Gaussian numeration systems

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Abstract

In this paper, we define an integer division for Gaussian integers which will permit us to link two objects apparently completely different: discrete affine applications and Gaussian numeration systems. Theoretical results of the study of the first ones will permit us to determine new Gaussian numeration systems.

Keywords: Gaussian integers, numeration systems, discrete affine transformations.

1 Introduction.

The first evidences of counting appeared about 50000 years ago. Since the way of representing numbers did not stop evolving. Some 4000 years ago, Babylonians invented the positional notation, using base 60. But the signs we use to do arithmetical operations like additions, substractions,..., only appeared at 1489 in a book written by Johann Widman. In 1484, in France, a manuscript was composed by Nicolas Chuquet, entitled "Triparty en la science des nombres", where the author mention equations that implied imaginary solutions; this manuscript has been printed only in 1880. Some years later, Cardan (1501-1576), Tartaglia (1500-1557), Bombelli (1526-1573), by studying cubic equations, had to deal with the square root of negative numbers but did not know how to manipulate them to solve cubic equations. It is only in 1777 that the symbol i for $\sqrt{-1}$ was adopted, firstly used by Euler.
Nowadays, the most widely used base is the base ten, called decimal notation, which is commonly used for counting, doing additions, substractions, multiplications. It is also well known that if A is a positive integer , greater than one then each positive integer N can be written uniquely in the form:

$$N = a_0 + a_1 A + ... + a_p A^p$$

where, for each $i = 0, 1, ..., p$, $0 \leq a_i < A$. The positive integers less then A are called the digits of the base. In this paper, we will generalise this result to Gaussian integers. More precisely, let $\beta = a + ib$ be a Gaussian integer, we will give conditions such as it exists a set of Gaussian integers $\mathcal{C}$ such as each Gaussian integer c can be written uniquely in the form:

$$c = c_0 + c_1 \beta + ... + c_p \beta^p$$

where, for each $i = 0, 1, ..., p$, $c_i \in C$. We will then say that $(\beta, \mathcal{C})$ is a numeration system.
Gilbert [2], [3] as well as Kata and Szabo [5] have studied this question n the particular case where the set $\mathcal{C}$ is the set of positive integers $C = \{0, 1, ..., a^2 + b^2\}$. They proved that each Gaussian integer can be expressed in the above form if and only if $a < 0$ and $b = \pm 1$.
In the following, we will define an integer division for Gaussian integers, which will induce a "quasi-affine transformation". Theoretical results about the behaviour under iteration of these transformations will allow us to determine new Gaussian numeration systems. We will then give an algorithm to determine the representation of a Gaussian integer in this new numeration system.

2 Definitions

Notations: [ ] will always represent the greatest integer function; if x and y are two integers, $\left[\frac{x}{y}\right]$ will denote the quotient of the integer division of x by y (with positive remainder).

If A is a positive integer greater than one, nd c a Gaussian integer written in the base A, the integer division of c by A can be done by shifting to the left the digits of the decomposition. That is to say, if $c = c_0 + c_1\beta + ... + c_p\beta^p$, then $\left[\frac{c}{A}\right] = c_1 + c_2\beta + ... + c_n\beta^{n-1}$. So, the digits of the base are the positive integers n such as $\left[\frac{n}{A}\right] = 0$ where [ ] represents the greatest integer function. In other words, the digits are the positive integers c such as the quotient when c is divided by A (with positive remainder) is 0.
We will define the digits of the Gaussian numeration systems in the same manner. For this, we firstly have to define the integer division for Gaussian integers.

