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Regular open or closed sets

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We consider regular open (or closed) sets in a topological space and possible generalizations for algebraic openings and closings in a complete lattice. We recall the well-known characterization of any complete Boolean lattice as the set of regular open sets in a topology. We show the relevance of such concepts for representating objects in continuous and digital space.

1. Introduction

Suppose that we represent a material object by a Euclidean set $X \subseteq \mathbb{R}^d$. Should X be topologically closed or open, that is, should it contain its border or not? In [2] it is claimed that known facts about the psychology of human vision "suggest that it is in the nature of human vision to include boundaries of perceived objects, i.e., objects are always seen as being closed." However, from a physical point of view, the material object is not really continuous, but made of molecules, themselves built from atoms, and we know from quantum mechanics that below the atomic level the classical notions of space and localisation break down. This should remind us that the Euclidean space $\mathbb{R}^d$ with its subsets is only a mathematical representation of the physical reality, but not that reality itself. It is chosen because it is more accurate than the digital space $\mathbb{Z}^d$ (limited in resolution), and offers more possibilities than $\mathbb{Q}^d$.

Now the set X cannot be both open and closed (because $\mathbb{R}^d$ is connected). In view of its link with a physical object, we can consider that *the border of X does not matter*. Or we can assume that *X and its complement X^c have nowhere zero thickness*. We will see in Section 4 that both conditions can be made precise in the same way in terms of regular open or closed sets. One says that a set A is *regular open* if $A = \overline{A}^\circ$, a set F is *regular closed* if $F = \overline{F^\circ}$; there is a one-to-one correspondence between regular open and regular closed sets, given by the equivalent relations $F = \overline{A}$ and $A = F^\circ$. Now our two assumptions mean both that for some regular open set A and the regular closed set F corresponding to it, $A \subseteq X \subseteq F$; moreover, all sets Y such that $A \subseteq Y \subseteq F$ will represent the same object as X.

A well-known fact (which we prove again in Section 3) is that the regular open sets form a complete Boolean lattice; moreover the same holds for regular closed sets, the one-to-one correspondence between regular open and regular closed sets given above being an isomorphism between the two complete lattices. Thus the usual set-theoretic operations of union, intersection, and complementation can be generalized to this framework. In Section 4 we show that the 'cellular representation' of digital space $\mathbb{Z}^d$ as a tesselation of $\mathbb{R}^d$ into cells corresponding to pixels (or voxels) can be defined coherently if these cells are treated as regular open or closed sets, and the two cellular representations of the lattice of digital sets (as a lattice either of regular open sets, or of regular closed sets) correspond to the two types of connectivity (axial and diagonal) that can be defined on digital sets. This removes the 'paradoxes' of digital connectivity, as we explained in [8] (but now with a more correct description of the cellular representation of digital space).

It is often thought that a complete Boolean lattice reduces by isomorphism to the case of a family of subsets of a set E which is closed under complementation and arbitrary unions and intersections; sometimes one even reduces it to the the case of the set $\mathcal{P}(E)$ of parts of E. For example in Chapter 2 of [11] Serra writes a misleading comment on the complete Boolean lattice $\mathcal{P}(E)$: "This case, more than example, will be an prototype. In fact, according to a classical result in algebra, for any complete Boolean lattice $\mathcal{P}$ there exists a set E such that $\mathcal{P}$ is isomorphic to a part of the set $\mathcal{P}(E)$ of subsets of E, with the empty set as null element $\emptyset$, and with E itself as universal element U." In [12] one finds an even more wrong characterization: "The most typical complete boolean lattice is that of the set $\mathcal{P}(S)$ of parts of a set S, equipped with the order relation of inclusion. Moreover, one can show that the study of complete boolean lattices can be reduced by isomorphism to the case of $\mathcal{P}(S)$." The error of both authors is to forget that in a complete lattice $\mathcal{L}$ of subsets of a set E, ordered by inclusion, the operations of supremum and infimum (defined on any family of elements of $\mathcal{L}$) will not necessarily coincide with the operations of union and intersection, even if it is true for the binary operations of supremum and infimum of *two* elements of that lattice $\mathcal{L}$ (for example in the Stone representation of a complete Boolean lattice, consisting of all sets both open and closed in a certain topology). In reaction to this confusion, we give in Section 3 the classical results of Stone and others on the characterization of Boolean lattices. It follows then that *the prototype of a complete Boolean lattice is given by the family of regular open sets (or alternately the family of regular closed sets) of an arbitrary topological space*. The set $\mathcal{P}(E)$ of parts of a set E corresponds to the particular case of the discrete topology on E, where every subset of E is both open and closed.

The concept of regular open or closed sets can be generalized in the digital and Euclidean spaces $\mathbb{Z}^d$ and $\mathbb{R}^d$, as well as other complete lattices, to other algebraic openings and closings than the topological ones. Let α be an opening and φ a closing. It is possible to envisage sets which are both open and closed, that is invariant under both α and φ. For example in Section V.C of [10] one considers the 'regular model' consisting of all compact sets in $\mathbb{R}^d$ which are invariant under both the morphological opening and closing by a compact ball of radius $r > 0$. However, while the domains of invariance of α and of φ respectively are both complete lattices, their intersection is generally not a lattice. This prevents the definition of the join and meet of open and closed sets. A better solution is to consider sets which are regular open (invariant under $\alpha\varphi$), or regular closed (invariant under $\varphi\alpha$); regular open sets form a complete lattice isomorphic to the one of regular closed sets (with the isomorphism given by $F = \varphi(A)$ or equivalently $A = \alpha(F)$). We will illustrate this approach in the digital case in Section 4.

In this work we will use several results about morphological filters, that is idempotent increasing operators on a complete lattice. We state them in Section 2.

2. Morphological filters and their domain of invariance

We assume the framework of [5,9] (more precisely, [5] up to and including Subsection 3.1, and [9] up to and including Subsection 2.2). Let $\mathcal{L}$ be a complete lattice for the partial order $\leq$, with least element O and greatest element I, and whose supremum and infimum operations are written $\bigvee$ and $\bigwedge$. Let $\mathbf{T}$ be a group of automorphisms of $\mathcal{L}$. Consider an operator $\psi : \mathcal{L} \to \mathcal{L}$. We recall several definitions from [5,9]. The *range* of ψ is the set $\mathrm{Ran}(\psi)$ of all $\psi(X)$ for $X \in \mathcal{L}$; an *invariant* of ψ is some $X \in \mathcal{L}$ such that $\psi(X) = X$; the *domain of invariance* of ψ is the set $\mathrm{Inv}(\psi)$ of all invariants of ψ. Clearly $\mathrm{Inv}(\psi) \subseteq \mathrm{Ran}(\psi)$; moreover

$$\psi^2 = \psi \iff \mathrm{Ran}(\psi) \subseteq \mathrm{Inv}(\psi) \iff \mathrm{Ran}(\psi) = \mathrm{Inv}(\psi); \tag{2.1}$$

we say then that ψ is *idempotent*. We say that ψ is *increasing* if for any $X, Y \in \mathcal{L}$, $X \leq Y$ implies $\psi(X) \leq \psi(Y)$. Finally ψ is called $\mathbf{T}$-invariant if $\psi\tau = \tau\psi$ for all $\tau \in \mathbf{T}$. A subset $\mathcal{B}$

of $\mathcal{L}$ is $\mathbf{T}$-invariant if $\tau(B) \in \mathcal{B}$ for all $B \in \mathcal{B}$ and $\tau \in \mathbf{T}$.

