. 3. Border of $P^{12}_O$.

4 Quasi-linear transformations and numeration systems

Let $\beta$ denote a complex number and $\mathcal{D}$ a finite set of elements of $\mathbb{Z}[\beta]$. $(\beta, \mathcal{D})$ is a valid base for $\mathbb{Z}[\beta]$ if each element $c$ of $\mathbb{Z}[\beta]$ can be written uniquely in the
form

\[ c = c_0 + c_1 \beta + c_2 \beta^2 \ldots + c_n \beta^n \]

with \( c_i \in D \) and \( n \in \mathbb{N} \); the length of the decomposition is then \( n + 1 \). We also say that \((\beta, D)\) is a numeration system of \( \mathbb{Z}[\beta] \) and \( D \) is the set of digits of this numeration system.

### 4.1 Case of Gaussian integers

The results below can be found in [8]: they allow us to determine numeration systems of \( \mathbb{Z}[i] \) by considering \( \beta = a + ib \) a Gaussian integer and \( D \) a set of Gaussian integers.

**Definition 7** Let \( c = x + iy \) and \( \beta = a + ib \) be two Gaussian integers. The integer division of \( c \) by \( \beta \), noted \( \left\lfloor \frac{c}{\beta} \right\rfloor \), is defined by:

\[
\left\lfloor \frac{c}{\beta} \right\rfloor = \left\lfloor \frac{ax + by}{a^2 + b^2} \right\rfloor + i \left\lfloor \frac{-bx + ay}{a^2 + b^2} \right\rfloor.
\]

This division corresponds to the usual division of complex numbers composed with the integer part function: so we have the following relation with QLTs.

**Proposition 4.** Let \( \beta = a + ib \) and \( c = x + iy \) be two Gaussian integers and let \( c' = x' + iy' = \left\lfloor \frac{c}{\beta} \right\rfloor \). The point \((x', y')\) is then the image of the point \((x, y)\) by the QLT \([g_\beta]\) defined on \( \mathbb{Z}^2 \) by:

\[
[g_\beta] : \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2
\]

\[
(x, y) \mapsto \begin{cases} 
  x' = \left\lfloor \frac{ax + by}{a^2 + b^2} \right\rfloor \\
  y' = \left\lfloor \frac{-bx + ay}{a^2 + b^2} \right\rfloor
\end{cases}
\]
Theorem 3. Let $\beta = a + ib$ be a Gaussian integer and $\mathcal{D}$ the set of Gaussian integers $c$ such that $\left\lfloor \frac{c}{\beta} \right\rfloor = 0$, the three following properties are then equivalent:

1. $(\beta, \mathcal{D})$ is a numeration system,
2. The QLT $\lfloor g \beta \rfloor$ is a consistent Quasi-Linear Transformation,
3. $a \leq 0$ and $|a| + |b| > 1$.

Remark 1. Consider the QLT $\lfloor g_\beta \rfloor$ and note $P_O$ and $P_p_O$ the tiles defined by this QLT. The set of digits is given by $D = \{c = x + iy | \lfloor c/\beta \rfloor = 0 \} = \{c = x + iy | (x, y) \in P_O \}$. In [7] and [9] we can find an algorithm that allows to determine $P_O$.

4.2 Case of algebraic integers of order 2

Now we consider $\beta$ an algebraic integer such that $\beta^2 + b\beta + a = 0$ with $a, b \in \mathbb{Z}$. We only consider the case where $\beta$ is a complex number, that is to say $b^2 - 4a < 0$. We will define numeration systems of $\mathbb{Z}[\beta]$ where the digits are elements of $\mathbb{Z}[\beta]$. First we will define an integer division by $\beta$, this integer division corresponds to a QLT that will define the digits of the numeration system. As in preceding section, the QLT has to be a consistent QLT. We will define the division by using another base of $\mathbb{Z}[\beta]$, the QLT associated to the division will depend on these base, a good choice of the base will raise to a consistent QLT and so defined a numeration system.