Définition 1 *Let $c = x + iy$ and $\beta = a + ib$ be two Gaussian integers. The integer division of c by β, noted $\left[\frac{c}{\beta}\right]$, is defined by:*

$$\left[\frac{c}{\beta}\right] = \left[\frac{ax+by}{a^2+b^2}\right] + i\left[\frac{-bx+ay}{a^2+b^2}\right].$$

Remark: let $c' = x' + iy' = \frac{c}{\beta}$ (x' and y' are rational numbers) where c and β are Gaussian integers. The point (x', y') is the image of the point (x, y) by the application g_β defined by :

$$g_\beta : \quad \mathbb{Z}^2 \longrightarrow \mathbb{Q}^2$$
$$(x, y) \longmapsto \begin{cases} x' &= \dfrac{ax+by}{a^2+b^2} \\ y' &= \dfrac{-bx+ay}{a^2+b^2} \end{cases}$$

We the deduce the following property.

Property 1 *The integer division by β corresponds to the composition of the function g_β with the usual integer part function. More precisely, let $\beta = a + ib$ and $c = x + iy$ be two Gaussian integers and let $c' = x' + iy' = \left[\frac{c}{\beta}\right]$. The point (x', y') is then the image of the point (x, y) by the application defined on the discrete plane $\mathbb{Z}^2$ by :*

$$g_\beta : \quad \mathbb{Z}^2 \longrightarrow \mathbb{Q}^2$$
$$(x, y) \longmapsto \begin{cases} x' &= \dfrac{ax+by}{a^2+b^2} \\ y' &= \dfrac{-bx+ay}{a^2+b^2} \end{cases}$$

The application $[g]$ defined above is a particular case of *quasi-affine transformations* or QAT which we define now.

Définition 2 *A quasi-affine transformation or QAT associated to a rational affine application g, is an application on the discrete plane $\mathbb{Z}^2$, defined by the composition of g with an integer part function noted int(). Thus, a QAT is defined by:*

$$[g] : \quad \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2$$
$$(x, y) \longmapsto \begin{cases} x' &= \left[\dfrac{ax+by+e}{\omega}\right] \\ y' &= \left[\dfrac{cx+dy+f}{\omega}\right] \end{cases}$$

where a, b, c, d, e, f are integers and ω is a positive integer.
We call $A = \frac{1}{\omega}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the matrix of the QAT and $\mathcal{V} = \begin{pmatrix} e \\ f \end{pmatrix}$ its translation vector.

A Quasi-Linear Transformation, or QLT, is a QAT whose vector is the vector $\mathcal{V} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

The application $[g_\beta]$ is the Quasi-linear Transformation associated to g_β, the integer part function being the usual integer part function. In the following we will only consider the case where the QATs are defined using the usual integer part function noted $[]$.

Notation Let $c = x + iy$, $\beta = a + ib$ and $c' = x' + iy' = \left[\frac{c}{\beta}\right]$, we will also note $c' = [g_\beta](c)$ which means that $\begin{pmatrix} x' \\ y' \end{pmatrix} = [g_\beta] \begin{pmatrix} x \\ y \end{pmatrix}$. A complex number will be noted (x, y) as well as $x + iy$.

Let us remind that a real linear application g is a contracting application on $\mathbb{R}^2$ if and only if, for each $X \in \mathbb{R}^2$, $\|g(X)\| < \|X\|$ where $\| \;\|$ is a norm on $\mathbb{R}^2$. Let g be e real linear contracting application, it is well known that $(0, 0)$ is the unique fixed point of g and that for each real point Y the sequence $(g^n(Y))_{n\geq 0}$ tends toward $(0, 0)$. It is not the same for the application $[g]$ which can have several fixed points or even cycles. We then call *real-like Quasi-Linear Transformations* the QLTs which have $(0, 0)$ as unique fixed point and such that for each discrete point Y the sequence $([g]^n(Y))_{n\geq 0}$ tends toward $(0, 0)$. We will say that Y converges to X when the sequence $([g]^n(Y))_{n\geq 0}$ tends toward X.

Remark : in the discrete space $\mathbb{Z}^2$, a sequence $(a_n)_{n\leq 0}$ converge to the limit a if and only if there exists a positive integer N such that $a_n = a$ for each $n \leq N$.

