Quasi-Affine Applications and Tiling of the Discrete Plane

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In [4] and [5], Nehlig introduced tilings generated by quasi-affine transformations (QAT). Let us remember that a Quasi-Affine Transformation, or QAT, is a transformation of the discrete plane obtained by composing a rational affine transformation with the usual integer function. So it is defined by:

$$[g]: \quad \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2$$
$$(x,y) \longmapsto \begin{cases} x' = \left[\dfrac{ax+by+e}{\omega}\right] \\ y' = \left[\dfrac{cx+dy+f}{\omega}\right] \end{cases}$$

where a, b, c, d, e, f and ω are integers, $\omega > 0$, and $[\,]$ denotes the usual integer function. The reciprocal image of a point (i,j) by a QAT can contain no, one or several points; this reciprocal image (the set of points leading to point (i,j)) is called paving or tile of index (i,j) or more simply paving or tile (i,j), and noted $P_{i,j}$. The pavings form a periodic tiling of the discrete plane. Nehlig defined the *supertile* which is a set of tiles containing all generic tiles: so the supertile tiles the discrete plane. He defined also the *generic strip* which is a part of the supertile sufficient to tile the discrete plane and proved that if $\omega = ad - bc$ then the paving $(0,0)$ tiles the plane. This study has been done assuming that $\gcd(a,b) = \gcd(b,d) = \gcd(c,d) = \gcd(c,a) = 1$ (where gcd denotes the greatest common divisor).

The aim of this paper is to go on with this study in the general case. We will determine a set of indices I and two vectors u and v such that each paving $P_{i,j}$ can be obtained by translating a paving whose index belongs to I, the translation vector depending on (i,j), u and v. We then call paving-cluster the set of pavings having an index belonging to I. When $\gcd(a,b) = \gcd(b,d) = \gcd(d,c) = \gcd(c,a) = 1$, the paving-cluster contains the same pavings as the generic strip defined by Nehlig.

We also prove that the number of neighbours of a paving varies from four to eight; when $\omega = ad - bc$ a paving has 4 or 6 neighbours. This last result is a particular case of Beauquier and Nivat's theorem [1].

Keywords discrete tilings, discrete geometry, quasi-affine applications.

1 Introduction.

In [5], P. Nehlig defined discrete plane pavings generated by quasi-affine applications, these being discretised rational affine applications. Following a summary of the definitions and principal results of this article, we will continue the study of these pavings.

Firstly, let us introduce the notation which will be used throughout:

- if x is a real number, $[x]$ represents the integer part of x, that is to say the largest integer less than or equal to x;
- if x and y are two integers $\left[\dfrac{x}{y}\right]$ and $\left\{\dfrac{x}{y}\right\}$ represent respectively the quotient and the remainder of the euclidian division of x by y;
- if $x_1, x_2, ..., x_n$ are integers, $\gcd(x_1, x_2, ..., x_n)$ is the greatest common divisor to $x_1, x_2, ..., x_n$.

1.1 Definitions

Definition 1 *A quasi-affine transformation or QAT is obtained by composing a rational affine application with the integer part. Therefore it is an application of the discrete plane which can be defined by:*

$$[g]: \quad \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2$$
$$(x,y) \longmapsto \begin{cases} x' = \left[\dfrac{ax+by+e}{\omega}\right] \\ y' = \left[\dfrac{cx+dy+f}{\omega}\right] \end{cases}$$

where $a, b, c, d, e, f,$ and ω are integers, ω being strictly positive.

We will say that the QAT $[g]$ is defined by its matrix $A = \frac{1}{\omega}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and its vector $\mathcal{V} = \begin{pmatrix} e \\ f \end{pmatrix}$.

Definition 2 *We will call the paving of index (i,j) which will be noted $P_{i,j}$ the set of antecedents of a point (i,j) by a QAT $[g]$. Therefore we have :*

$$P_{i,j} = \{(x,y) \mid \quad [g](x,y) = (i,j)\}.$$

The set of pavings forms a periodic tiling of the discrete plane. Nehlig studied these pavings assuming that the coefficients of the QAT were such that $\gcd(a,b) = \gcd(b,d) = \gcd(d,c) = \gcd(c,a) = 1$. Remember the definitions of the supertile and the arithmetically identical pavings introduced by Nehlig. We will note $\delta = ad - bc$.

Definition 3

- *the first remainder of paving $P_{i,j}$ is :* $\left\{\dfrac{di - bj}{\delta}\right\}$*;*
- *the second remainder of paving $P_{i,j}$ is :* $\left\{\dfrac{-ci + aj}{\delta}\right\}$*;*
- *the supertile of index (i,j) is the set of points (x,y) of the discrete plane verifying*

$$S([g](x,y)) = (i,j)$$

  where S is the QAT defined by: $S(i,j) = \left(\left[\dfrac{di-bj}{\delta}\right], \left[\dfrac{-ci+aj}{\delta}\right]\right)$*;*
- *two pavings are said to be arithmetically identical if they have the same first remainder; the generic strip is the subsequent set of pavings :*

$$\left\{P_{0,i} \quad \mid \quad i \in \left[0, \frac{\delta}{gcd(\omega,\delta)}\right]\right\}.$$

Figure 1 illustrates the different definitions; these are the pavings of the QAT associated with the matrix $A = \frac{1}{6}\begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix}$ and the vector $\mathcal{V} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. The thin lines represent the pavings which denote the pair (first remainder, second remainder). The medium lines indicate the two generic strips and the thick lines the supertile of index $(0,0)$.

The following paragraph summarises the principal results demonstrated in [5].

1.2 Properties.

The following results have been obtained assuming the hypothesises, noted $H1$: $gcd(a,b) = gcd(b,d) = gcd(d,c) = gcd(c,a) = 1$.

Theorem 1 *1. Two pavings have the first same remainder if and only if they have the same second remainder.*

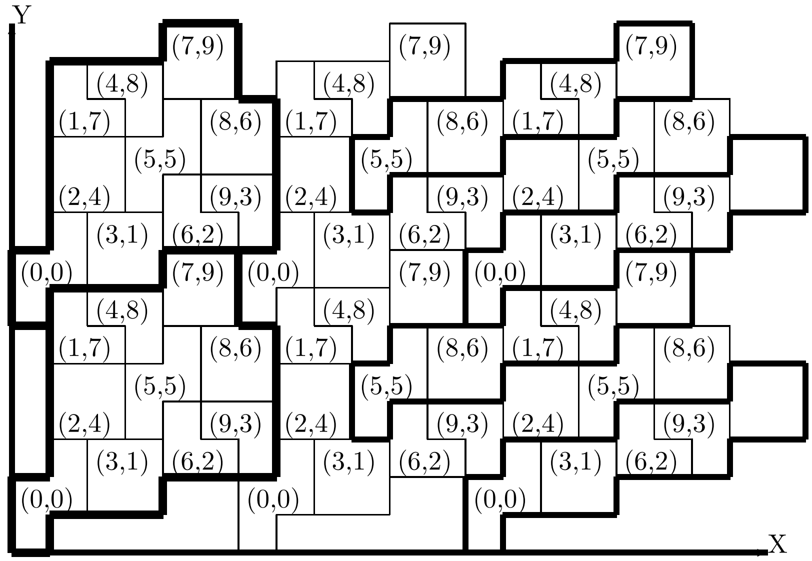

Figure 1: Example of pavings, supertile and generic strip

2. *An ordinary supertile contains once and once only all arithmetically distinct pavings.*
3. *Two arithmetically identical pavings are geometrically identical (the reciprocal is false).*
4. *if $\omega = ad - bc$ then all of the pavings are arithmetically identical to the paving $(0,0)$ which therefore forms a discrete plane paving.*
5. *The set of tiles for a generic strip contains all of the geometrically distinct pavings of a QAT.*

The above results show us that the set of pavings forms a periodic tiling of the discrete plane. A first period is obtained by considering the arithmetically distinct pavings having re-grouped in the supertile. However, the number of pavings of this period may be reduced thanks to the generic strip which constitutes a pattern for the pavings.

In the following study we assume that the hypothesises $H1$ are no-longer verified (the coefficients are not relatively prime numbers); the generic strip no-longer contains all of the geometrically distinct pavings. The supertile always tiles the plane, however it is possible to find a set of pavings containing less tiles and which always forms a plane paving. In order to find this group, we will define a new remainder which we will call remainder modulo $[g]$. Two pavings having the same first and second remainders will be identical modulo $[g]$, the reciprocal being true if the hypothesises $H1$ are verified and if ω and δ are relatively prime. Equally, we will define a hypertile which contains all of the distinct pavings modulo $[g]$ and forms a paving of the discrete plane. When the hypothesises $H1$ are verified and if ω and δ are relatively prime, the hypertile and the generic strip defined by Nehlig contain the same pavings (modulo $[g]$).

Finally we will study the connectivity of a paving. We show that the number of neighbours of a paving varies between 4 and 8 and, if $\omega = ad - bc$, then the number of neighbours of a paving equals 4 or 6. These results are extracts from [3] where other results concerning the study of pavings and QATs can be found.

2 The remainder modulo [g].

2.1 Definitions.

Definition 4 *$P_{i,j} = \{(x_0, y_0), ..., (x_n, y_n)\}$ being a non-empty paving; we will call remainder modulo $[g]$, noted $R_{i,j}$, the set:*

$$R_{i,j} = \{(r_0, r'_0), ..., (r_n, r'_n)\}$$

where $r_k = ax_k + by_k + e \ mod \ \omega$ and $r'_k = cx_k + dy_k + f \ mod \ \omega$, $k = 0, ..., n$.