For any $\mathcal{B} \subseteq \mathcal{L}$, we define the $\mathbf{T}$-opening $\alpha_{\mathcal{B}}^{\mathbf{T}}$ and the $\mathbf{T}$-closing $\varphi_{\mathcal{B}}^{\mathbf{T}}$ as respectively the least and greatest $\mathbf{T}$-invariant operators whose domain of invariance contain $\mathcal{B}$ (see [9]).

We define a *morphological filter* (or in brief a *MF*) as an increasing and idempotent operator, and a $\mathbf{T}$-*morphological filter* (or in brief a $\mathbf{T}$-*MF*) as a $\mathbf{T}$-invariant morphological filter. In the case where $\mathbf{T}$-invariance is not taken into account, all results on $\mathbf{T}$-MFs apply to MFs by taking $\mathbf{T} = \{\mathbf{id}\}$. A detailed study of morphological filters (without $\mathbf{T}$-invariance) has been made by Matheron in Chapter 6 of [11]. In particular, the following two results characterising $\mathbf{T}$-MFs in terms of $\mathbf{T}$-invariant complete lattices embedded in $\mathcal{L}$ are simply extensions to $\mathbf{T}$-invariance of results from Section 6.2 of [11]:

Proposition 2.1. *Let ψ be a $\mathbf{T}$-MF. Then $\mathrm{Inv}(\psi)$ is a $\mathbf{T}$-invariant complete lattice, with least element $\psi(O)$ and greatest element $\psi(I)$; given $\mathcal{S} \subseteq \mathrm{Inv}(\psi)$, the supremum and infimum of $\mathcal{S}$ in $\mathrm{Inv}(\psi)$ are $\psi(\bigvee \mathcal{S})$ and $\psi(\bigwedge \mathcal{S})$.*

Proof. For $S \in \mathrm{Inv}(\psi)$ and $\tau \in \mathbf{T}$ we have $\psi\big(\tau(S)\big) = \tau\big(\psi(S)\big) = \tau(S)$, so that $\tau(S) \in \mathrm{Inv}(\psi)$; hence $\mathrm{Inv}(\psi)$ is $\mathbf{T}$-invariant.

For $X \in \mathcal{L}$, $O \leq X \leq I$, and as ψ is increasing, $\psi(O) \leq \psi(X) \leq \psi(I)$, so that $\psi(O)$ and $\psi(I)$ are the least and greatest elements of $\mathrm{Ran}(\psi) = \mathrm{Inv}(\psi)$. Let $\mathcal{S} \subseteq \mathrm{Inv}(\psi)$ and suppose that $U \in \mathrm{Inv}(\psi)$ is an upper bound of $\mathcal{S}$: for all $S \in \mathcal{S}$, $U \geq S$. Thus for $S \in \mathcal{S}$ we have $S \leq \bigvee \mathcal{S} \leq U$, and as ψ is increasing,

$$S = \psi(S) \leq \psi(\bigvee \mathcal{S}) \leq \psi(U) = U;$$

this means that $\psi(\bigvee \mathcal{S})$ is the least upper bound of $\mathcal{S}$ in $\mathrm{Ran}(\psi) = \mathrm{Inv}(\psi)$. We prove in the same way that $\psi(\bigwedge \mathcal{S})$ is the greatest lower bound of $\mathcal{S}$ in $\mathrm{Inv}(\psi)$. ∎

Proposition 2.2. *Let $\mathcal{B}$ be a $\mathbf{T}$-invariant complete lattice included in $\mathcal{L}$, with supremum and infimum operations written $\bigvee^{\mathcal{B}}$ and $\bigwedge^{\mathcal{B}}$. The set of $\mathbf{T}$-MFs having $\mathcal{B}$ as domain of invariance is not empty; its least element is $\varphi_{\mathcal{B}}^{\mathbf{T}}\alpha_{\mathcal{B}}^{\mathbf{T}}$ and its greatest element is $\alpha_{\mathcal{B}}^{\mathbf{T}}\varphi_{\mathcal{B}}^{\mathbf{T}}$. Moreover, for any $X \in \mathcal{L}$, we have*

$$\varphi_{\mathcal{B}}^{\mathbf{T}}\alpha_{\mathcal{B}}^{\mathbf{T}}(X) = \bigvee^{\mathcal{B}}\{B \in \mathcal{B} \mid B \leq X\}$$

and $$\alpha_{\mathcal{B}}^{\mathbf{T}}\varphi_{\mathcal{B}}^{\mathbf{T}}(X) = \bigwedge^{\mathcal{B}}\{B \in \mathcal{B} \mid B \geq X\}.$$

Proof. We show only the half of the statement concerning $\varphi_{\mathcal{B}}^{\mathbf{T}}\alpha_{\mathcal{B}}^{\mathbf{T}}$. The other half about $\alpha_{\mathcal{B}}^{\mathbf{T}}\varphi_{\mathcal{B}}^{\mathbf{T}}$ follows by duality. As $\varphi_{\mathcal{B}}^{\mathbf{T}}$ and $\alpha_{\mathcal{B}}^{\mathbf{T}}$ are $\mathbf{T}$-invariant, so is $\varphi_{\mathcal{B}}^{\mathbf{T}}\alpha_{\mathcal{B}}^{\mathbf{T}}$. Now $\varphi_{\mathcal{B}}^{\mathbf{T}}\alpha_{\mathcal{B}}^{\mathbf{T}}$ is a MF by Criterion 6.6 in Section 6.1 of [11]. Let $X \in \mathcal{L}$ and $\mathcal{B}(X)$ the set of $B \in \mathcal{B}$ such that $B \leq X$. As $\mathcal{B}$ is $\mathbf{T}$-invariant, $\alpha_{\mathcal{B}}^{\mathbf{T}}(X) = \bigvee \mathcal{B}(X)$ (see [9]). For any $C \in \mathcal{B}$, $C \geq \alpha_{\mathcal{B}}^{\mathbf{T}}(X) = \bigvee \mathcal{B}(X)$ if and only if $C \geq B$ for every $B \in \mathcal{B}(X)$, in other words if and only if $C \geq \bigvee^{\mathcal{B}} \mathcal{B}(X)$; hence $\bigvee^{\mathcal{B}} \mathcal{B}(X)$ is the least element of $\mathcal{B}$ which is $\geq \alpha_{\mathcal{B}}^{\mathbf{T}}(X)$. As $\mathcal{B}$ is $\mathbf{T}$-invariant, we have (see [9]):

$$\varphi_{\mathcal{B}}^{\mathbf{T}}\alpha_{\mathcal{B}}^{\mathbf{T}}(X) = \bigwedge\{C \in \mathcal{B} \mid C \geq \alpha_{\mathcal{B}}^{\mathbf{T}}(X)\} = \bigwedge\{C \in \mathcal{B} \mid C \geq \bigvee^{\mathcal{B}} \mathcal{B}(X)\} = \bigvee^{\mathcal{B}} \mathcal{B}(X).$$

In particular $\varphi^{\mathbf{T}}_{\mathcal{B}}\alpha^{\mathbf{T}}_{\mathcal{B}}(X) \in \mathcal{B}$ and for $B \in \mathcal{B}$ we have $B = \varphi^{\mathbf{T}}_{\mathcal{B}}\alpha^{\mathbf{T}}_{\mathcal{B}}(B)$; hence $\mathrm{Ran}(\varphi^{\mathbf{T}}_{\mathcal{B}}\alpha^{\mathbf{T}}_{\mathcal{B}}) = \mathrm{Inv}(\varphi^{\mathbf{T}}_{\mathcal{B}}\alpha^{\mathbf{T}}_{\mathcal{B}}) = \mathcal{B}$. Let ψ be a $\mathbf{T}$-MF such that $\mathrm{Inv}(\psi) = \mathcal{B}$. For all $B \in \mathcal{B}(X)$ we have $B \leq X$ and so $B = \psi(B) \leq \psi(X)$, since ψ is increasing; as $\psi(X) \in \mathcal{B}$, we have thus $\varphi^{\mathbf{T}}_{\mathcal{B}}\alpha^{\mathbf{T}}_{\mathcal{B}}(X) = \bigvee^{\mathcal{B}} \mathcal{B}(X) \leq \psi(X)$. Therefore $\varphi^{\mathbf{T}}_{\mathcal{B}}\alpha^{\mathbf{T}}_{\mathcal{B}}$ is the least $\mathbf{T}$-MF having $\mathcal{B}$ as domain of invariance. ∎