In the following section we will see the relation between numeration systems and real-like qats. In the section 4 we will see some properties of these qats which will allow us to determine new Gaussian numeration systems. A theoretical study of general QATs can be found in [6] and [4].

3 Gaussian numeration systems and real-like QATs.

Let us now define the set $\mathcal{C}$ of Gaussian integers c such that $\left[\frac{c}{\beta}\right] = 0$. In other words:

$$\mathcal{C} = \{x' + iy' | [g_\beta](x', y') = (0, 0)\}.$$

The results of this section and the following section will give us necessary and sufficient conditions such that $(\beta, \mathcal{C})$ is a numeration system.

Remark : it seams clear that 0, ± 1 and $\pm i$ can not be bases of the Gaussian integers, so we will only consider the cases where $\beta \neq 0, \pm 1, \pm i$.

Theorem 1 *Let $\beta = a + ib$ be a Gaussian integer, the three following properties are then equivalent:*

1. *$(\beta, \mathcal{C})$ is a numeration system,*
2. *The qat $[g_\beta]$ is a real-like Quasi-Linear Transformation,*
3. *The points $\{(\pm 1, 0), (0, \pm 1), (-1, -1)\}$ converge to $(0, 0)$.*

Lemma 1 *Let $X = x + iy$, it exists an unique decomposition $X = \lambda\beta + C$ where λ is a Gaussian integer and $c \in \mathcal{C}$. Moreover, $\lambda = [g_\beta](X)$.*

Proof : Let us suppose that it exists two decompositions $c = c_0 + c'_0\beta = c_1 + c'_1\beta$. We then have $c_0 = c_1 + \beta Q$ where Q is a Gaussian integer, and so $[g_\beta](c_0) = [g_\beta](c_1) + Q$. But c_0 and c_1 are digits and so $[g_\beta](c_0) = [g_\beta](c_1) = 0$. It follows that $Q = 0$ and $c_0 = c_1$.
Let $X = x + iy$, $\lambda = [g_\beta](X)$ and $c = X - \lambda\beta$, we have:

$$[g_\beta](c) = \left[\frac{c}{\beta}\right] = \left[\frac{X - \lambda\beta}{\beta}\right] = \left[\frac{X}{\beta}\right] - \lambda = 0$$

We can conclude that $X = \lambda\beta + c$ with $c \in \mathcal{C}$ and $\lambda = [g_\beta](c)$.

Remark : this lemma proves that if the decomposition of a Gaussian integer exists, it is always unique.

Lemma 2 *Let D be the set of points $D = \{(\pm 1, 0), (0, \pm 1), (\pm 1, \pm 1)\}$, let c be a point of $\mathcal{C}$ and d a point of D, then $c + d$ can be written in the form $c' + d'\beta$ with $c' \in \mathcal{C}$ and $d' \in D$.*

Proof : Let $X = x + iy$, X belongs to $\mathcal{C}$ if and only if :

$$\begin{cases} 0 \leq \dfrac{ax+by}{a^2+b^2} < 1 \\ 0 \leq \dfrac{-bx+ay}{a^2+b^2} < 1 \end{cases}$$

Let now $Y = x' + iy'$ be a point of D, we then have:

$$\begin{cases} -(|a|+|b|) \leq ax'+by' < |a|+|b| \\ -(|a|+|b|) \leq -bx'+ay' < |a|+|b| \end{cases}$$

Thus

$$\begin{cases} -1 \leq -\dfrac{|a|+|b|}{a^2+b^2} \leq \dfrac{ax+by}{a^2+b^2} + \dfrac{ax'+by'}{a^2+b^2} < 1 + \dfrac{|a|+|b|}{a^2+b^2} \leq 2 \\ -1 \leq -\dfrac{|a|+|b|}{a^2+b^2} \leq \dfrac{-bx+ay}{a^2+b^2} + \dfrac{-bx'+ay'}{a^2+b^2} < 1 + \dfrac{|a|+|b|}{a^2+b^2} \leq 2 \end{cases}$$

It follows that $[g_\beta](X+Y) = (r,s)$ with $r, s = 0, \pm 1$, that is to say $(r,s) \in D$. We can conclude, using lemma 1, that $X + Y = \lambda\beta + c$ with $\lambda \in D$ and $c \in \mathcal{C}$.