Definition 5 *We will say that two pavings P and P' are identical modulo $[g]$ if they have the same remainder modulo $[g]$. So we will note that $P \equiv P'$.*

Property 1 *Two non-empty pavings P and P' are identical modulo $[g]$ if and only if two poins (x, y) and (x', y') exist belonging respectively to P and P' such that:*

$$\mathcal{C}\left\{\begin{array}{lcll} ax + by & = & ax' + by' & \mod \omega \\ cx + dy & = & cx' + dy' & \mod \omega. \end{array}\right.$$

Proof: Following the definition, the existence of two points verifies that the conditions $\mathcal{C}$ are necessary, so that P and P' are identical modulo $[g]$. We show that it is sufficient.

As (x, y) and (x', y') belong respectively to P and P' and verify conditions $\mathcal{C}$, we have :

$$\left\{\begin{array}{lcl} a(x - x') + b(y - y') & = & (i - i')\omega \\ c(x - x') + d(y - y') & = & (j - j')\omega \end{array}\right.$$

where (i, j) and (i', j') are the respective indices of pavings P and P'.
$(x", y")$ being another point of P, we have:

$$\left\{\begin{array}{lcl} ax" + by" & = & i\omega + r \\ cx" + dy" & = & j\omega + r' \\ & 0 \leq r, r' < \omega & \end{array}\right.$$

hence:

$$\left\{\begin{array}{lcl} a(x" - x + x') + b(y" - y + y') & = & i'\omega + r \\ c(x" - x + x') + d(y" - y + y') & = & j'\omega + r' \\ & 0 \leq r, r' < \omega & \end{array}\right.$$

The point $(x" - x + x', y" - y + y')$ is a point of P' having the same remainder as $(x", y")$. So, for all points of P there exists a point of P' having the same remainder. We can conclude that P and P' are identical modulo $[g]$.

Property 2 *Two pavings identical modulo $[g]$ are geometrically identical.*

Proof: $P_{i,j} = \{(x_0, y_0), ..., (x_n, y_n)\}$ and $P_{i',j'} = \{(x'_0, y'_0), ..., (x'_n, y'_n)\}$ being two pavings identical modulo $[g]$, we have therefore for all values of m between 0 and n :

$$\left\{\begin{array}{lcl} ax_m + by_m & = & i\omega + r_m \\ cx_m + dy_m & = & j\omega + r'_m \\ ax'_m + by'_m & = & i'\omega + r_m \\ cx'_m + dy'_m & = & j'\omega + r'_m \end{array}\right.$$

hence

$$\left\{\begin{array}{lcl} \delta(x_m - x'_m) & = & \omega(d(i' - i) - b(j' - j)) \\ \delta(y_m - y'_m) & = & \omega(a(j' - j) - c(i' - i)) \end{array}\right.$$

with $\delta = ad - bc$.
So we can formulate $u = \omega(d(i' - i) - b(j' - j))$ and $v = \omega(a(j' - j) - c(i' - i))$, u and v are multiples of δ. Therefore, there exists integers u' and v' such that:

$$\left\{\begin{array}{lcl} u & = & u'\delta \\ v & = & v'\delta \\ x_m & = & x'_m + u' \\ y_m & = & y'_m + v' \end{array}\right.$$

$P_{i,j}$ is therefore the image of $P_{i',j'}$ by the translation of the vector $\overrightarrow{v} = (u', v')$.

Remark : It is possible to have two geometrically equal pavings which are not equal modulo $[g]$. To validate this we can re-examine the example given by Nehlig in figure 1.

2.2 Comparison with Nehlig's arithmetical paving.

Property 3 *Let note* $R_{i,j} = \left(\left\{\frac{di-bj}{\delta'}\right\}, \left\{\frac{-ci+aj}{\delta'}\right\}\right)$ *where* $\delta' = \frac{\delta}{gcd(\omega,\delta)}$, *then* $P_{i,j}$ *and* $P_{i',j'}$ *are identical modulo* $[g]$ *if and only if* $R_{i,j} = R_{i',j'}$.

Proof: $P_{i,j}$ and $P_{i',j'}$ being two identical pavings such as $R_{i,j} = R_{i',j'}$:

$$\left\{ \begin{array}{lcl} \left\{\frac{di-bj}{\delta}\right\} & = & \left\{\frac{di'-bj'}{\delta}\right\} \\ \left\{\frac{-ci+aj}{\delta}\right\} & = & \left\{\frac{-ci'+aj'}{\delta}\right\}. \end{array} \right.$$

Therefore there exists two integers k and k' such as :

$$\left\{ \begin{array}{lcl} di-bj & = & di'-bj'+k\delta \\ -ci+aj & = & -ci'+aj'+k'\delta. \end{array} \right.$$

hence

$$\left\{ \begin{array}{lcl} \delta i & = & \delta i' + \delta(ak+bk') \\ \delta j & = & \delta j' + \delta(dk'+ck) \end{array} \right.$$

which is equivalent to

$$\left\{ \begin{array}{ll} i & = i' + ak + bk' \\ j & = j' + ck + dk'. \end{array} \right.$$

(x, y) being a point of $P_{i,j}$, we therefore have :

$$\left\{ \begin{array}{lclcl} ax+by & = & i\omega + r & = & i'\omega + \omega(ak+bk') + r \\ cx+dy & = & j\omega + r' & = & j'\omega + \omega(ck+dk') + r' \end{array} \right.$$

hence

$$\left\{ \begin{array}{lcl} a(x-k\omega)+b(y-k'\omega) & = & i'\omega + r \\ c(x-k\omega)+c(y-k'\omega) & = & j'\omega + r'. \end{array} \right.$$

The point $(x-k\omega, y-k'\omega)$ is a point of $P_{i',j'}$ having the same remainder modulo $[g]$ as (x, y). We can therefore conclude that the two pavings are equal modulo $[g]$. To prove the reciprocal: $P_{i,j}$ and $P_{i',j'}$ being two equal pavings modulo $[g]$, there exists two points (x, y) and (x', y') such that :

$$\left\{ \begin{array}{lcl} ax+by & = & ax'+by'+(i'-i)\omega \\ cx+dy & = & cx'+dy'+(j'-j)\omega \end{array} \right.$$

hence

$$\left\{ \begin{array}{lcl} \delta x & = & \delta x' + \omega(di'-bj'-(di-bj)) \\ \delta y & = & \delta y' + \omega(aj'-ci'-(aj-ci)). \end{array} \right.$$

By simplifying by $gcd(\omega,\delta)$ and formulating $\omega' = \frac{\omega}{gcd(\omega,\delta)}$ and $\delta' = \frac{\delta}{gcd(\omega,\delta)}$ we obtain:

$$\left\{ \begin{array}{lcl} \delta'(x-x') & = & \omega'(di'-bj'-(di-bj)) \\ \delta'(y-y') & = & \omega'(aj'-ci'-(aj-ci)) \end{array} \right.$$

We have $gcd(\omega', \delta') = 1$, hence:

$$\left\{ \begin{array}{lcll} di'-bj' & = & di-bj & \quad mod\ \delta' \\ aj'-ci' & = & aj-ci & \quad mod\ \delta' \end{array} \right.$$

and so $R_{i,j} = R_{i',j'}$.

Remark : we can state that $R_{i,j}$ does not depend on $\mathcal{V} = \begin{pmatrix} e \\ f \end{pmatrix}$. The number of distinct pavings modulo $[g]$ is independent of the values of e and f. Therefore we assume that $e = f = 0$ in the following.

Corollary 1

If $\gcd(a,b) = \gcd(b,d) = \gcd(d,c) = \gcd(c,a) = \gcd(\omega,\delta) = 1$, *then two pavings are arithmetically equal if and only if they are equal modulo* $[g]$.

Proof: If $\gcd(a,b) = \gcd(b,d) = \gcd(d,c) = \gcd(c,a) = \gcd(\omega,\delta) = 1$, then $\delta' = \delta$ and $\omega' = \omega$: $R_{i,j}$ is the pair formed from the first and the second remainders. Two pavings are therefore identical modulo $[g]$ if and only if they have the same first and second remainders. But, we know from theorem 1, that two pavings have the same first remainders if and only if they have the same second remainders, hence corollary 1.

3 The Hypertile

3.1 Periodicity of arithmetical pavings.

In the following we do not consider the coefficients of the QAT to be relatively prime numbers. However, we can without loss of generality assume that there is no common divisor (other than 1 or -1) for all the coefficients. Indeed, if an integer q exists such that $a = qa', b = qb', c = qc', d = qd'$ and $\omega = q\omega'$, we can easily demonstrate that the pavings of the QAT defined by the matrix $A = \frac{1}{\omega}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ are the same as the pavings of the QAT $A = \frac{1}{\omega'}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$.

Introducing the notation used in the following:

$c' = \dfrac{c}{gcd(c,d)}$, $\quad d' = \dfrac{d}{gcd(c,d)}$, $\quad \delta_1 = \dfrac{\delta}{gcd(c,d)}$, $\quad \delta_1' = \dfrac{\delta_1}{gcd(\delta_1,\omega)}$,

$\omega' = \dfrac{\omega}{gcd(\delta_1,\omega)}$, $\quad \omega" = \dfrac{\omega}{gcd(c,d,\omega)}$, $\quad \delta_1" = \dfrac{\delta 1}{gcd(\delta_1,\omega")}$.

Equally, remember Bezout's theorem: if x and y are two relatively prime integers, there exists two integers u and v such that $ux + vy = 1$. If u' and v' are the solutions to this equation, all other solutions are given by $u = u' + ky, v = v' - kx$ where k is any integer. The integers u' and v' can be determined thanks to Blankinship's algorithm [2].