As shown first by Matheron (see Section 6.2 of [11]), for any $\mathcal{S} \subseteq \mathcal{B}$ and any $\mathbf{T}$-MF ψ having $\mathcal{B}$ as domain of invariance, $\psi(\bigvee \mathcal{S}) = \varphi^{\mathbf{T}}_{\mathcal{B}}\alpha^{\mathbf{T}}_{\mathcal{B}}(\bigvee \mathcal{S}) = \varphi^{\mathbf{T}}_{\mathcal{B}}(\bigvee \mathcal{S})$ and $\psi(\bigwedge \mathcal{S}) = \alpha^{\mathbf{T}}_{\mathcal{B}}\varphi^{\mathbf{T}}_{\mathcal{B}}(\bigwedge \mathcal{S}) = \alpha^{\mathbf{T}}_{\mathcal{B}}(\bigwedge \mathcal{S})$. Let us derive this result for $\bigvee \mathcal{S}$ (the corresponding one for $\bigwedge \mathcal{S}$ follows by duality). As $\mathcal{B} = \mathrm{Inv}(\psi) = \mathrm{Inv}(\varphi^{\mathbf{T}}_{\mathcal{B}}\alpha^{\mathbf{T}}_{\mathcal{B}})$, by Proposition 2.1 we have $\psi(\bigvee \mathcal{S}) = \bigvee^{\mathcal{B}} \mathcal{S} = \varphi^{\mathbf{T}}_{\mathcal{B}}\alpha^{\mathbf{T}}_{\mathcal{B}}(\bigvee \mathcal{S})$; as $\mathcal{S} \subseteq \mathrm{Inv}(\alpha^{\mathbf{T}}_{\mathcal{B}})$ and $\mathrm{Inv}(\alpha^{\mathbf{T}}_{\mathcal{B}})$ is sup-closed (see [9]), $\alpha^{\mathbf{T}}_{\mathcal{B}}(\bigvee \mathcal{S}) = \bigvee \mathcal{S}$, so that $\varphi^{\mathbf{T}}_{\mathcal{B}}\alpha^{\mathbf{T}}_{\mathcal{B}}(\bigvee \mathcal{S}) = \varphi^{\mathbf{T}}_{\mathcal{B}}(\bigvee \mathcal{S})$.

Given two MFs ψ and ξ, we have $\psi\xi = \xi$ if and only if $\mathrm{Inv}(\xi) \subseteq \mathrm{Inv}(\psi)$. In this case we can prove the following:

Proposition 2.3. *Let ψ and ξ be two MFs such that* $\mathrm{Inv}(\xi) \subseteq \mathrm{Inv}(\psi)$. *For any $\mathcal{S} \subseteq \mathrm{Inv}(\xi)$ we have:*

(*i*) $\psi(\bigvee \mathcal{S}) \leq \xi(\bigvee \mathcal{S})$, *and if $\psi \geq \xi$, $\psi(\bigvee \mathcal{S}) = \xi(\bigvee \mathcal{S})$.*

(*ii*) $\psi(\bigwedge \mathcal{S}) \geq \xi(\bigwedge \mathcal{S})$, *and if $\psi \leq \xi$, $\psi(\bigwedge \mathcal{S}) = \xi(\bigwedge \mathcal{S})$.*

Proof. We show only (*i*), since (*ii*) follows by duality. By Proposition 1 $\xi(\bigvee \mathcal{S})$ is the supremum of $\mathcal{S}$ in the complete lattice $\mathrm{Inv}(\xi)$, and $\psi(\bigvee \mathcal{S})$ is the supremum of $\mathcal{S}$ the complete lattice in $\mathrm{Inv}(\psi)$. As $\mathrm{Inv}(\xi) \subseteq \mathrm{Inv}(\psi)$, the supremum of $\mathcal{S}$ in $\mathrm{Inv}(\xi)$ is an upper bound of $\mathcal{S}$ in $\mathrm{Inv}(\psi)$, and hence it is greater than or equal the supremum of $\mathcal{S}$ in $\mathrm{Inv}(\psi)$. Therefore $\psi(\bigvee \mathcal{S}) \leq \xi(\bigvee \mathcal{S})$. When $\psi \geq \xi$, we get the converse inequality $\psi(\bigvee \mathcal{S}) \geq \xi(\bigvee \mathcal{S})$, and the equality follows. ∎

Proposition 2.4. *Given two increasing operators ζ, η such that $\zeta\eta$ and $\eta\zeta$ are MFs, then* $\mathrm{Inv}(\zeta\eta)$ *and* $\mathrm{Inv}(\eta\zeta)$ *are isomorphic complete lattices: $Z \in \mathrm{Inv}(\zeta\eta)$ and $Y \in \mathrm{Inv}(\eta\zeta)$ correspond under this isomorphism by the equivalent relations $Z = \zeta(Y)$ and $Y = \eta(Z)$.*

The proof is straightforward and is left to the reader. In particular, for an adjunction (ε, δ), $\mathrm{Inv}(\varepsilon\delta)$ and $\mathrm{Inv}(\delta\varepsilon)$ are isomorphic.

When ζ and η are MFs and $\zeta \geq \eta$, then $\zeta\eta$ and $\eta\zeta$ are MFs by Criterion 6.6 in Section 6.1 of [11], $\zeta \geq \zeta\eta$ and $\mathrm{Inv}(\zeta\eta) = \mathrm{Ran}(\zeta\eta) \subseteq \mathrm{Ran}(\zeta) = \mathrm{Inv}(\zeta)$, $\eta \leq \eta\zeta$ and $\mathrm{Inv}(\eta\zeta) = \mathrm{Ran}(\eta\zeta) \subseteq \mathrm{Ran}(\eta) = \mathrm{Inv}(\eta)$, and we can thus apply Propositions 2.1, 2.3, and 2.4:

Corollary 2.5. *Let ζ and η be two $\mathbf{T}$-MFs such that $\zeta \geq \eta$. Then* $\mathrm{Inv}(\eta\zeta)$ *and* $\mathrm{Inv}(\zeta\eta)$ *are isomorphic $\mathbf{T}$-invariant complete lattices, where $Y \in \mathrm{Inv}(\eta\zeta)$ corresponds to $Z \in \mathrm{Inv}(\zeta\eta)$ by the equivalent relations $Z = \zeta(Y)$ and $Y = \eta(Z)$. For $\mathcal{Y} \subseteq \mathrm{Inv}(\eta\zeta)$, its supremum and infimum in* $\mathrm{Inv}(\eta\zeta)$ *are given by*

$$\bigvee^{\mathrm{Inv}(\eta\zeta)} \mathcal{Y} = \eta\zeta(\bigvee \mathcal{Y}) \qquad \text{and} \qquad \bigwedge^{\mathrm{Inv}(\eta\zeta)} \mathcal{Y} = \eta\zeta(\bigwedge \mathcal{Y}) = \eta(\bigwedge \mathcal{Y}).$$