Proof of the theorem: Let us remark that $(0,0)$ is a fixed,point of the QLT $[g_\beta]$, so $[g_\beta]$ is a real-like transformation if and only if, for each point (x,y) it exists a positive integer N such that, for each $n \geq N$, $[g_\beta]^n(x,y) = (0,0)$. We will note $ord(X, [g_\beta]) = \inf(k \quad | \quad [g_\beta]^k(X) = (0,0))$.

- Let us first proof that 1) implies 2).
  If $(\beta, \mathcal{C})$ is a numeration system, then each Gaussian integer c can be written uniquely in the form:

  $$c = c_0 + c_1\beta + ... + c_p\beta^p$$

  with $c_i \in \mathcal{C}$ for $i = 0, 1, \ldots, p$. So we have $[g_\beta](c) = c_1 + ... + c_p\beta^{p-1}$ and, recurrently, $[g_\beta]^{p+1} = \left[\dfrac{c_p}{\beta}\right] = 0$. So for each point $Y = (x,y)$ the sequence $([g_\beta]^n(Y))_{n\geq 0}$ tends toward the unique fixed point $(0,0)$: $[g_\beta]$ is a real-like QLT.

  If a Gaussian integer c is written in the base β, the integer division by β corresponds to a shifting to the left of its digits. So, the Gaussian integer c is a digit if and only if $\left[\dfrac{c}{\beta}\right] = 0$. In other words: $\mathcal{C} = \{x' + iy' | [g](x', y') = (0,0)\}$.

- Let us now prove that 2) implies 1).
  We will prove recurrently that for each c the decomposition exists (if it exists it is unique by lemma 1).
  Let c be a Gaussian integer, if $[g_\beta](c) = 0$ that is to say $ord(c, [g_\beta]) = 1$, then c is a digit: its decomposition exists and is unique.
  Let us suppose that the decomposition exists for each Gaussian integer c such that $ord(c, [g_\beta]) \leq n$ and let c' be a Gaussian integer such that $ord(c', [g_\beta]) = n + 1$. Then $c' = c_0 + [g_\beta](c')\beta$ (lemma 1) with $ord([g_\beta](c'), [g_\beta]) = ord(c'[g_\beta]) - 1 = n$ so that $[g_\beta](c')$ has a unique decomposition. By multiplying this decomposition by β and adding c_0 we obtain the decomposition of c'.

- The second assertion implies trivialy the third.

- Let us now prove that 3) implies 1).
  For this we will prove that, if 3) is verified, it exists for each Gaussian integer c, an algorithm to decompose $c \pm 1$ and $c \pm i$ if we know the decomposition of c.
  Let $c = c_0 + c_1\beta + ... + c_p\beta^p +$ with $c_i \in \mathcal{C}$, let $x = (\pm 1, 0)$ or $x = (0, \pm 1)$, x belongs to D so $c + x = c_0 + x + c_1\beta + ... + c_p\beta^p = c'_0 + \beta((c_1 + d_1) + \ldots + c_p\beta^{p-1})$ with $c'_1 \in \mathcal{C}$ and $d_1 \in D$ (lemma 2). By recurrence over p we have:

  $$c + x = c'_0 + c'_1\beta + \ldots + c'_p\beta^p + d_{p+1}\beta^{p+1}$$

  with $c'_i \in \mathcal{C}$ and $d_{p+1} \in D$. If the elements of D converge to $(0,0)$, then d_{p+1} can be expressed in the form : $d_{p+1} = d'_0 + d'_1\beta + \ldots + d'_{p'}\beta^{p'}$ with $d'_i \in D$. So we have the decomposition of $x + c$.