Theorem 2

- *We have* $P_{i,j} \equiv P_{i',j}$ *if and only if an integer* k *exists such that* $i' = i + k\delta_1'$. *In this case* $P_{i',j}$ *is the image of* $P_{i,j}$ *by the translation of vector* $k\overrightarrow{u}$ *with* $\overrightarrow{u} = k(-\omega'd', \omega'c')$.
- α *being the smallest non-zero integer so that an integer* β *exists verifying* $P_{i+\beta,j} \equiv P_{i,j+\alpha}$ *whatever* (i,j), *then:*

$$\begin{cases} \alpha &= \dfrac{gcd(\omega,c,d,\delta_1")gcd(c,d)}{gcd(c,d,\omega)} \\ \beta &= -s(av_0 + bv_1) \end{cases}$$

where s *and* s' *are such that* $s\ gcd(\omega,c,d) + s'\delta_1" = gcd(\omega,c,d,\delta_1")$ *and* v_0 *and* v_1 *are such that* $c'v_0 + d'v_1 = 1$.
$P_{i,j+\alpha}$ *is the the image of* $P_{i+\beta,j}$ *by the translation of the vector* $\overrightarrow{v} = (v_x, v_y)$ *with:*

$$\begin{cases} v_x &= v_0\dfrac{\alpha\omega}{gcd(c,d)} - d's'(av_0+bv_1)\dfrac{\omega"}{gcd(\delta_1,\omega")} \\ v_y &= v_1\dfrac{\alpha\omega}{gcd(c,d)} + c's'(av_0+bv_1)\dfrac{\omega"}{gcd(\delta_1",\omega")}. \end{cases}$$

- *The number of distinct pavings modulo* $[g]$ *is* $N = \dfrac{\delta}{gcd(\omega,\delta)}$.

Remark: In all of the following we use the notation I when considering the set of indices defined by $\{(i,j)$ *such that* $0 \le i < \delta_1'$ *and* $0 \le j < \alpha\}$. The previous theorem tells us that the set of pavings defined by $\mathcal{H} = \{P_{i,j}, (i,j) \in I\}$ forms a representative system of pavings of a QAT: It contains exactly once all of the arithmetically distinct pavings and all pavings can be obtained by the translation of a paving belonging to this set. The set $\mathcal{H}$ forms a periodic tiling of the discrete plane by the pavings of a QAT. More precisely,(i,j) being a normal index, note that $j_0 = \left\{\dfrac{j}{\alpha}\right\}$, $j_1 = \left[\dfrac{j}{\alpha}\right]$, $i_0 = \left\{\dfrac{i+j_1\beta}{\delta_1'}\right\}$ and $i_1 = \left[\dfrac{i+j_1\beta}{\delta_1'}\right]$, then the index (i_0, j_0) belongs to I, the pavings $P_{i,j}$ and P_{i_0,j_0} are arithmetically identical and $P_{i,j}$ is the image of P_{i_0,j_0} by the translation of the vector $i_1\overrightarrow{u} + j_1\overrightarrow{v}$.

The figures 2 and 3 illustrate the previous theorem. Two pavings having the same texture are two equal pavings modulo $[g]$. We have surrounded the set of distinct pavings modulo $[g]$ in bold type. In figure 2 we consider the QAT defined by the matrix $A = \dfrac{1}{217}\begin{pmatrix} 12 & -13 \\ 42 & 63 \end{pmatrix}$; we therefore have $\delta_1' = 2, \alpha = 3$ and $\beta = 1$. In this example we can point out that the joining of the tiles $(i, j+k)$ for k going from 0 to 5, contains equally all of the distinct pavings modulo $[g]$. Equally, this joining therefore forms a discrete plane paving. In figure 3 we consider the QAT defined by the matrix $A = \dfrac{1}{84}\begin{pmatrix} 12 & -11 \\ 18 & 36 \end{pmatrix}$, we have therefore $\delta_1' = 5, \alpha = 3$ and $\beta = 2$.

Before proving this theorem, we will prove the following lemma.

Lemma 1

- *v_0 and v_1 geing such that $c'v_0 + d'v_1 = 1$; we therefore have $gcd(\omega, c, d, av_0 + bv_1) = 1$.*
- *we have $\dfrac{\delta_1 gcd(\omega, c, d, \delta_1'')gcd(c,d)}{gcd(\omega,\delta_1)gcd(c,d,\omega)} = \dfrac{\delta}{gcd(\omega,\delta)}$.*

Proof:

- Let $u = gcd(c, d, \omega, av_0 + bv_1)$, therefore the integers $\omega_1, c_1, d_1, \delta_2$ and k_1 exist such that: $\omega = u\omega_1, c = uc_1, d = ud_1, \delta_1 = u\delta_2$ et $av_0 + bv_1 = uk_1$. We therefore have:

$$\left\{\begin{array}{lcl} ad' - bc' & = & u\delta_2 \\ av_0 + bv_1 & = & uk_1 \end{array}\right.$$

  and so

$$\left\{\begin{array}{lcl} a & = & u(c'k_1 + v_1\delta_2) \\ b & = & u(d'k_1 - v_0\delta_2) \end{array}\right.$$

  so u divides a, b, c, d and ω. But, by hypothesis, $gcd(a, b, c, d, \omega) = 1$; we can conclude that $u = 1$.

- Note $u = gcd(c, d, \omega)$ and $v = \dfrac{gcd(c,d)}{gcd(c,d,\omega)}$.

  So we have $\dfrac{\delta}{gcd(\omega,\delta)} = \dfrac{uv\delta_1}{gcd(u\omega'', uv\delta_1)} = \dfrac{v\delta_1}{gcd(\omega'', v\delta_1)}$. But we have $gcd(v, \omega'') = 1$.

  We can conclude that $\dfrac{\delta}{gcd(\omega,\delta)} = \dfrac{v\delta_1}{gcd(\omega'',\delta_1)} = v\delta_1''$

  On the other hand, we have

$$\begin{array}{rcl} \dfrac{\delta_1 gcd(\omega, c, d, \delta_1'')gcd(c,d)}{gcd(\omega,\delta_1)gcd(c,d,\omega)} & = & \dfrac{\delta_1 gcd(u, \delta_1'')v}{gcd(u\omega'', \delta_1'' gcd(\delta_1, \omega''))} \\ & = & \dfrac{v\delta_1 gcd(u, \delta_1'')}{gcd(\delta_1, \omega'')gcd(u\dfrac{\omega''}{gcd(\delta_1,\omega'')}, \delta_1'')} \end{array}$$

  But we have $gcd(\delta_1'', \dfrac{\omega''}{gcd(\delta_1,\omega'')}) = 1$.