Lemma 1. Let $\beta_1 = n + \beta$ with $n \in \mathbb{Z}$, we have $\mathbb{Z}[\beta] = \mathbb{Z}[\beta_1]$.

Proof. Let $x = x_1 + x_2\beta \in \mathbb{Z}[\beta]$, we have:

$\exists x_1, x_2 \in \mathbb{Z} | x = x_1 + x_2\beta$

$\Rightarrow \exists x_1, x_2 \in \mathbb{Z} | x = x_1 - x_2n + x_2\beta_1$

$\Rightarrow \exists x_1', x_2' \in \mathbb{Z} | x = x_1' + x_2'\beta_1$

$\Rightarrow x \in \mathbb{Z}[\beta_1]$

Let now define the integer division using this base. We have $\beta^2 + b\beta + a = 0$, so $\frac{\beta + b}{a} = \frac{-\beta_1 + n - b}{a}$ and $\beta_1^2 = \beta_1(2n - b) - n^2 + nb - a$. Let $x = x_1 + x_2\beta_1$, we have:

$$\frac{x}{\beta} = \frac{(x_1 + x_2\beta_1)(-\beta_1 + n - b)}{a}$$

$$= \frac{x_2(\beta_1(b - 2n) + n^2 - nb + a) + \beta_1(x_2(n - b) - x_1) + x_1(n - b)}{a}$$

$$= \frac{x_1(n - b) + x_2(n^2 - nb + a) + (x_1 - x_2n)\beta_1}{a}.$$

We define the integer division of $x$ by $\beta$ by the composition of the division above with the integer part function.
Definition 1. Let \( x = x_1 + x_2\beta_1 \) and \( G® \) the QLT defined by the matrix \( \frac{1}{a} \begin{pmatrix} n-b & a-nb+n^2 \\ -1 & -n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \). The quotient of the integer division of \( x \) by \( \beta \), noted \( \lfloor x / \beta \rfloor \), is defined by \( x_1' + x_2'\beta_1 \), where \((x_1', x_2') = G®(x_1, x_2) \).

The set of digits is

\[ \mathcal{D} = \{ x = x_1 + x_2\beta_1 \mid \lfloor x / \beta \rfloor = 0 \} = \{ x = x_1 + x_2\beta_1 \mid G(x_1, x_2) \in P® \} \]

where \( P® \) is the tile of order one associated to the QLT \( G® \).

Theorem 4. Let \( \beta \) be an algebraic integer such that \( \beta^2 + b\beta + a = 0 \) and \( \mathcal{D} \) the set of Gaussian integers \( c \) such that \( \lfloor c / \beta \rfloor = 0 \), the three following properties are then equivalent:

1. It exists \( n \) such that \((\beta, \mathcal{D})\) is a numeration system,
2. It exists \( n \) such that the QLT \( \lfloor g® \rfloor \) (defining the integer division) is a consistent QLT,
3. \( (b > 2) \) or \( (b = 2 \) and

Proof. \((\beta, \mathcal{D})\) is then a numeration if and only if for all \( c \in \mathbb{Z}[\beta] \), there exist \( c_0, c_1, \ldots, c_p \in \mathcal{D} \) such that

\[ c = c_0 + c_1\beta + \ldots + c_p\beta^p \]

Moreover, this decomposition has to be unique. It is easy to see that if this decomposition exists then \( c_i = x_i + y_i\beta_1 \) with \( \begin{pmatrix} x_i \\ y_i \end{pmatrix} = G® \begin{pmatrix} x \\ y \end{pmatrix} \) where \( x + y\beta_1 = c \). We conclude that the decomposition exists and is unique if there exists \( p \in \mathbb{N} \) such that \( G® \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \). Finally \((\mathcal{D}, \beta)\) is a numeration system if and only if \( G® \) is a consistent QLT. Theorem 1 gives the necessary and sufficient conditions such that \( G® \) is a consistent QLT but only for QLTs such that \( ||g®||_\infty < 1 \). The first and the third case of this theorem can not be satisfied. Indeed, in these two cases we have the condition \( n^2 - nb + a \geq 0 \) and we have assumed that \( b^2 - 4a < 0 \) so that \( n^2 - nb + a \) is always positive. The second case corresponds to the conditions