In the following section we will give the conditions for a QLT to be a real-like QLT. We will also give an algorithm to determine the set of Gaussian integers $\mathcal{C}$.

4 QLTs properties.

4.1 Real-like QLTs.

In [4], we have done a theoretical study of the behaviour of quasi-affine transformations under iteration. In particular, we gave the conditions for a QLT, defined by the matrix $A = \frac{1}{\omega}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and such that $\| A \|_\infty \leq 1$, to be a real-like QLT. Let us remind that the infinite norm of a matrix $A = \frac{1}{\omega}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is defined by: $\| A \|_\infty = \frac{1}{\omega} max(|a|+|b|, |c|+|c|)$.
We will only give these conditions for a QLT corresponding to the integer division by a Gaussian integer $\beta = a+ib$, that is to say for $[g_\beta]$.
If $\beta = \pm i, \pm 1, 0$ then $(\beta, \mathcal{C})$ can not be a numeration system and $[g_\beta]$ is not a real-like QLT. So, in the following, we will suppose that $|a|+|b| > 1$. Let us remark that such a QLT verifies allways $\| A \|_\infty \leq 1$. More precisely, if $|a|+|b| = 1$ then $\| A \|_\infty = 1$, else $\| A \|_\infty < 1$.

Theorem 2 *A QLT defined by the matrix* $A = \frac{1}{a^2+b^2}\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ *is real-like if and only if:*

$$a \leq 0 \qquad \text{and} \qquad |a|+|b| > 1.$$

Proof: Let $X = x+iy$ be a point on $\mathbb{Z}^2$, we have :

$$\begin{cases} -\| X \|_\infty \frac{|a|+|b|}{a^2+b^2} & \leq & \frac{ax+by}{a^2+b^2} & < & \| X \|_\infty \frac{|a|+|b|}{a^2+b^2} \\ -\| X \|_\infty \frac{|a|+|b|}{a^2+b^2} & \leq & \frac{-bx+ay}{a^2+b^2} & < & \| X \|_\infty \frac{|a|+|b|}{a^2+b^2} \end{cases}$$

We deduce that $\| [g_\beta](X) \|_\infty \leq \| X \|_\infty$, so, if X belongs to a cycle (i.e it exists an integer n such that $[g_\beta]^n(X) = X$), then it exists a point Y such that $\| Y \|_\infty = \| X \|_\infty$ and $[g_\beta](Y) = X$. As we have seen in the preceding section, $[g_\beta]$ is a real-like QLT if and only if the points of $D = \{(-1,-1), (0,\pm 1), (\pm 1, 0)\}$ converge to $(0,0)$, that is to say if they do not belong to a cycle. We will now prove that these points do not belong to a cycle if and only if $a \leq 0$ and $|a|+|b| > 1$.

- Let, in a first time, consider the case of QLT such that $\| A \|_\infty < 1$, that is to say $|a| > 1$ or $|b| > 1$. Let X be a point of D, we have :

$$\begin{cases} -1 & < & \frac{|a|+|b|}{a^2+b^2} & \leq & \frac{ax+by}{a^2+b^2} & \leq & \frac{|a|+|b|}{a^2+b^2} & < & 1 \\ -1 & < & -\frac{|a|+|b|}{a^2+b^2} & \leq & \frac{-bx+ay}{a^2+b^2} & < & \| X \|_\infty \frac{|a|+|b|}{a^2+b^2} & < & 1 \end{cases}$$