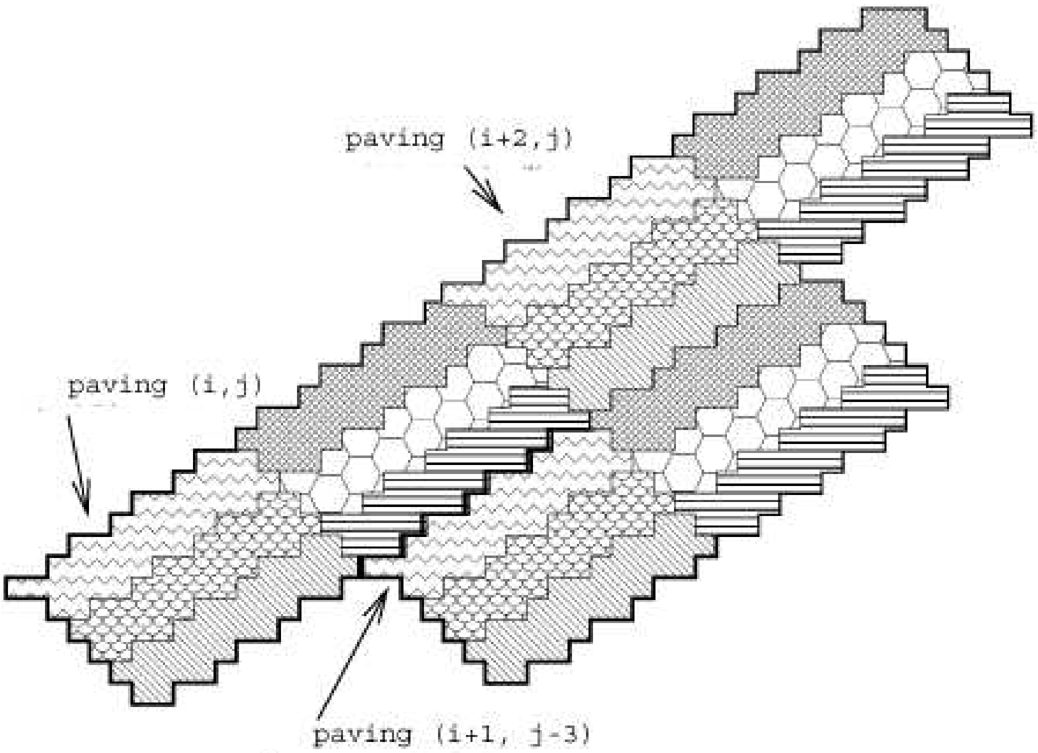

Figure 2: The first example of distinct pavings modulo $[g]$

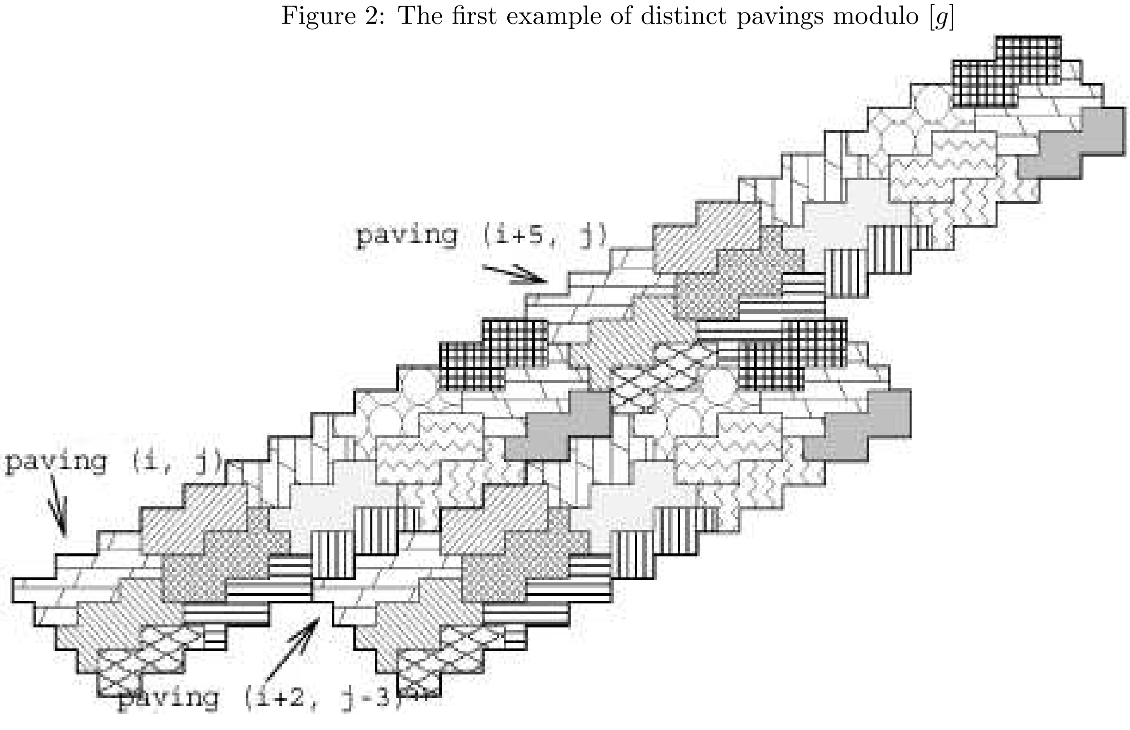

Figure 3: The second example of distinct pavings modulo $[g]$

Hence the equalities

$$\begin{aligned}
\frac{\delta_1 gcd(\omega, c, d, \delta_1'')gcd(c,d)}{gcd(\omega,\delta_1)gcd(c,d,\omega)} &= \frac{v\delta_1 gcd(u,\delta_1'')}{gcd(\delta_1,\omega'')gcd(u,\delta_1'')} \\
&= \frac{v\delta_1}{gcd(\delta_1,\omega'')} \\
&= v\delta_1'' \\
&= \frac{\delta}{gcd(\omega,\delta)}.
\end{aligned}$$

Proof of the theorem:

- Two pavings $P_{i,j}$ and $P_{i',j}$ are equal modulo $[g]$ if and only if two points (x,y) and (x',y') exist such that:

$$\left\{\begin{array}{lcl} ax+by &=& ax'+by'+(i'-i)\omega \\ cx+dy &=& cx'+dy' \end{array}\right.$$

$$Let \quad \left\{\begin{array}{lcl} \delta(x-x') &=& \omega d(i'-i) \\ \delta(y-y') &=& -\omega c(i'-i) \end{array}\right.$$

$$or\ again \quad \left\{\begin{array}{lcl} \delta_1'(x-x') &=& \omega' d'(i'-i) \\ \delta_1'(y-y') &=& -\omega' c'(i-i') \end{array}\right.$$

  but $gcd(\omega' d', \delta_1') = gcd(\omega' c', \delta_1') = 1$. So the previous equations are verified if and only if an integer k exists such that:

$$\left\{\begin{array}{lcl} i'-i &=& k\delta_1' \\ x-x' &=& k\omega' d' \\ y-y' &=& -k\omega' c' \end{array}\right.$$

  Hence the first assertion of the theorem.

- Now we find the smallest integer α such that an integer β (independent of i and j) exists verifying the identity:

$$P_{i,j+\alpha} \equiv P_{i+\beta,j}.$$

  In order for these pavings to be equal modulo $[g]$, it is necessary and sufficient that two points (x,y) and (x',y') exist (belonging to the pavings $P_{i,j+\alpha}$ and $P_{i+\beta,j}$ respectively) such that:

$$\left\{\begin{array}{lcl} ax+by &=& ax'+by'-\beta\omega \\ cx+dy &=& cx'+dy'+\alpha\omega \end{array}\right.$$

$$\text{therefore} \quad \left\{\begin{array}{lcl} a(x-x')+b(y-y') &=& -\beta\omega \\ \dfrac{gcd(c,d)}{gcd(c,d,\omega)}\left(c'(x-x')+d'(y-y')\right) &=& \alpha\omega". \end{array}\right.$$

  So an integer α' exists such that $\alpha = \alpha' \dfrac{gcd(c,d)}{gcd(c,d,\omega)}$. Furthermore, if we state $k' = x - x'$ and $k" = y' - y"$, the solutions of the second equation are of the form: $k' = v_0\alpha'\omega" + kd'$ and $k" = v_1\alpha'\omega" - kc'$ where k is an ordinary integer and v_0, v_1 are such that $c'v_0 + d'v_1 = 1$. Therefore it is necessary and sufficient to find k so that the first equation is verified, that is to say $ak' + bk" = -\beta\omega$ and so that α' is minimal.
  On replacing k' and k'' by their definition and by factorising, we obtain:

$$\alpha'\omega"(av_0+bv_1) + k(ad'-bc') = -\beta\omega" gcd(c,d,\omega).$$

  From this we can deduce that an integer k_1 exists such that $k\delta_1 = k_1\delta_1"\omega"$ and so:

$$\alpha'(av_0+bv_1) + k_1\delta_1" = -\beta gcd(c,d,\omega)$$

which is equivalent to:

$$\begin{array}{ll} & \alpha'(av_0+bv_1) = -(\beta r + k_1 r')gcd(\omega,c,d,\delta_1") \\ \text{with} & r = \dfrac{gcd(c,d,\omega)}{gcd(\omega,c,d,\delta1")} \text{ and } r' = \dfrac{\delta_1"}{gcd(\omega,c,d,\delta_1")}. \end{array}$$

To verify this equation α' must be a multiple of gcd$(\omega, c, d, \delta_1")$ (because we can deduce from the lemma that gcd$(c, d, \omega, \delta_1", av_0 + bv_1) = 1$).
But r and r' are relatively prime numbers, so the integers s and s' exist such that $sr + sr' = 1$. Stating $\beta = -s(av_0 + bv_1)$ and $k_1 = -s'(av_0 + bv_1)$, then $\alpha' = gcd(\omega, c, d, \delta_1")$. We can deduce the equalities:

$$\alpha = \alpha' \frac{gcd(c,d)}{gcd(c,d,\omega)} = \frac{gcd(\omega,c,d,\delta_1")gcd(c,d)}{gcd(c,d,\omega)}.$$

Equally we have:

$$\begin{array}{ll} & \left\{ \begin{array}{lclcl} x - x' & = & k' & = v_0\alpha'\omega" + kd' \\ y - y' & = & k" & = v_1\alpha'\omega" - kc' \end{array} \right. \\ \Longleftrightarrow & \left\{ \begin{array}{lcl} x & = & x' + v_0 \dfrac{\alpha\omega}{gcd(c,d)} - d's'(av_0+bv_1)\dfrac{\omega"}{gcd(\delta_1,\omega")} \\ y & = & y' + v_1 \dfrac{\alpha\omega}{gcd(c,d)} + c's'(av_0+bv_1)\dfrac{\omega"}{gcd(\delta_1,\omega")} \end{array} \right. \end{array}$$

Hence the second assertion of the theorem.

- The number of distinct pavings modulo $[g]$ is therefore:

$$\begin{array}{lcll} N & = & \alpha\delta'_1 & \\ & = & \dfrac{\delta_1 gcd(\omega,c,d,\delta_1")gcd(c,d)}{gcd(\omega,\delta_1)gcd(c,d,\omega)} & \\ & = & \dfrac{\delta}{gcd(\omega,\delta)} & \text{following the lemma.} \end{array}$$

In the following we will note I the set $\{(i,j) \mid 0 \leq i < \delta'_1, \quad 0 \leq j < \alpha\}$.

Definition 6 *The set of distinct pavings modulo $[g]$, defined below, will be called a hypertile and noted $\mathcal{H}$,*

$$\mathcal{H} = \bigcup_{(i,j)\in I} P_{i,j}.$$

Remark: we can demonstrate easily that the hypertile of a QAT defined by the matrix $A = \frac{1}{\omega}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and the vector $\mathcal{V} = \begin{pmatrix} e \\ f \end{pmatrix}$ is equally the paving $(0,0)$ of the QAT defined by the matrix $A = \dfrac{1}{scm(\alpha,\delta'_1)\omega}\begin{pmatrix} \alpha'a & \alpha'b \\ \delta_2 c & \delta_2 d \end{pmatrix}$ and the vector $\mathcal{V} = \begin{pmatrix} \alpha'e \\ \delta_2 f \end{pmatrix}$ where $scm(\alpha,\delta'_1)$ is the smallest common multiple to α and δ'_1 and where α' and δ_2 are defined by: $\alpha' = \dfrac{\alpha}{gcd(\alpha,\delta'_1)}$ and $\delta_2 = \dfrac{\delta'_1}{gcd(\alpha,\delta'_1)}$.