For $\mathcal{Z} \subseteq \mathrm{Inv}(\zeta\eta)$, its supremum and infimum in $\mathrm{Inv}(\zeta\eta)$ *are given by*

$$\bigvee^{\mathrm{Inv}(\zeta\eta)} \mathcal{Z} = \zeta\eta(\bigvee \mathcal{Z}) = \zeta(\bigvee \mathcal{Z}) \qquad \text{and} \qquad \bigwedge^{\mathrm{Inv}(\zeta\eta)} \mathcal{Z} = \zeta\eta(\bigwedge \mathcal{Z}).$$

In particular if we take for η an opening α and for ζ a closing φ, we get $\varphi\alpha(\bigvee \mathcal{S}) = \varphi(\bigvee \mathcal{S})$ for $\mathcal{S} \subseteq \mathrm{Inv}(\varphi\alpha)$, while $\alpha\varphi(\bigwedge \mathcal{T}) = \alpha(\bigwedge \mathcal{T})$ for $\mathcal{T} \subseteq \mathrm{Inv}(\alpha\varphi)$. This generalizes what we said after Proposition 2.2 about $\varphi_{\mathcal{B}}^{\mathbf{T}}$ and $\alpha_{\mathcal{B}}^{\mathbf{T}}$ for a $\mathbf{T}$-invariant complete lattice $\mathcal{B} \subseteq \mathcal{L}$.

Our next result uses the same framework as Corollary 2.5:

PROPOSITION 2.6. *Let ζ and η be two $\mathbf{T}$-MFs such that $\zeta \geq \eta$. Then: (a) The following two statements are equivalent for all $Y, Z \in \mathcal{L}$:*

(i) Y and Z are corresponding elements of $\mathrm{Inv}(\eta\zeta)$ and $\mathrm{Inv}(\zeta\eta)$, in other words $Y \in \mathrm{Inv}(\eta\zeta)$, $Z \in \mathrm{Inv}(\zeta\eta)$, $Y = \eta(Z)$, and $Z = \eta(Y)$.

(ii) $Y \leq Z$ and there is some $S \in \mathcal{L}$ such that $Y = \eta\zeta(S)$ and $Z = \zeta\eta(S)$.

(b) For any $M \in \mathcal{L}$, the following two statements are equivalent:

(iii) There are $Y, Z \in \mathcal{L}$ satisfying (i) such that $Y \leq M \leq Z$.

(iv) $\eta\zeta(M) \leq M \leq \zeta\eta(M)$.

Moreover in (iii) Y and Z are uniquely determined by $Y = \eta(M)$ and $Z = \zeta(M)$.

PROOF. (a) *(i) implies (ii)*: As $\eta \leq \zeta$, $Y = \eta(Y) \leq \zeta(Y) = Z$. We have $Y = \eta\zeta(Y)$ and $Z = \zeta(Y) = \zeta\eta(Y)$, so we take $S = Y$.

(ii) implies (i): As $Y \in \mathrm{Ran}(\eta\zeta)$ and $Z \in \mathrm{Ran}(\zeta\eta)$, we get $Y \in \mathrm{Inv}(\eta\zeta)$ and $Z \in \mathrm{Inv}(\zeta\eta)$. As $Y \leq Z$, $\zeta(Y) \leq \zeta(Z) = Z$; now $\zeta\eta\zeta \geq \zeta\eta$, and so $\zeta(Y) = \zeta\eta\zeta(S) \geq \zeta\eta(S) = Z$; combining both inequalities, $Z = \zeta(Y)$. Thus Y corresponds to Z and $Y = \eta(Z)$.

(b) *(iii) uniquely determines $Y = \eta(M)$ and $Z = \zeta(M)$*: Indeed, applying η to the inequality $Y \leq M \leq Z$ gives $Y = \eta(Y) \leq \eta(M) \leq \eta(Z) = Y$, that is $Y = \eta(M)$, and applying ζ to that inequality gives $Z = \zeta(M)$.

(iii) implies (iv): We have $\eta\zeta(M) = \eta(Z) = Y \leq M$ and $\zeta\eta(M) = \zeta(Y) = Z \geq M$.

(iv) implies (iii): We set $Y = \eta\zeta(M)$ and $Z = \zeta\eta(M)$, and (ii) is satisfied with $S = M$, so that Y, Z verify (i). ∎

In the next two sections, we will apply Corollary 2.5 and Proposition 2.6 in the case where η is the topological interior and ζ the topological closure.

3. Characterization of complete Boolean algebras

A complete Boolean lattice can be characterized as the lattice of regular open sets in a topology. We show this directly in the first subsection. In the second one we recall Stone's representation theorem and the refinement it gives to that characterization.

3.1. Direct analysis

Let $\mathcal{E}$ be a topological space; we momentarily drop the question of $\mathbf{T}$-invariance. For any $X \subseteq \mathcal{E}$, write X^c for its complement in $\mathcal{E}$. The operator $\alpha : \mathcal{E} \to \mathcal{E} : X \mapsto X^\circ$, associating to a set its interior, is an opening, while the operator $\varphi : \mathcal{E} \to \mathcal{E} : X \mapsto \overline{X}$, associating to a set its closure, is a closing. The set of regular open sets is $\mathrm{Inv}(\alpha\varphi)$, while the set of regular closed sets is $\mathrm{Inv}(\varphi\alpha)$. Corollary 2.5 can be expressed as follows:

The set of regular open sets and the set of regular closed sets are isomorphic complete lattices, where a regular open set A corresponds to a regular closed set F by the equivalent *relations $F = \overline{A}$ and $A = F^\circ$. Given a family of regular open sets A_j ($j \in J$), its supremum and infimum are given by*

$$\Big(\overline{\bigcup_{j\in J} A_j}\Big)^\circ \qquad \text{and} \qquad \Big(\overline{\bigcap_{j\in J} A_j}\Big)^\circ = \Big(\bigcap_{j\in J} A_j\Big)^\circ. \tag{3.1}$$

Given a family of regular closed sets F_j ($j \in J$), its supremum and infimum are given by

$$\overline{\Big(\bigcup_{j\in J} F_j\Big)^\circ} = \overline{\bigcup_{j\in J} F_j} \qquad \text{and} \qquad \overline{\Big(\bigcap_{j\in J} F_j\Big)^\circ}. \tag{3.2}$$

Note that $\emptyset$ and $\mathcal{E}$ are both regular open and regular closed; they are thus the universal bounds of both complete lattices. Furthermore the infimum of two regular open sets is their intersection:

$$A_1 \wedge A_2 = (A_1 \cap A_2)^\circ = A_1 \cap A_2. \tag{3.3}$$

Similarly the supremum of two regular closed sets is their union.

In addition to the isomorphism $A \mapsto \overline{A}$ between the complete lattice of regular open sets and that of regular closed sets, we have also the dual isomorphism $A \mapsto A^c$. Together they form a dual automorphism $A \mapsto (A^c)^\circ = \big(\overline{A}\big)^c$ of the complete lattice of regular open sets. Thanks to the underlying topology, we can prove the following:

Proposition 3.1. *The complete lattice of regular open sets is Boolean, that is distributive and complemented. The complement of a regular open set A is $(A^c)^\circ = \big(\overline{A}\big)^c$.*

Proof. Given an *open* set A and any $T \subseteq \mathcal{E}$, we have

$$\begin{aligned} A \cap T^\circ &= (A \cap T)^\circ \\ \text{and} \quad A \cap \overline{T} &\subseteq \overline{A \cap T}. \end{aligned} \tag{3.4}$$

The first equality follows from the fact that $(A \cap T)^\circ = A^\circ \cap T^\circ$, while the second is proven as follows: let $x \in A \cap \overline{T}$, and take any open neighbourhood $V(x)$ of x; as $x \in A$ (an open set), $A \cap V(x)$ is an open neighbourhood of x, and as $x \in \overline{T}$, $A \cap V(x)$ contains some $t \in T$; thus any $V(x)$ contains $t \in A \cap T$, and so $x \in \overline{A \cap T}$.