  It follows that $[g_\beta](X) = (-1,0), (0,-1), (-1,-1)$ or $(0,0)$ for all X in D. The other points of D cannot be the image of a point of norm 1 : they do not belong to a cycle.
  Now, if $a > 0$ and $b \geq 0$, $[g_\beta](-1,0) = (\left\lfloor \frac{-a}{a^2+b^2} \right\rfloor, \left\lfloor \frac{b}{a^2+b^2} \right\rfloor) = (-1,0)$: $[g_\beta]$ is not a real-like QLT.
  Now, if $a > 0$ and $b < 0$, $[g_\beta](0,-1) = (\left\lfloor \frac{-b}{a^2+b^2} \right\rfloor, \left\lfloor \frac{-a}{a^2+b^2} \right\rfloor) = (0,-1)$: $[g_\beta]$ is not a real-like QLT.
  If $a \leq 0$ and $b > 0$, $[g_\beta](0,-1) = (-1,0)$, $[g_\beta](-1,0) = (0,0)$ and $[g_\beta](-1,-1) = (-1,0), (0,-1)$ or $(0,0)$: $[g_\beta]$ is a real-like QLT.
  If $a \leq 0$ and $b \leq 0$, $[g_\beta](-1,0) = (0,-1)$ or $(0,0)$, $[g_\beta](0,-1) = (0,0)$ and $[g_\beta](-1,-1) = (0,-1)$ or $(0,0)$: $[g_\beta]$ is a real-like QLT.

- Let us now suppose that $|a| = |b| = 1$.
  If $a = b = 1$, $[g_\beta](-1,0) = (-1,0)$: $[g_\beta]$ is not a real-like QLT.

If $a=1, b=-1$, $[g_\beta](0,-1)=(0,-1)$: $[g_\beta]$ is not a real-like QLT.
If $a=-1, b=1$, $[g_\beta](-1,1)=(1,0)$, $[g_\beta](1,0)=(-1,-1)$, $[g_\beta](-1,-1)=(0,1)$, $[g_\beta](0,1)=(0,-1)$, $[g_\beta](0,-1)=(-1,0)$, $[g_\beta](-1,0)=(0,0)$, $[g_\beta](1,1)=(0,-1)$ and $[g_\beta](1,-1)=(-1,0)$. Each point of D converges to $(0,0)$: $[g_\beta]$ is a real-like QLT.
If $a=b=-1$, $[g_\beta](1,-1)=(0,1)$, $[g_\beta](0,1)=(-1,-1)$, $[g_\beta](-1,-1)=(1,0)$, $[g_\beta](1,0)=(-1,0)$, $[g_\beta](-1,0)=(0,-1)$, $[g_\beta](0,-1)=(0,0)$, $[g_\beta](1,1)=(-1,0)$ and $[g_\beta](-1,1)=(0,-1)$. Each point of D converges to $(0,0)$: $[g_\beta[$ is a real-like QLT.

This end the proof of the theorem.

We then deduce the following theorem:

Theorem 3 *Let $\beta=a+ib$ be a Gaussian integer, $(\beta,\mathcal{C})$ is a numeration system if and only if:*

$$a \leq 0 \qquad \text{and} \qquad (|a|+|b|>1).$$

Proof: it can be deduced from the two preceding theorems.

4.2 Determination of $\mathcal{C}$.

Let us remember the theorem of Bezout. Let x and y be two intergers, it exists two integers u and v such as $xu+yv=\gcd(x,y)$; an algorithm to determine u and v can be found in [1].
In the following we will denote:

$$a'=\frac{a}{\gcd(a,b)} \qquad b'=\frac{b}{\gcd(a,b)} \qquad \delta=a^2+b^2 \qquad \delta'=\frac{\delta}{\gcd(a,b)} \qquad \delta''=\frac{\delta'}{\gcd(a,b)}$$