3.2 Determinantal case and pavings of order n.

If we re-look at the previous theorem, we see that in the determinantal case ($\omega = ad - bc$) there is one unique paving which forms a discrete plane paving. Likewise, we show that in this case there is one unique paving of the order n and a paving (i,j) of the order n can be defined by:

$$P^n_{i,j} = \{(x,y) \mid [g]^n(x,y) = (i,j)\}.$$

We do not prove this result here. This proof is similar to the previous proof and can be found in [3]. The figures 4 and 5 give us two examples of discrete plane pavings of the order n. Figure 4 corresponds to pavings of order 2 of the QAT associated to the matrix $A = \frac{1}{3}\begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$, and figure 5 corresponds to the pavings of order 2 of the QAT associated to the matrix $A = \frac{1}{5}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$.

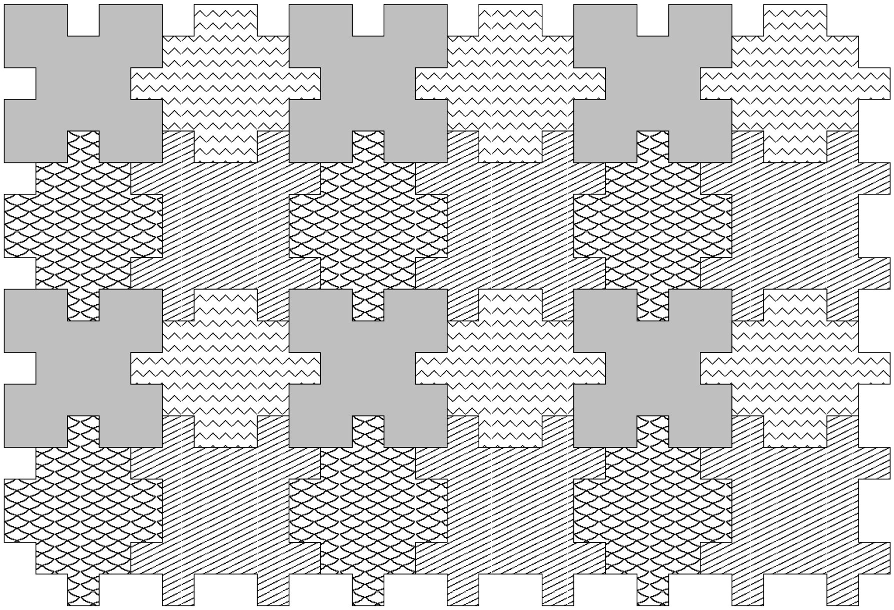

Figure 4: There are 4 distinct pavings of the order 2.

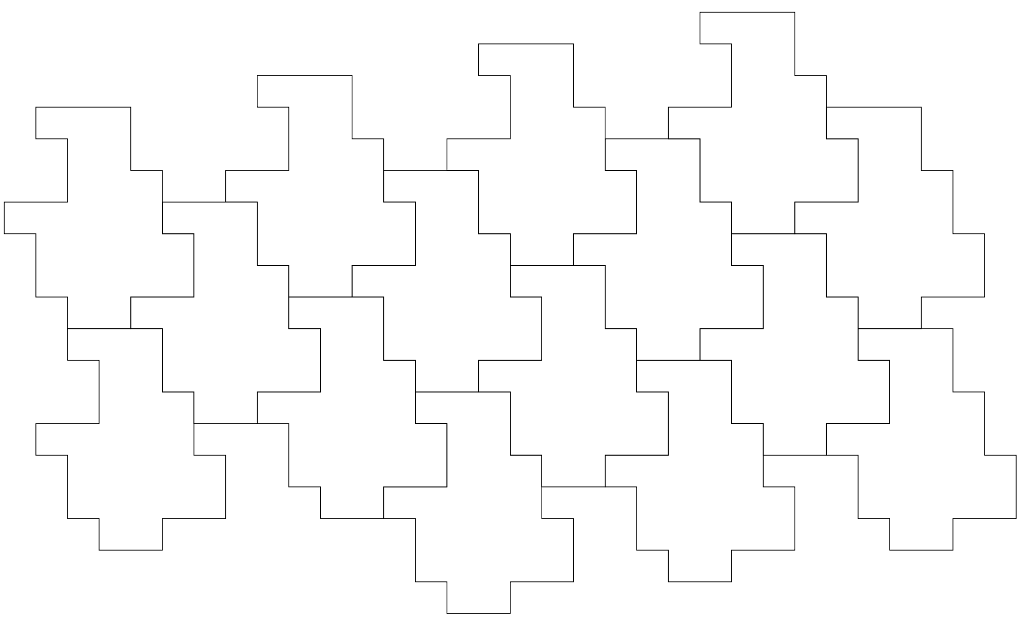

Figure 5: The QAT being determinantal, there is only one paving of the order 2.

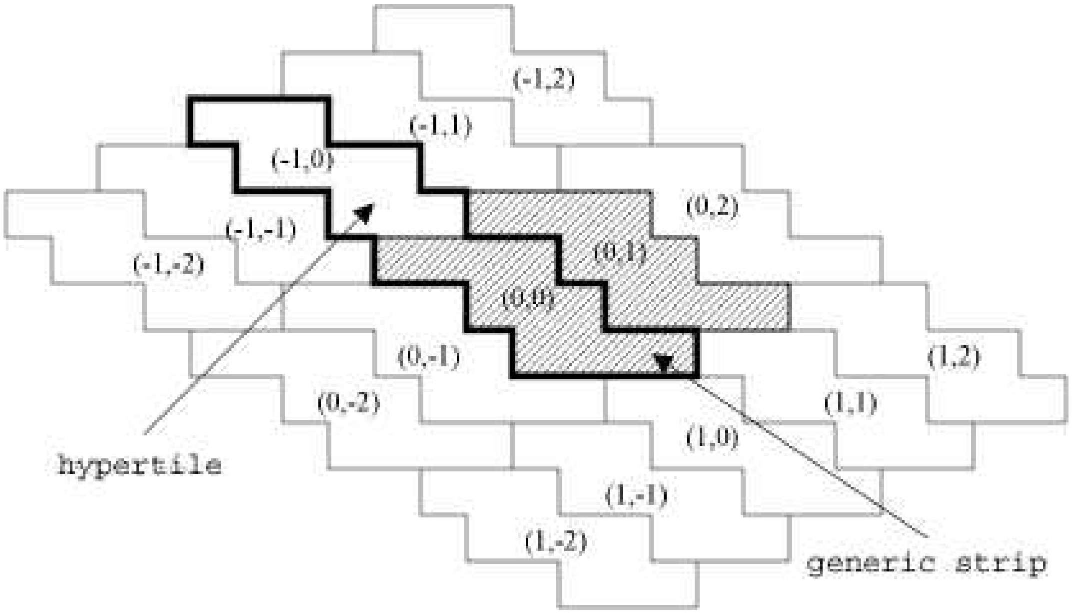

Figure 6: Distinction between the generic strip, the hypertile and the supertile.

3.3 Relation whith the generic strip.

We observe that the number of pavings of the hypertile is equal to the number of pavings of the generic strip, but the generic strip can contain the same paving several times. Therefore, in this case, it will not contain all of the distinct pavings modulo $[g]$. The generic strip contains all of the distinct pavings when the hypothesises H1 are verified. Indeed, $P_{0,i}$ and $P_{0,i'}$ being two pavings of the generic strip, according to previous theorem, for the two pavings to be equal modulo $[g]$, an integer k must exist such that $i = i' + k\delta'$. But we have $0 \leq i, i' < \delta'$; we can deduce that $i' = i$. So the generic strip contains all of the distinct pavings modulo $[g]$. The supertile contains all of the distinct pavings modulo $[g]$ but it does not contain them only once. In figure 6, we see an example where the hypothesises H1 are not verified; the generic strip contains the same pavings twice.

4 Neighbouring of a paving.

Definition 7 *Let E and E' be two sets of points of the discrete plane. We will say that E and E' are 4-connected (resp. 8-connected) if two points (x, y) and (x', y') exist belonging respectively to E and E' such that $(x = x'$ and $y = y' \pm 1) or (x = x' \pm 1$ and $y = y')$ (resp $x = x' \pm 1$ and $y = y' \pm 1$).*

Definition 8 *We will say that the discrete line D is 4-connected if, for all points X of that line, there exists a point Y belonging to D such that X and Y are 4-connected.*

4.1 General case

Theorem 3 *Let the QAT be defined by the matrix $A = \frac{1}{\omega}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and the vector $\mathcal{V} = \begin{pmatrix} e \\ f \end{pmatrix}$, such that:*

$$\omega \geq \max(\mid a \mid + \mid b \mid, \mid c \mid + \mid d \mid).$$

The number of neighbours of a paving varies between 4 and 8.

Proof:

- To show that any paving has at most 8 neighbouring pavings.