Let us now show that our complete lattice is distributive. Given regular open sets A, B, C we obtain from (3.3) and (3.4):

$$\begin{aligned} A \wedge (B \vee C) = A \cap (B \vee C) = A \cap \big(\overline{B \cup C}\big)^\circ = \big[A \cap \big(\overline{B \cup C}\big)\big]^\circ \\ \subseteq \big[\overline{A \cap (B \cup C)}\big]^\circ = \big[\overline{(A \cap B) \cup (A \cap C)}\big]^\circ = (A \wedge B) \vee (A \wedge C). \end{aligned}$$

Thus $A \wedge (B \vee C) \subseteq (A \wedge B) \vee (A \wedge C)$; the converse inequality is trivial, and so distributivity follows.

It remains to show that our lattice is complemented. For a regular open set A, $A' = (A^c)^\circ = \big(\overline{A}\big)^c$ is also regular open. We have

$$\begin{aligned} A \wedge A' \subseteq A \cap A' = A \cap (A^c)^\circ \subseteq A \cap A^c = \emptyset, \\ \text{and} \quad A \vee A' = \big(\overline{A \cup A'}\big)^\circ \supseteq \big(\overline{A} \cup A'\big)^\circ = \big(\overline{A} \cup (\overline{A})^c\big)^\circ = \mathcal{E}^\circ = \mathcal{E}. \end{aligned}$$

Hence $A \wedge A' = O$ and $A \vee A' = I$. ∎

We will now show that every complete Boolean lattice is isomorphic to the complete lattice of regular open sets in a suitable topology. The property (3.3) and the fact that $\emptyset$ and $\mathcal{E}$ are regular open sets are sufficient to prove that characterization in the case of a complete lattice of sets:

THEOREM 3.2. *Let $\mathcal{E}$ be a set and let $\mathcal{L}$ be a complete Boolean lattice of subsets of $\mathcal{L}$, ordered by inclusion, such that $\emptyset, \mathcal{E} \in \mathcal{L}$, and for any $X, Y \in \mathcal{L}$, $X \cap Y \in \mathcal{L}$. Let $\mathcal{A}$ be the family of subsets of $\mathcal{E}$ generated by arbitrary unions of elements of $\mathcal{L}$. Then $\mathcal{E}$ is a topological space with $\mathcal{A}$ as set of open sets and $\mathcal{L}$ as set of regular open sets.*

PROOF. $\mathcal{A}$ is the set of all $\bigcup \mathcal{X}$ for $\mathcal{X} \subseteq \mathcal{L}$. As $\emptyset, \mathcal{E} \in \mathcal{L}$, $\emptyset, \mathcal{E} \in \mathcal{A}$. Clearly a union of elements of $\mathcal{A}$ remains in $\mathcal{A}$. Given $A = \bigcup \mathcal{X}$ and $B = \bigcup \mathcal{Y}$ for $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{L}$, infinite distributivity implies that

$$(\bigcup \mathcal{X}) \cap (\bigcup \mathcal{Y}) = \bigcup_{Y \in \mathcal{Y}} \Big((\bigcup \mathcal{X}) \cap Y\Big) = \bigcup_{Y \in \mathcal{Y}} \bigcup_{X \in \mathcal{X}} (X \cap Y),$$

which is again in $\mathcal{A}$, since each $X \cap Y \in \mathcal{L}$. Thus the intersection of two elements of $\mathcal{A}$ is in $\mathcal{A}$. Therefore $\mathcal{A}$ endows $\mathcal{E}$ with a topology.

As $\mathcal{L} \subseteq \mathcal{A}$, elements of $\mathcal{L}$ are open. Consider a family X_j ($j \in J$) of elements of $\mathcal{L}$. Let $U = \bigcup_{j \in J} X_j$ and $V = \bigvee_{j \in J} X_j$. Clearly $V \in \mathcal{L}$ and $U \in \mathcal{A}$. For every $Y \in \mathcal{L}$ we have the following equivalences:

$$\begin{aligned}
Y \subseteq V' &\iff Y \wedge \Big(\bigvee_{j \in J} X_j\Big) = Y \wedge V = \emptyset; \\
&\iff \forall j \in J,\ Y \cap X_j = Y \wedge X_j = \emptyset \qquad \text{(by infinite distributivity in } \mathcal{L}\text{);} \\
&\iff Y \cap U = Y \cap \Big(\bigcup_{j \in J} X_j\Big) = \emptyset; \\
&\iff Y \subseteq U^c; \\
&\iff Y \subseteq U^{c\circ} \qquad \text{(since } Y \text{ is open).}
\end{aligned}$$

Thus V' is the greatest $Y \in \mathcal{L}$ such that $Y \subseteq U^{c\circ}$, and as $U^{c\circ}$ is the union of such sets Y, we have $V' = U^{c\circ}$, that is

$$\Big(\bigvee_{j \in J} X_j\Big)' = \Big(\bigcup_{j \in J} X_j\Big)^{c\circ}.$$

Applying this equality to the family consisting only of V', where $V = \bigvee_{j \in J} X_j$, we find that $V = (V')' = (V')^{c\circ} = (U^{c\circ})^{c\circ}$, in other words

$$\bigvee_{j \in J} X_j = \Big(\bigcup_{j \in J} X_j\Big)^{c\circ c\circ} = \Big(\overline{\bigcup_{j \in J} X_j}\Big)^{\circ}.$$

Now any $U \in \mathcal{A}$ takes the form $U = \bigcup_{j \in J} X_j$, where $X_j \in \mathcal{L}$, and clearly $U \in \mathcal{L}$ if and only if $U = \bigvee_{j \in J} X_j$, which by the preceding equation is equivalent to $U = \overline{U}^{\circ}$. Therefore $\mathcal{L}$ is the set of regular open sets of $\mathcal{A}$. ∎

Now this result can be applied to any complete Boolean lattice:

Corollary 3.3. *Every complete Boolean lattice is isomorphic to the complete lattice of regular open sets of some topological space.*

Proof. Let $\mathcal{L}$ be a complete Boolean lattice. Let $\mathcal{E} = \mathcal{L} \setminus \{O\}$, and define the map $\psi : \mathcal{L} \to \mathcal{P}(\mathcal{E})$ by setting

$$\psi(X) = \{Y \in \mathcal{E} \mid Y \leq X\} \qquad \text{for} \quad X \in \mathcal{L}.$$

Let $\mathcal{M} = \mathrm{Ran}(\psi)$, the set of all $\psi(X)$, where $X \in \mathcal{L}$. For any $X, Y \in \mathcal{L}$, we have $\psi(X) \subseteq \psi(Y) \iff X \leq Y$, and so ψ is an isomorphism $\mathcal{L} \to \mathcal{M}$. Thus $(\mathcal{M}, \subseteq)$ is a complete Boolean lattice isomorphic to $(\mathcal{L}, \leq)$. Now $\emptyset = \psi(O)$ and $\mathcal{E} = \psi(I)$, so that $\emptyset, \mathcal{E} \in \mathcal{M}$. Moreover, for any $X, Y \in \mathcal{L}$ we have $\psi(X \wedge Y) = \psi(X) \cap \psi(Y)$. Thus $\mathcal{M}$ is closed under binary intersection. By Theorem 3.2, $\mathcal{E}$ can be endowed with a topology such that $\mathcal{M}$ is the set of regular open sets. ∎

Note that for any family X_j $(j \in J)$ we have

$$\psi\Big(\bigwedge_{j \in J} X_j\Big) = \bigwedge_{j \in J} \psi(X_j),$$

so that $\mathcal{M}$ is closed under arbitrary intersection.