\[ n-b \leq 0, \quad n^2 - nb + a > 0, \quad -1 \leq 0 \] and \(-1 - n \leq 0 \)

which is equivalent to \(-1 \leq n \leq b \). If we choose \( n \) such that \( ||g®||_\infty < 1 \), we have

\[ ||G®||_\infty = \frac{1}{a} \max(b - n + n^2 - nb + a; 1 + |n|) < 1 \]

\[ \Leftrightarrow \begin{cases} n^2 - (b+1)n + b < 0 & (1) \\ 1 + |n| < a & (2) \end{cases} \]

If \( b \leq 2 \), the conditions \(-1 \leq n \leq b \) and \( n^2 - (b+1)n + b < 0 \) are conflicting. But if \( b > 2 \), the condition (1) is satisfied for \( 1 < n < b \) and the condition (3) \( 1 + n < a \) can always be satisfied (because \( b^2 < 4a \)).
Remark 2. when $b = 2$ the QLT associated with the integer division is of norm 1. In [7] the conditions for such a QLT to be a consistent QLT are also given, we can choose $n = 1$ or $n = 2$ to obtain a consistent QLT and so to define a numeration system. In the same way, if $n = b$ or $1 + n = a$ the QLT is of norm 1; the conditions given in [7] will determine the conditions such that $(D, \beta)$ is a numeration system.

Examples: let $\beta^2 + 3\beta + 3 = 0$, and $\beta_1 = 1 + \beta$, the QLT corresponding to the integer division by $\beta$ is defined by the matrix $\frac{1}{3} \begin{pmatrix} -2 & 1 \\ -1 & -1 \end{pmatrix}$ and the set of digits is $D = \{0, -1, -2 - \beta\}$, if we choose $\beta_1 = 2 + \beta$, the QLT corresponding to the integer division by $\beta$ is defined by the matrix $\frac{1}{3} \begin{pmatrix} -1 & 1 \\ -1 & -2 \end{pmatrix}$ and the set of digits is $D = \{0, -1, -2\}$.

5 Quasi-linear transformations and fractals

5.1 Border of tiles in 2-dimension

Gilbert [6] proved that if $(\beta, D)$ is a valid base for $\mathbb{Z}[\beta]$, then every complex number $c \in \mathbb{C}$ has an infinite representation (not necessarily unique) in the form:

$$c = \sum_{j=-\infty}^{\infty} c_j \beta^j, \quad c_j \in D.$$

Let us consider the set of complex numbers with zero integer part in this numeration system (also called "fundamental domain" in the literature [12]), that is to say the set $K$ of complex numbers $z$ such that:

$$c = \sum_{j=-\infty}^{-1} c_j \beta^j, \quad c_j \in D.$$

In [5] Gilbert determined the fractal dimension of the border of this set considering the base $\beta = -n + i$ with $n \in \mathbb{N}$ and $D = 0, 1, \ldots, n^2$. Note $K^p = \left\{ c \in \mathbb{C} | c = \sum_{j=1}^{p} \frac{c_j}{\beta^j} \right\}$ the set $K$ is the limit of $K_p$ when $p$ tends toward infinity.