Let $X = (x, y)$ being a point of a paving (i, j), we will show that the neighbours of X can only belong to the paving with indices $(i, j), (i \pm 1, j), (i, j \pm 1)$ or $(i \pm 1, j \pm 1)$.
As X belongs to the paving (i, j), we have:

$$\begin{cases} ax + by = i\omega + r \\ cx + dy = j\omega + r' \\ 0 \leq r, r' < \omega. \end{cases}$$

The neighbouring points of X are of the form $(x + \varepsilon_1, y + \varepsilon_2)$ with $\varepsilon_1, \varepsilon_2 = \pm 1$ or 0. So we have:

$$\begin{cases} a(x + \varepsilon_1) + b(y + \varepsilon_2) = i\omega + r + a\varepsilon_1 + b\varepsilon_2 \\ c(x + \varepsilon_1) + d(y + \varepsilon_2) = j\omega + r' + c\varepsilon_1 + d\varepsilon_2 \\ 0 \leq r, r' < \omega. \end{cases}$$

Since $\omega \geq \max(\mid a \mid + \mid b \mid, \mid c \mid + \mid d \mid)$, we have:

$$\begin{cases} -\omega & \leq & a\varepsilon_1 + b\varepsilon_2 & \leq & \omega \\ -\omega & \leq & c\varepsilon_1 + d\varepsilon_2 & \leq & \omega \end{cases}$$

hence:

$$\begin{cases} -\omega & \leq & a\varepsilon_1 + b\varepsilon_2 + r & < & 2\omega \\ -\omega & \leq & c\varepsilon_1 + d\varepsilon_2 + r' & < & 2\omega. \end{cases}$$

We can deduce that:

$$\begin{cases} \left[\dfrac{a(x + \varepsilon_1) + b(y + \varepsilon_2)}{\omega}\right] & = & i \pm 1 \text{ or } i \\ \left[\dfrac{c(x + \varepsilon_1) + d(y + \varepsilon_2)}{\omega}\right] & = & j \pm 1 \text{ or } j. \end{cases}$$

So the paving (i, j) has at most 8 neighbours.

- To show that the number of neighbours of any paving is at least 4.

For this we will show that the pavings $(i \pm 1, j)$ or $(i, j \pm 1)$ are always neighbours of the paving (i, j). Remember (see [5], [3], [6]) that the paving (i, j) is the intersection of the discrete lines:

$$D_i : \left[\frac{ax + by}{\omega}\right] = i \qquad \text{and} \qquad D'_j : \left[\frac{cx + dy}{\omega}\right] = j.$$

Following the first part of the proof, the only pavings being able to be neighbours of $P_{i,j}$, are the pavings $P_{i+\varepsilon,j}, P_{i,j+\varepsilon'}$ and $P_{i+\varepsilon,j+\varepsilon'}$ with $\varepsilon, \varepsilon' = \pm 1$. These pavings are the intersections of discrete lines $D_i, D_j, D_{i+\varepsilon}, D_{j+\varepsilon'}$ and $D_{i+\varepsilon,j+\varepsilon'}$, represented in figure 7.

Considering a 4-connected line $\mathcal{D}$ included in the line D_i and ...X_{-m}, X_{-m+1} ,..., X_{-1}, X_0, X_1,..., X_n,... (one such line exists because D_i is at least 4-connected following hypothesis of Theorem 3 and a theorem of [6]). We number this series so that for any k, X_k and X_{k+1} are 4-connected and X_0 belongs to D'_j (the two lines being 4-connected, there must be an intersection point). The number of points belonging to the intersection of two non-parallel lines being finite, a positive index k exists such that X_k does not belong to D'_j. Let k_0 be the smallest of these indices, X_{k_0-1} belongs therefore to $\mathcal{D} \cap D'_j$ and X_{k_0} belongs $D'_{j\pm1}$ because D'_j is 4-connected. Similarly, if k_1 is the smallest of these indices k such that X_{-k} does not belong to D'_j so X_{-k_1+1} belongs to $\mathcal{D} \cap D'_j$ and X_{-k_1} belongs to $D'j \pm 1$. If X_{k_0} belongs to D'_{j-1} then X_{k_1} belongs to D'_{j+1} and inversely. We can conclude that there exists two points X_{k_0} and X_{k_1} such as one of them belongs to $\mathcal{D} \cap D'_{j+1}$, and the other to $\mathcal{D} \cap D'_{j-1}$ and such that each one has a 4-connected point belonging to $\mathcal{D} \cap D'_j$. But $\mathcal{D}$ is included in D_i, we can deduce that $D_i \cap D'_j$ is 4-connected to $D_i \cap D'_{j-1}$ and to $D_i \cap D'_{j+1}$.

In figure 8 we have an example where we show that all of the cases (4,5,6,7 or 8 neighbours) are possible.

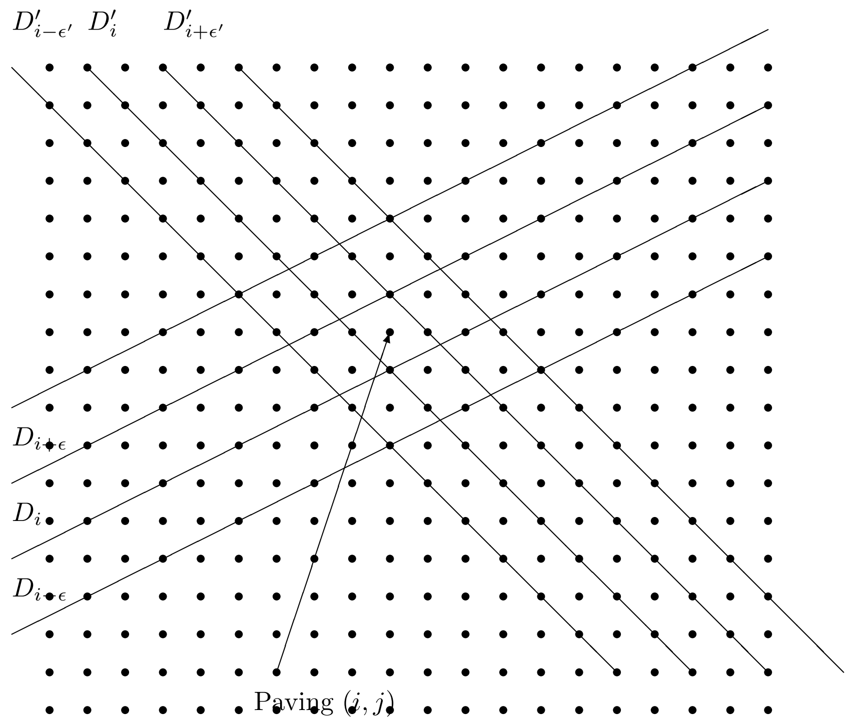

Figure 7: Visualisation of pavings as intersection of discrete lines.

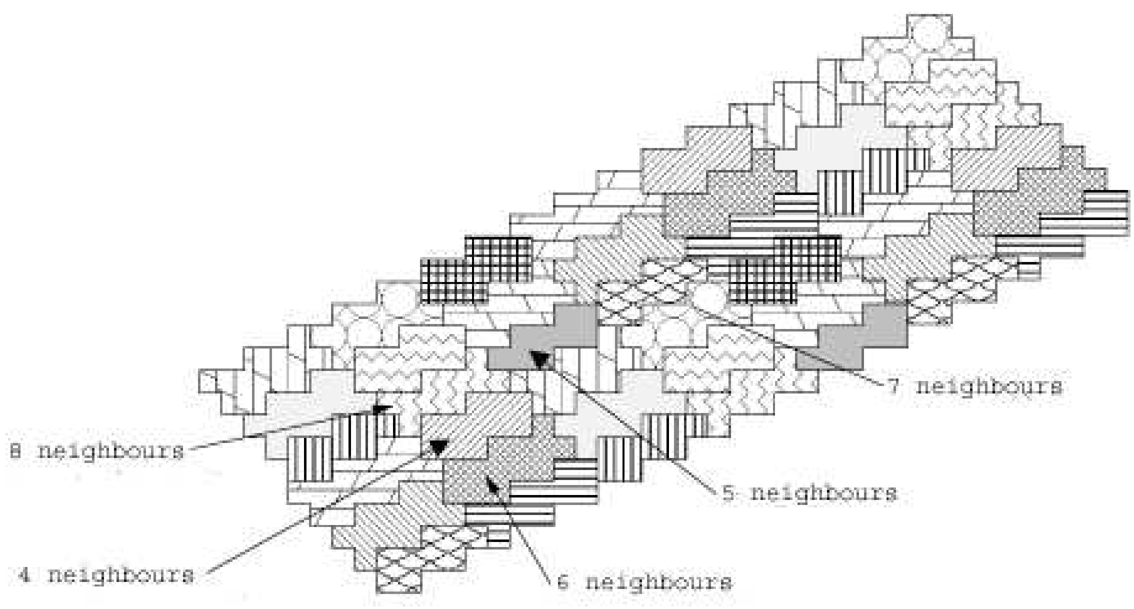

Figure 8: Neighbouring of a paving

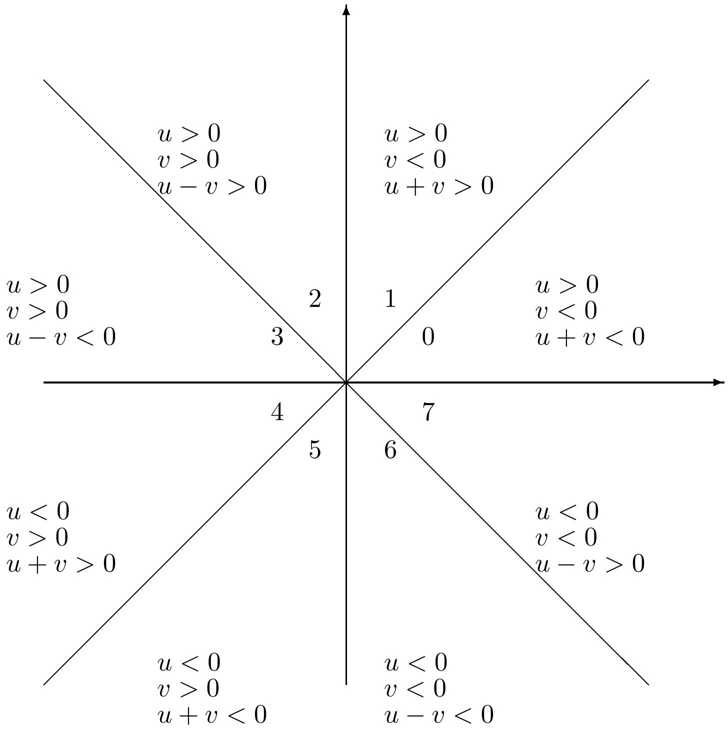

Figure 9: Numeration of octants.