We have shown that a complete lattice is Boolean if and only if its is isomorphic to the complete lattice of regular open sets in some topological space. The complete Boolean lattice $(\mathcal{P}(\mathcal{E}), \subseteq)$ is a particular case, where $\mathcal{E}$ is endowed with the *discrete* topology; here every subset of $\mathcal{E}$ is regular open.

3.2. Stone's representation theory

We end this section by stating the classical results of Stone and others on the topological characterization of Boolean lattices. For this purpose we must recall a few definitions.

Let $\mathcal{T}$ be a topological space. $\mathcal{T}$ is *Hausdorff* or *T2* if every two distinct points in $\mathcal{T}$ have disjoint neighbourhoods: for $x, y \in \mathcal{T}$, $x \neq y$, there exists $V(x), V(y) \subseteq \mathcal{T}$, with $x \in V(x)^\circ$, $y \in V(y)^\circ$, and $V(x) \cap V(y) = \emptyset$. One calls $\mathcal{T}$ *compact* if every family of open sets covering $\mathcal{T}$ contains a finite subfamily covering it. $\mathcal{T}$ is *connected* if $\emptyset$ and $\mathcal{T}$ are the only open-and-closed subsets of $\mathcal{T}$. A subset $\mathcal{S}$ of $\mathcal{T}$ is said to be connected if the topology induced on it by $\mathcal{T}$ makes it a connected space, in other words: given $A, F \subseteq \mathcal{T}$ such that A is open and F is closed, if $A \cap \mathcal{S} = F \cap \mathcal{S} = X$, then either $X = \emptyset$ or $X = \mathcal{S}$. It is well-known that $\mathcal{T}$ can be partitioned into its *connected components*, in other words its maximal connected subsets. $\mathcal{T}$ is *disconnected* if it is not connected. $\mathcal{T}$ is *totally disconnected* if every subset of it having at least two points is disconnected, in other words if its connected components are the singletons. We say that $\mathcal{T}$ is *Boolean* if it is totally disconnected, compact, and Hausdorff.

Consider now the following six statements:

— Regular closed sets coincide with regular open sets.

— Regular closed sets coincide with open-and-closed sets.

— Regular open sets coincide with open-and-closed sets.

— The interior of a closed set is closed.

— The closure of an open set is open.

— The closure of the interior of a set is contained in the interior of the closure of that set.

They are equivalent. Indeed, given the opening $\alpha : X \mapsto X^\circ$ and the closing $\varphi : X \mapsto \overline{X}$, they can be expressed respectively as: $\mathrm{Inv}(\varphi\alpha) = \mathrm{Inv}(\alpha\varphi)$, $\mathrm{Inv}(\varphi\alpha) = \mathrm{Inv}(\varphi) \cap \mathrm{Inv}(\alpha)$, $\mathrm{Inv}(\alpha\varphi) = \mathrm{Inv}(\varphi) \cap \mathrm{Inv}(\alpha)$, $\varphi\alpha\varphi = \alpha\varphi$, $\alpha\varphi\alpha = \varphi\alpha$, and $\varphi\alpha \leq \alpha\varphi$, which are equivalent formulas by Criterion 6.6 in Section 6.1 of [11]. When anyone of these six statements holds, we say that $\mathcal{T}$ is *extremally disconnected* (cfr. [4], where the fifth statement is used in the definition). The choice of this terminology is justified by the following property:

If $\mathcal{T}$ is extremally disconnected and Hausdorff, then for any two distinct points $x, y \in \mathcal{T}$, there is an open-and-closed set S such that $x \in S$ and $y \notin S$. In particular $\mathcal{T}$ is totally disconnected.

Indeed, as $\mathcal{T}$ is Hausdorff, there are disjoint open sets A_x, A_y with $x \in A_x$ and $y \in A_y$, and so $y \notin \overline{A_x}$. As $\mathcal{T}$ is extremally disconnected, $S = \overline{A_x}$ is open-and-closed.

Now we come to Stone's representation theorem (see [1], Chapter 9):

There is a one-to-one correspondence between Boolean lattices $\mathcal{L}$ and Boolean topological spaces $\mathcal{T}$, under which the elements of $\mathcal{L}$ correspond to the open-and-closed subsets of $\mathcal{T}$, and the points of $\mathcal{T}$ to the maximal ideals of $\mathcal{L}$.

The Boolean space $\mathcal{T}$ corresponding to the Boolean lattice $\mathcal{L}$ is called the *Stone space* of $\mathcal{L}$. In this representation of $\mathcal{L}$ as the lattice of open-and-closed subsets of $\mathcal{T}$, the binary join and meet are the binary union and intersection (since the union and intersection of two open-and-closed sets is open-and-closed), and the lattice complementation reduces to the set complementation (since the complement of an open-and-closed set is open-and-closed).

For complete lattices, we have the following characterization (see [4]):

A Boolean space is the Stone space of a complete Boolean lattice if and only if it is extremally disconnected.

One of the definitions we gave of an extremally disconnected space is that open-and-closed sets coincide with regular open sets. Thus $\mathcal{L}$ is isomorphic to the complete lattice of regular open sets of its Stone space $\mathcal{T}$. In this representation, although the binary join and meet are the union and intersection, this is not so for infinitary supremum and infimum operations. Given an infinite family C_j ($j \in J$) of open-and-closed subsets of an extremally disconnected Stone space $\mathcal{T}$, their supremum and infimum are respectively

$$\overline{\bigcup_{j \in J} C_j} \qquad \text{and} \qquad \bigcap_{j \in J} C_j^{\ \circ}.$$

Thus the operations of complete Boolean lattices do generally not correspond to union and intersection. This justifies our criticism of the assertions of Serra and Vincent quoted in the Introduction.

4. Practical interpretations

In the first subsection we show how the cellular representation of digital pixels as regular open or closed sets eliminates the so-called 'paradoxes' of the two types of digital connectivities on square grids. In the second subsection we explain that regular open sets and the corresponding regular closed sets can be used to represent objects so as to satisfy certain common-sense physical requirements. The third subsection discusses briefly similar digital models, where algebraic openings and closings dual under complementation take the role of topological interior and closure.