Let u and v be two integers such that $-b'u+a'v=1$, we will denote $p=au+bv$. It is easy to prove that u and v can be chosen such that $0 \leq p < \delta'$, we will choose them such that this condition is verified.
Let us consider the matrix $U=\begin{pmatrix} a' & u \\ b' & v \end{pmatrix}$; U is an unimodular matrix. So, for each point $\begin{pmatrix} x \\ y \end{pmatrix}$ it exists an unique point $\begin{pmatrix} X \\ Y \end{pmatrix}$ such as $\begin{pmatrix} x \\ y \end{pmatrix}=U\begin{pmatrix} X \\ Y \end{pmatrix}$.
Let $c=x+iy$ be a Gaussian integer, c belongs to $\mathcal{C}$ if and only if:

$$\begin{cases} \left\lfloor \dfrac{ax+by}{\delta} \right\rfloor & = & 0 \\ \left\lfloor \dfrac{-bx+ay}{\delta} \right\rfloor & = & 0 \end{cases}$$

Which is equivalent to

$$\begin{cases} 0 & \leq & ax+by & < & \delta \\ 0 & \leq & -bx+ay & < & \delta \end{cases}$$

Dividing by $\gcd(a,b)$ we obtain:

$$\begin{cases} 0 & \leq & a'x+b'y & < & \delta' \\ 0 & \leq & -b'x+a'y & < & \delta' \end{cases}$$

Replacing x by $a'X+uY$ and y by $-b'X+vY$, we obtain:

$$\begin{cases} 0 & \leq & \delta''X+pY & < & \delta' \\ 0 & \leq & Y & < & \delta' \end{cases}$$

Which is equivalent to

$$\begin{cases} 0 & \leq & Y & < & \delta' \\ \dfrac{-pY}{\delta''} & \leq & X & < & \dfrac{-pY}{\delta''}+\gcd(a,b) \end{cases}$$

That is to say: $X=-\left\lfloor \dfrac{pY}{\delta''} \right\rfloor + k$ with $k=0,...,\gcd(a,b)-1$.

Corollary 1 *The set $\mathcal{C}$ contains a^2+b^2 integers.*

Remark : Gilbert [2] has already proved that, if $(\beta, \mathcal{S})$ with $\beta = a+ib$ is a numeration system, then $\mathcal{S}$ contains a^2+b^2 digits.

We can deduce the following algorithm to determine X and Y which verify these inequalities:

$X = 0$
$Y = 0$
$R = 0$ (R denotes the remainder when pY is divided by δ'')
for $Y = 0$ to $\delta' - 1$
  for $k = 0$ to $\gcd(a, b) - 1$
    (X+k,Y) verify the inequalities
  endfor
  $R = R + p$
  if $R \geq \delta''$
    $R = R - \delta''$
    $X = X - 1$
  endif
endfor

From this algorithm and from the equations $x = a'X + uY$ and $y = -b'X + vY$, we obtain the following algorithm, to determine the set $\mathcal{C}$:

$x = 0$
$y = 0$
$R = 0$
for $k' = 0$ to $\delta' - 1$
  for $k = 0$ to $\gcd(a, b) - 1$
    $x + ka' + i(y - kb')$ belongs to $\mathcal{C}$
  endfor
  $R = R + p$
  if $R \geq \delta''$
    $R = R - \delta''$
    $x = x - a'$
    $y = y + b'$
  endif
  $x = x + u$
  $y = y + v$
endfor

5 Generalization for other divisions.

We have determined a lot of new Gaussian numeration systems $(\beta, \mathcal{C})$ where $\beta = a+ib$ is a Gaussian integer of module greater than one and such that $a \leq 0$. The set of digits $\mathcal{C}$ has been defined with the help of the definition of an integr division for Gaussian divisions. If we use another integer division, we will obtain more new numeration systems. For example, if we replace the usual integer part function by the rounding function we obtain, in the same manner, numerations systems for each Gaussian integer $\beta = a + ib$ such that $(a+1)^2 + (b+1)^2 > 2$. The case of Gaussian integers which do not verify this condition has not been studyed yet. It would be interesting to study these cases as well as the numerations systems obtained for other divisions. Indeed, it seems possible to define numeration systems for each Gaussian integer.

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