4.2 Determinantal case

We have already seen that all of the pavings of a determinantal QAT are equal modulo $[g]$. If we note u and v the vectors $(d, -c)$ et $(-b, a)$, we have:

$$P_{i,j} = \mathcal{T}_{iu+jv} P_{0,0}$$

To know the number of neighbours of a paving, it is sufficient to know the number of neighbours of the paving $(0, 0)$.

Firstly we can say that a paving of a determinantal QAT always has an even number of neighbours. Indeed, if $P_{i,j}$ is a neighbour of $P_{0,0}$, the image of these two pavings by translation of the vector $-iu - jv$ is the pavings $P_{0,0}$ and $P_{-i,-j}$, which are therefore also neighbours. We can conclude that if $P_{i,j}$ is the neighbour of $P_{0,0}$ then $P_{-i,-j}$ is equally the neighbour of $P_{0,0}$.

Remember that the paving $(0, 0)$ is the intersection of two discrete lines which we will note as D and D'. The following theorem shows us that the number of neighbours of a paving depends on the octant in which the two lines are situated. The numeration of the octants is defined by the coordinates of the direction vector $(-v, u)$ of the discrete line $\left[\dfrac{ux + vy}{\omega}\right] = 0$ as we can see in figure 9.

Theorem 4 *Taking a determinantal QAT such as $\delta \geq \max(\mid a \mid + \mid b \mid, \mid c \mid + \mid d \mid)$. So the original paving has 6 4-connected neighbours except in the following cases:*

- *one of the two lines is horizontal and the other vertical*
- *one of the two lines is horizontal or vertical and the other is situated in the third or seventh octant (without being a bisector)*
- *one of the two lines belongs to the octant 0, 1, 2 or 3 and the other to the octant 5, 4, 7 or 6.*

In these cases, the paving $(0, 0)$ has 4 4-connected neignbours and 4 other 8-connected neighbours.

Firstly we will prove two lemmas.

Lemma 2 *The geometric conditions of theorem 4 are equivalent to the arithmetical conditions following:*

- $a = d = 0$
- $c = b = 0$
- $a = 0$, $bcd \neq 0$ *and* c, d *and* $d - c$ *of the same sign and non-zero*
- $b = 0$, $acd \neq 0$ *and* c, d *and* $d - c$ *of the same sign and non-zero*
- $c = 0$, $abd \neq 0$ *and* a, b *and* $b - a$ *of the same sign and non-zero*
- $d = 0$, $abc \neq 0$ *and* a, b *and* $b - a$ *of the same sign and non-zero*
- $abcd \neq 0, ac < 0, bd < 0, (a + b)(c + d) > 0$ *or* $(a - b)(c - d) > 0$.

Proof: This can be deduced directly from the definition of the octants.

Lemma 3 *If we have a determinantal QAT such as* $\delta \geq \max(| a | + | b |, | c | + | d |)$*; if the paving* $(-1, -1)$ *is either 4-connected or non-connected to the paving* $(0, 0)$ *then the latter has 6 4-connected neighbours. If it is 8-connected to the paving* $(0, 0)$ *then the latter has 4-connected neighbours and 4 8-connected neighbours.*

Proof: In previous paragraph, we showed that the pavings $(1, 0)$ and $(0, 1)$ are always 4-connected to the paving $(0, 0)$. The other pavings susceptible to be neighbours of the latter are the pavings $(\pm 1, \pm 1)$. We will show that only the three following cases are possible:

- the pavings $(-1, -1)$ and $(1, 1)$ are 4-connected to $P_{0,0}$ and the pavings $(-1, 1)$ and $(1, -1)$ are not neighbours of the paving $(0, 0)$.
- the pavings $(-1, 1)$ and $(1, -1)$ are 4-connected to $P_{0,0}$ and the pavings $(-1, -1)$ and $(1, 1)$ not neighbours of the paving $(0, 0)$.
- the 4 pavings are 8-connected to the paving $(0, 0)$

Given that D and D' are two distinct secant lines, they separate the discrete plane in 4 regions $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$ and $\mathcal{R}_4$ as visualised in figure 10 (we consider that the integer points situated on a line belong to the region situated under the line).

If two opposite regions are 4-connected then the two others are not neighbours. Indeed, if (x, y) belongs to $\mathcal{R}_4$ and is such as $(x, y + 1)$ belongs to $\mathcal{R}_2$ then all of the points of the form $(x, y + k)$ (resp. $(x, y - k)$) with k positive belong to $\mathcal{R}_4$ (resp. $\mathcal{R}_2$). The two other regions are therefore separeted by a 4-connected line: they are not neighbours. Inversely, if two opposite regions are not neighbours, the two others are 4-connected. Indeed, two non-neighbours regions can be separated by a 4-connected line belonging to two other regions which are therefore 4-connected.

Considering the euclidien lines of equations $ax + by = 0$ and $cx + dy = 0$. These two lines separate the discrete plane in 4 regions, the pavings $(0, 0)$ and $(-1, -1)$ belonging to two opposite regions and the pavings $(-1, 0)$ and $(0, -1)$ belonging to the two other regions. If the first two pavings are 4-connected, the last two are therefore not neighours and vice versa. The translation of these last two pavings by the vector $(d, -c)$ gives us the pavings $(0, 0)$ and $(1, -1)$. Furthermore, we have shown that the paving (i, j) is the neighbour of the paving $(0, 0)$ if and only if the paving $(-i, -j)$ is the neighbour of the paving $(0, 0)$. We can conclude that:

- if $P_{-1,-1}$ is 4-connected to $P_{0,0}$ then $P_{1,1}$ is 4-connected to $P_{0,0}$ and neither $P_{1,-1}$ nor $P_{-1,1}$ are neighbours of $P_{0,0}$.
- if $P_{-1,-1}$ is not a neighbour of $P_{0,0}$ then $P_{1,1}$ is not a neighbour of $P_{0,0}$ and $P_{1,-1}$ and $P_{-1,1}$ are 4-connected to $P_{0,0}$.

If $P_{-1,-1}$ and $P_{0,0}$ are 8-connected then $P_{0,-1}$ and $P_{-1,0}$ are also 8-connected, indeed:
- if they were not neighbours $P_{-1,-1}$ and $P_{0,0}$ would be 4-connected.
- if they were 4-connected, $P_{-1,-1}$ and $P_{0,0}$ would not be neighbours.

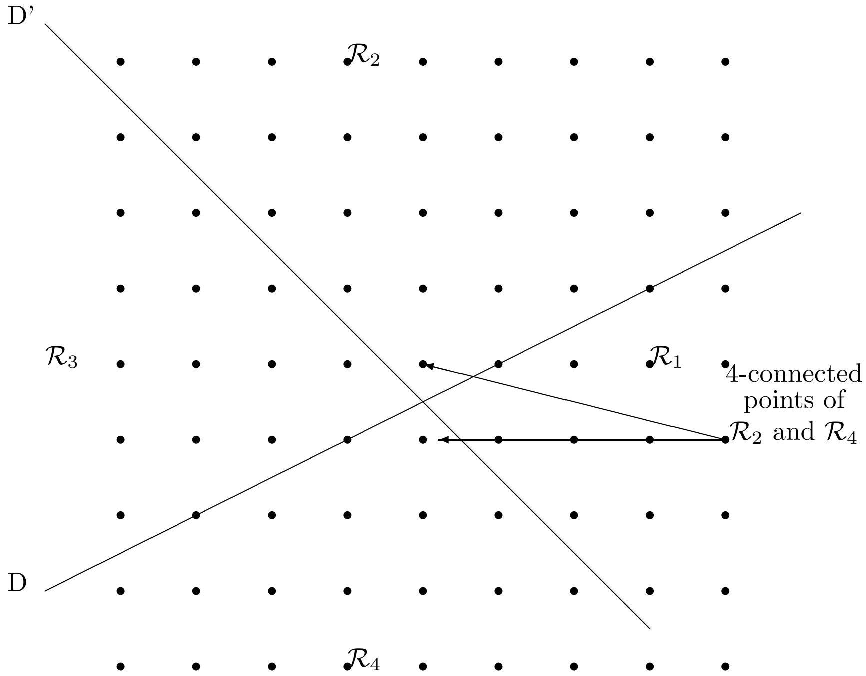

Figure 10: Regions of the plane defined by two concurrent lines.

By translation, we can deduce that $P_{0,0}$ has 4 8-connected neighbours which are the pavings $(\pm 1, \pm 1)$.
Proof of the theorem : We will show that if one of the arithmetical conditions of the lemma is verified (which are equivalents to the geometric conditions of the theorem), then the original paving has 4 4-connected pavings and 4 other 8-connected pavings, otherwise it has 6 4-connected neighbours.