4.1. Digital and cellular pixels

Digital pixels are elements of $\mathbb{Z}^2$. The *cellular representation* of $\mathbb{Z}^2$ associates to each pixel $(i,j) \in \mathbb{Z}^2$ a subset $C(i,j)$ of $\mathbb{R}^2$, called a *cellular pixel*, in such a way that the union of all $C(i,j)$ for $(i,j) \in \mathbb{Z}^2$ covers the whole of $\mathbb{R}^2$. One assumes in general that this representation is translation-invariant, in other words each $C(i,j)$ is the translate of $C(0,0)$ by (i,j). A natural choice for $C(i,j)$ is the closed square of size 1 centered about (i,j):

$$C(i,j) = \{(x,y) \in \mathbb{R}^2 \mid |x-i|, |y-j| \leq 1/2\}. \tag{4.1}$$

Given a digital set $S \subseteq \mathbb{Z}^2$, we define its cellular representation $C(S)$ by

$$C(S) = \bigcup_{(i,j)\in S} C(i,j).$$

Thus $C(i,j) = C(\{(i,j)\})$. Note that $C(\mathbb{Z}^2) = \mathbb{R}^2$. From the definition we have for any family of subsets S_r of $\mathbb{Z}^2$ $(r \in R)$:

$$C\big(\bigcup_{r\in R} S_r\big) = \bigcup_{r\in R} C(S_r). \tag{4.2}$$

However the equality does not hold for intersection:

$$C\big(\bigcap_{r\in R} S_r\big) \subseteq \bigcap_{r\in R} C(S_r).$$

Indeed, given two vertically adjacent pixels (i,j) and $(i+1,j)$, $\{(i,j)\} \cap \{(i+1,j)\} = \emptyset$, but $C(i,j) \cap C(i+1,j) \neq \emptyset$, since it contains the closed segment spanned by the points $(i+1/2, j-1/2)$ and $(i+1/2, j+1/2)$ (see Figure 1). Moreover the cellular representation does not commute with complementation; for any $S \subseteq \mathbb{Z}^2$,

$$\mathbb{R}^2 \setminus C(S) = \big(C(\mathbb{Z}^2 \setminus S)\big)^\circ \qquad \text{and} \qquad C(\mathbb{Z}^2 \setminus S) = \overline{\mathbb{R}^2 \setminus C(S)}. \tag{4.3}$$

Thus the cellular representation of digital sets does not transpose the complete Boolean lattice structure of $(\mathcal{P}(\mathbb{Z}^2), \subseteq, \bigcup, \bigcap, {}^c)$ into that of $(\mathcal{P}(\mathbb{R}^2), \subseteq, \bigcup, \bigcap, {}^c)$.

The solution to this defect is to consider the cellular representation of digital sets as a map from $\mathcal{P}(\mathbb{Z}^2)$ into the complete lattice of regular closed sets. Indeed, it is not hard to check that for each $S \subseteq \mathbb{Z}^2$, $C(S)$ is regular closed, $C(S^c) = \overline{C(S)^c}$, and for any family of subsets S_r of $\mathbb{Z}^2$ $(r \in R)$ we have

$$C\big(\bigcup_{r\in R} S_r\big) = \overline{\bigcup_{r\in R} C(S_r)}$$

$$\text{and} \qquad C\big(\bigcap_{r\in R} S_r\big) = \overline{\Big(\bigcap_{r\in R} C(S_r)\Big)^\circ}.$$

By (3.2) and the dual version of Proposition 3.1, this means that the cellular representation of digital sets satisfies the following properties for $S, T, S_r \subseteq \mathbb{Z}^2$ $(r \in R)$:

$$\begin{aligned} S \subseteq T \ &\Leftrightarrow\ C(S) \subseteq C(T), \\ C\Big(\bigcup_{r\in R} S_r\Big) &= \bigvee_{r\in R} C(S_r), \\ C\Big(\bigcap_{r\in R} S_r\Big) &= \bigwedge_{r\in R} C(S_r), \\ C(S^c) &= C(S)', \end{aligned} \tag{4.4}$$

where the supremum $\bigvee$, infimum $\bigwedge$, and complementation $'$ are taken in the complete Boolean lattice of regular closed sets. In other words this map is an isomorphism from $(\mathcal{P}(\mathbb{Z}^2), \subseteq, \bigcup, \bigcap, ^c)$ to a complete Boolean sublattice of the one of regular closed sets.

As the isomorphism from the lattice of regular closed sets to the one of regular open sets is given by map $F \mapsto F^\circ$, the map $S \mapsto C(S)^\circ$ is an isomorphism from $(\mathcal{P}(\mathbb{Z}^2), \subseteq, \bigcup, \bigcap, ^c)$ to a complete Boolean sublattice of the one of regular open sets. The interest of considering both $C(S)$ and $C(S)^\circ$ lies in their relation between the two types of digital connectivity in $\mathbb{Z}^2$, called the 4-connectivity and 8-connectivity. As remarked in [8]:

For any $S \subseteq \mathbb{Z}^2$, S is 8-connected if and only if $C(S)$ is connected, and S is 4-connected if and only if $C(S)^\circ$ is connected.

Now by (4.3) we have $C(S)^c = C(S^c)^\circ$ (where the first complementation is in $\mathbb{Z}^2$, and the second one in $\mathbb{R}^2$). Thus if we consider 8-connectivity on $S \subseteq \mathbb{Z}^2$, we must consider 4-connectivity on its complement S^c. The so-called 'paradoxes' of digital connectivity on a square grid disappear, thanks to the two distinct cellular representations of an object and its background in $\mathbb{Z}^2$ as subsets of $\mathbb{R}^2$ which are respectively regular closed and regular open.

In this exposition, we restricted ourselves to the 2-dimensional case. Of course everything generalizes to the d-dimensional space. Here the two types of digital connectivity on $\mathbb{Z}^d$ can be called *axial* and *diagonal*; they correspond to two types of adjacency relations, where a point has respectively $2d$ and $d^3 - 1$ neighbours (namely 4 and 8 for $d = 2$, or 6 and 26 for $d = 3$).

4.2. *Admissible representation of objects*

We will give here a few conditions on a subset X of $\mathbb{R}^d$ which correspond intuitively to the fact that X represents a real physical object or phenomenon. We will see that they are equivalent to the existence of a regular open set A and a regular closed set F which correspond each other (that is, $A = F^\circ$ or equivalently $F = \overline{A}$), such that the set X is comprised between A and F; moreover, all sets comprised between A and F will represent the same physical phenomenon.

For the sake of brievity, let us write $\mathcal{R}$ for the set of pairs (A, F) such that A and F are corresponding regular open and regular closed sets that is, $A = \overline{A}^\circ$, $F = \overline{F^\circ}$, and $F = \overline{A}$ or equivalently $A = F^\circ$. The mathematical basis for this subsection is provided by Proposition 2.6, whose interpretation in terms of regular open and regular closed sets is the following:

(a) *Given* $A, F \in \mathcal{P}(\mathbb{R}^d)$, $(A,F) \in \mathcal{R}$ *if and only if* $A \subseteq F$ *and there exists* $S \in \mathcal{P}(\mathbb{R}^d)$ *such that* $A = \overline{S}^\circ$ *and* $F = \overline{S^\circ}$.
(b) *Given* $M \in \mathcal{P}(\mathbb{R}^d)$, *there exist* A *and* F *such that* $(A,F) \in \mathcal{R}$ *and* $A \subseteq M \subseteq F$, *if and only if* $\overline{M}^\circ \subseteq M \subseteq \overline{M^\circ}$. *Moreover,* A *and* F *are then uniquely determined by* $A = M^\circ$ *and* $F = \overline{M}$.

Let us now come to our consideration of mathematical representations of physical phenomena. A key idea is that we do not physically distinguish between open and closed sets, and that observable phenomena must have a non-zero 'magnitude'. Thus there are classes of sets which are equivalent, because they represent the same materially measurable phenomenon.

A first expresson of this idea is to say that we consider only sets X *whose border does not matter*. Thus isolated points (in X or X^c) are physically meaningless. An interpretation of the lack of value of the border $\overline{X} \setminus X^\circ$ of X is that we can always reconstruct it from $\overline{X}$ or X°, in other words

$$\overline{X}^\circ = X^\circ \qquad \text{and} \qquad \overline{X^\circ} = \overline{X}. \tag{4.5}$$

It is easily seen by statement (b) above that this is equivalent to $A \subseteq X \subseteq F$, where $(A,F) \in \mathcal{R}$; moreover we have $A = X^\circ$ and $F = \overline{X}$.