Denote $\rho$ and $\theta$ the module and argument of $\beta$: $\beta = \rho e^{i\theta}$. We have $\rho = \sqrt{\delta}$ where $\delta$ is the determinant of the matrix associated to the division, and so:

$$K^p = \left\{ c \in \mathbb{C} | c = \sum_{j=1}^{p} \frac{c_j}{\beta^j} \right\}$$

$$= \left\{ c \in \mathbb{C} | c = \frac{1}{\beta^p} \sum_{j=1}^{p} c_j \beta^{-j} \right\}$$

$$= \frac{1}{\beta^p} \left\{ c \in \mathbb{C} | c = \sum_{j=0}^{p-1} c_j \beta^j \right\}$$
If $\beta$ is a Gaussian integer $D = P_O$, if $\beta$ is an algebraic integer
\[ D = \{x = x_1 + x_2\beta_1 | (x_1, x_2) \in P_O\}. \]

In [8] we studied particular determinantal QLTs associated with matrices of the form
\[ \frac{1}{s^2 + \tau^2} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \]
and showed how the border of the p-tiles of these QLTs can be generated by a substitution. This study has been generalized to every 2D determinantal QLT, so we define substitutions that generate the border of the p-tiles $P_O^p$. We don’t give the method to generate the substitutions in this paper, it can be found in [\]. The substitution associated with the QLT (associated with the numeration system), determines the border of $P_O$. If at each step of the substitution we divide the length of these segments $\sqrt{\delta}$, we obtain the border of a fractal set noted $F$ (which corresponds exactly to the fundamental domain in case of Gaussian integers). Let design by $N_p$ the number of segments of $P_O^p$, the substitution allows to determine $N_p$. Using the counting box dimension we can determine the fractal dimension of the border which is given by
\[ d = \lim_{p \to +\infty} \frac{\log N_p}{\log((m^n\delta^{n-1} - N)^p)} = \frac{\log(m^n\delta^{n-1} - N)}{\log(m\delta^{n-1})}. \]

If we consider numeration systems, $P_O^p$ corresponds to a subset $D'$ of the set of digits $D$, so that the fractal obtained corresponds to the set of numbers with zero integer part and whose decomposition uses only the digits of $D'$.
Example 1. In figures 5, 6 and 7 we see $P_O$ and the fractal generated. The black points of $P_0$ are removed to obtain $P'_O$ (which corresponds to the grey points).

The QLT in figure 5 is defined by $\begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix}$, the fractal dimension of the set is $\frac{\log(7)}{\log(3)} = 1.7712$, it corresponds to the numeration system $(-3i, \{u + iv | u = 0, 1, 2, v = 0, -1, -2\})$ and represents the set of numbers with zero integer part and whose decomposition don’t use the digits 0 and $2 - 2i$. In figure 6 the QLT is defined by $\begin{bmatrix} -2 & 3 \\ -3 & -2 \end{bmatrix}$, the fractal dimension of the set is $2^{\frac{\log(8)}{\log(13)}} = 1.6214$, it corresponds to the numeration system $(-2 + 3i, \{0, -1, -2, -3, -4, -1 + i, -2 + i, -3 + i - 4 + i, -2 + 2i, -3 + 2i, -2 - i, -3 - i\})$ and represents the set of numbers with zero integer part and whose decomposition don’t use the digits 0, $-4, -1 + i, -3 + 2i$ and $-2 - 2i$. In figure 7 the QLT is defined by $\begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$ and the fractal dimension of the set is $2^{\frac{\log(5)}{\log(8)}} = 1.5479$.

![Fig. 5. Fractal associated with numeration systems](image)

6 Conclusion

We have seen relations that exists between QLTs, substitutions, numeration systems and fractals. Quasi-linear transformations generates tilings of $\mathbb{Z}^n$ and $n$-dimensional fractals, we give the fractal dimension of these fractals. We defined substitutions that generate the border of the tiles in 2 dimension and allow to compute the fractal dimension of the border. In a future work we will show how these results can be extended in dimension $n$. We also used consistent 2D-QLTs (see definition 1) to define new numeration systems of $\mathbb{Z}/\beta$ where $\beta$ is an algebraic integer of order 2. In [2], the author studied fractals associated with shift radix...
Fig. 6. Another fractal associated with numeration systems

Fig. 7. Fractal associated with QLT
systems (which generalize numeration systems). Some of these fractals can be obtained using QLTs, one of our future works is to study the relations between QLTs and these numeration systems.

References