The paving $(0,0)$ is the intersection of the discrete lines:

$$D : \left[\frac{ax+by}{\omega}\right] = 0 \qquad \text{and} \qquad D' : \left[\frac{cx+dy}{\omega}\right] = 0.$$

When $a = d = 0$ or $b = c = 0$, one of the two lines is horizontal and the other vertical. It is clear that in this case the paving $(0,0)$ has 4 4-connected pavings and 4 other 8-connected pavings.

The following 4 cases ($a = 0$ or $b = 0$ or $c = 0$ or $d = 0$) correspond to the case where one of the lines and only one is horizontal or vertical. We will only process the case where $a = 0$ and $bcd \neq 0$, the other cases being prooved analogically. The point $(0,0)$ always belongs to the paving $(0,0)$. We will determine the pavings to which belong the neighbouring points of $(0,0)$, we will deduce the neighbouring pavings of $P_{0,0}$. Since $\delta = ad - bc > 0$, we have $-bc > 0$: b and c are of opposite sign. So we have:

$$\begin{aligned}
[g](1,0) &= \left(0, \left[\frac{c}{\omega}\right]\right) &&= (0, j) \text{ with } j = 0 \text{ or } -1, \\
[g](-1,0) &= \left(0, \left[\frac{-c}{\omega}\right]\right) &&= (0, -1-j), \\
[g](0,1) &= \left(\left[\frac{b}{\omega}\right], \left[\frac{d}{\omega}\right]\right) &&= (-1-j, j') \text{ with } j' = 0 \text{ or } -1 \\
[g](0,-1) &= \left(\left[\frac{-b}{\omega}\right], \left[\frac{-d}{\omega}\right]\right) &&= (j - 1 - j')
\end{aligned}$$

If $-1-j = j'$ (that is to say d and c of opposite sign), then one of the two points $(0,1)$ and $(0,-1)$ belongs to the paving $(-1,-1)$.The latter is therefore 4-connected to the paving $(0,0)$. Following the previous lemma, we conclude that in this case $P_{0,0}$ has 6 4-connected neighbours.

If $j = j'$ (c and d of the same sign), the 4-connected points of $(0,0)$ all belong to $P_{0,-1}, P_{-1,0}$ or $P_{0,0}$, the two first pavings being at least 8-connected. Finding therefore the images of the 8-connected neighbours of

$(0,0)$:

$$\begin{array}{lllll} [g](-1,-1) &=& \left(\left[\frac{-b}{\omega}\right],\left[\frac{-c-d}{\omega}\right]\right) &=& (-1-j,j),\\ [g](1,1) &=& \left(\left[\frac{b}{\omega}\right],\left[\frac{c+d}{\omega}\right]\right) &=& (j,-1-j),\\ [g](1,-1) &=& \left(\left[\frac{-b}{\omega}\right],\left[\frac{c-d}{\omega}\right]\right) &=& (-1-j,j") \text{ with } j"=0 \text{ or } -1,\\ [g](-1,1) &=& \left(\left[\frac{b}{\omega}\right],\left[\frac{-c+d}{\omega}\right]\right) &=& \left\{\begin{array}{lll}(j,-1-j") & \text{if} & c-d\neq 0\\ (j,0) & \text{if} & c-d=0.\end{array}\right. \end{array}$$

If $j"=-1-j$ and $c-d\neq 0$ (that is to say $c-d$ non-zero and of opposite sign to b, therefore the same sign as c and d), then $(1,-1)$ or $(-1,1)$ belongs to the paving $(-1,-1)$ which is therefore at least 8-connected to the paving $(0,0)$. As $P_{-1,0}$ and $P_{0,-1}$ are also 8-connected to the paving $(0,0)$, we can conclude that the paving $(0,0)$ has 4 8-connected neighbours and 4 4-connected neighbours.

If $j"=j$ (that is to say $c-d$ and b of the same sign) or $j"=-1-j$ and $c-d=0$ (that is to say $c=d$ and b negative), then $P_{-1,0}$ and $P_{0,-1}$ are 4-connected. We can conclude, following the previous lemma, that the paving $(0,0)$ has 6 4-connected neighbours. In this case the paving $(0,0)$ has 4 4-connected neighbours and 4 8-connected neighbours.

There remains only the case where $abcd\neq 0$. We determine again the image of the 4-connected points to $(0,0)$:

$$\begin{array}{lllll} [g](1,0) &=& \left(\left[\frac{a}{\omega}\right],\left[\frac{c}{\omega}\right]\right) &=& (i,j) \text{ with } i,j=0 \text{ or } -1,\\ [g](-1,0) &=& \left(\left[\frac{-a}{\omega}\right],\left[\frac{-c}{\omega}\right]\right) &=& (-1-i,-1-j),\\ [g](0,1) &=& \left(\left[\frac{b}{\omega}\right],\left[\frac{d}{\omega}\right]\right) &=& (i',j') \text{ with } i',j'=0 \text{ or } -1,\\ [g](0,-1) &=& \left(\left[\frac{-b}{\omega}\right],\left[\frac{-d}{\omega}\right]\right) &=& (-1-i',-1-j'). \end{array}$$

If $i=j$ or $i'=j'$ (that is to say a and c of the same sign or b and d of the same sign), one of the 4-connected points to $(0,0)$ belongs to the paving $(-1,-1)$ which is therefore 4-connected to the paving $(0,0)$. In this case, following the lemma, the paving $(0,0)$ has 6 4-connected neighbours.

If $i\neq j$ and $i'\neq j'$ (that is to say a and c of opposite sign and b and d of opposite sign), the pavings $P_{0,-1}$ and $P_{-1,0}$ are at least 8-connected and we have:

$$\begin{array}{lllll} [g](-1,-1) &=& \left(\left[\frac{-b}{\omega}\right],\left[\frac{-c-d}{\omega}\right]\right) &=& (x,y) \text{ with } x,y=0 \text{ or } -1,\\ [g](1,1) &=& \left(\left[\frac{b}{\omega}\right],\left[\frac{c+d}{\omega}\right]\right) &=& \left\{\begin{array}{lll}(-1-x,-1-y) & \text{if} & a\neq -b \text{ and } c\neq -d\\ (-1-x,0) & \text{if} & a\neq -b \text{ and } c=-d\\ (0,-1-y) & \text{if} & a=-b \text{ and } c\neq -d\\ (0,0) & \text{if} & a=-b \text{ and } c=-d\end{array}\right.\\ [g](1,-1) &=& \left(\left[\frac{-b}{\omega}\right],\left[\frac{c-d}{\omega}\right]\right) &=& (x',y') \text{ with } x',y'=0 \text{ or } -1\\ [g](-1,1) &=& \left(\left[\frac{b}{\omega}\right],\left[\frac{-c+d}{\omega}\right]\right) &=& \left\{\begin{array}{lll}(-1-x',-1-y') & \text{if} & a\neq b \text{ and } c\neq d\\ (-1-x',0) & \text{if} & a\neq b \text{ and } c=d.\\ (0,-1-y') & \text{if} & a=b \text{ and } c\neq d\\ (0,0) & \text{if} & a=b \text{ and } c=d.\end{array}\right. \end{array}$$

If $x=y, a\neq -b$ and $c\neq -d$ (that is to say $a+b$ and $c+d$ of the same sign and non-zero) or $x'=y', a\neq b$ and $c\neq d$ (that is to say $a-b$ and $c-d$ of the same sign and non-zero), the one of the 4 8-connected points to $(0,0)$ belongs to the paving $(-1,-1)$ which is therefore at least 8-connected to the paving $(0,0)$. As $P_{0,-1}$ and $P_{-1,0}$ we can deduce that the paving $(0,0)$ has 4 4-connected neighbours and 4 other 8-connected neighbours.

The cases $a = -b$ and $c = -d$ or $a = b$ and $c = d$ are impossible because the determinant is estimated to be different from zero.

If $x \neq y$ or $a = -b$ or $c = -d$ and $x' \neq y'$ or $a = b$ or $c = d$ (that is to say $a+b$ and $c+d$ of opposite sign or zero and $a-b$ and $c-d$ of opposite sign or zero), then $P_{0,-1}$ and $P_{-1,0}$ are 4-connected. We conclude that the paving $(0,0)$ has 6 4-connected neighbours.

To summarise: if $abcd \neq 0$, the paving $(0,0)$ has 4 4-connected neighbours and 4 other 8-connected neighbours if a and c are of opposite sign, b and d are of opposite sign, $a+b$ and $c+d$ non-zero and of the same sign or $a-b$ and $c-d$ non-zero and of the same sign.

5 Conclusion

Thanks to quasi-affine transformations we have a simple method of generating an infinite number of periodic pavings of the discrete plane. The pavings are the intersections of discrete lines, that is to say discrete parallelograms. The period of the pavings depends on the choice of coefficients of the QAT: if the QAT is determinantal, we obtain a periodic tiling with one unique paving.

The number of neighbours of a paving can vary between 4 and 6. In the determinantal case there are either 6 4-connected neighbours or 4 4-connected and 4 8-connected neighbours.

The definition of pavings of the order n allows us to obtain a periodic tiling where the pavings have a more general form than the discrete parallelograms. In the determinantal case there is once again one unique paving of the order n.

We have not shown the algorithm which permits us to determine the set of points of a paving as it can be found in [3]. Equally we can find some other results on pavings like a study of the pavings of the order n, a grammar that generates the border of these pavings in the determinantal case and a method to generate fractals with the help of these pavings.

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