A second expresson is to say that X and X^c *have nowhere zero thickness*. For example X and X^c do not contain isolated portions of dimension $d-1$. A possible meaning of the non-zero thickness of a set S is that every point of that set is adherent to an open ball completely contained in that set, in other words $S \subseteq \overline{S^\circ}$. Hence we require $X \subseteq \overline{X^\circ}$ and $X^c \subseteq \overline{X^{c\circ}} = \overline{X}^{\circ c}$, or simply

$$\overline{X}^\circ \subseteq X \subseteq \overline{X^\circ}. \tag{4.6}$$

Again, we find by statement (b) above that (4.6) is equivalent to (4.5), that is $A \subseteq X \subseteq F$, where $(A,F) \in \mathcal{R}$ (with $A = X^\circ$ and $F = \overline{X}$).

Given X such that $A \subseteq X \subseteq F$, where $(A,F) \in \mathcal{R}$, the fact that the border of X does not count implies that X is equivalent to $X^\circ = A$ and to $\overline{X} = F$. Thus all sets Y such that $A \subseteq Y \subseteq F$ are equivalent. We obtain thus a family of equivalence classes of Euclidean sets, corresponding to all pairs $(A,F) \in \mathcal{R}$; each such class is closed under non-empty unions and intersections. As the correspondence between A and F gives the isomorphism between the complete Boolean lattice of regular open sets and the one of regular closed sets, these equivalence classes form themselve a complete Boolean lattice isomorphic to the previous two. The complement of the class corresponding to $(A,F) \in \mathcal{R}$ is the one corresponding to (F^c, A^c), and is obtained by taking all complements of elements of that class.

Thus we have a representation of physical phenomena or objects by equivalence classes of Euclidean sets, and the corresponding structure of a complete Boolean lattice provides an analogue of the one of sets with operations of union, intersection, and complementation. However, the resulting lattice is not *atomic* (generated by points), and so we cannot apply arguments of the form "a point x belonging to $\sup_j X_j$ must belong to at least one X_j".

We can now consider a weaker condition, that *the border of* X *is negligible*. Here X can have isolated points, so that we do not necessarily have $X \subseteq \overline{X^\circ}$, and hence X does not satisfy (4.6), that is $A \subseteq X \subseteq F$ for $(A,F) \in \mathcal{R}$. If by negligible we mean of Lebesgue measure zero, a negligible set must have empty interior, since a non-empty open set has positive Lebesgue measure. Let us thus assume that *the border of X has empty interior.* The border of X is $\overline{X} \setminus X^\circ$, and for two sets V, W we have $(V \setminus W)^\circ = V^\circ \setminus \overline{W}$; hence the interior of the border of X is $\left(\overline{X} \setminus X^\circ\right)^\circ = \overline{X}^\circ \setminus \overline{X^\circ}$, and the condition that it is empty simply means:

$$\overline{X}^\circ \subseteq \overline{X^\circ}. \tag{4.7}$$

By statement (a) above this means that $(\overline{X}^\circ, \overline{X^\circ}) \in \mathcal{R}$. Note that every $(A, F) \in \mathcal{R}$ arises in this way, because $A = \overline{A}^\circ = \overline{F}^\circ$ and $F = \overline{A^\circ} = \overline{F^\circ}$. Moreover (4.7) is satisfied whenever X is closed or open.

However in (4.7) X does not necessarily lie between the two $\overline{X}^\circ$ and $\overline{X^\circ}$, as in (4.6). Here X has a physically meaningless part $X \setminus \overline{X^\circ}$, and a meaningful part $\overline{X^\circ}$. Moreover all sets Y such that $\overline{Y}^\circ = \overline{X}^\circ$ and $\overline{Y^\circ} = \overline{X^\circ}$ are equivalent to X. In particular, as $(\overline{X}^\circ, \overline{X^\circ}) \in \mathcal{R}$, we have

$$\overline{\overline{X}^\circ} = \overline{X^\circ} \qquad \text{and} \qquad \overline{X^\circ}^\circ = \overline{X}^\circ.$$

This mean that X° and $\overline{X}$ are equivalent to X. We get thus again a complete Boolean lattice of equivalence classes of Euclidean sets, corresponding to all pairs $(A, F) \in \mathcal{R}$, each one being closed under the operations of topological closure and interior. The classes obtained here are strictly wider than those considered before. In particular they cover together all closed and all open sets. Moreover, they are not closed under non-empty unions and intersections (for example the one corresponding to $(\emptyset, \emptyset)$ contains all discrete sets, but the union of all discrete sets is $\mathbb{R}^d$, which belongs to the class corresponding to $(\mathbb{R}^d, \mathbb{R}^d)$). Note that X belongs to the equivalence class corresponding to $(A, F) \in \mathcal{R}$ if and only if X^c belongs to the one corresponding to (F^c, A^c). Thus the complementation of classes is straightforward, but their supremum or infimum is not so easily built.

For another type of topological representation of physically meaningful objects, see also the 'hit-or-miss topology' [6,7,10].

4.3. Digital models of objects

The problem that two sets may represent the same physical object, or that a too small set may be physically insignifiant, is even more present in a digital framework, due to the loss of resolution arising from quantization.

Consider subsets of a digital space $\mathcal{E}$. We can say that an isolated pixel in a figure or its background does not count, and that there is a digital structuring element B such that B and its translates are the smallest significant portions in a set or its complement. Let $\mathbf{T}$ be the group of translations of the digital space. Let α_B the *structural* $\mathbf{T}$*-opening by* B and φ_B the *structural* $\mathbf{T}$*-closing by* B^c [9]; clearly φ_B is the *dual by complementation* of α_B, which applies α_B to the complement of a set: for $X \subseteq \mathcal{E}$, $\varphi_B(X) = \left(\alpha_B(X^c)\right)^c$ (see [9]). More generally we can consider any opening α and the closing φ which is its dual by complementation: $\varphi(X) = \left(\alpha(X^c)\right)^c$.

It is possible to restrict ourselves to sets X such that both X and X^c are invariant under α, that is $X = \alpha(X) = \varphi(X)$. This is the analogue for algebraic openings and closings of open-and-closed sets in a topological space (see Subsection 3). Note also that in continuous space Serra (see [10], Section V.C) defines the *regular model* consisting of all compact sets invariant under both α and φ, where α is the structural $\mathbf{T}$-opening by a compact ball of positive radius. The problem with such a restriction is that the family $\mathrm{Inv}(\alpha) \cap \mathrm{Inv}(\varphi)$ in general does not constitute a complete lattice, and is relatively small and difficult to construct.

On the other hand $\mathrm{Inv}(\alpha\varphi) = \mathrm{Ran}(\alpha\varphi)$ and $\mathrm{Inv}(\varphi\alpha) = \mathrm{Ran}(\varphi\alpha)$ are isomorphic complete lattices for the ordering by inclusion (see Corollary 2.5), where $A \in \mathrm{Inv}(\alpha\varphi)$ corresponds to $F \in \mathrm{Inv}(\varphi\alpha)$ by $A = \alpha(F)$ and $F = \varphi(A)$. As φ is the dual of α by complementation, for $A \in \mathrm{Inv}(\alpha\varphi)$ we have $A^c \in \mathrm{Inv}(\varphi\alpha)$, and so $\alpha(A^c) \in \mathrm{Inv}(\alpha\varphi)$; as $\mathrm{Inv}(\alpha\varphi)$ is a lattice, it has an element included in both A and $\alpha(A^c) \subseteq A^c$, that is $\emptyset \in \mathrm{Inv}(\alpha\varphi)$. Thus the least and greatest elements of $\mathrm{Inv}(\alpha\varphi)$ are $\emptyset$ and $\alpha(\mathcal{E})$. Moreover if for any $A \in \mathrm{Inv}(\alpha\varphi)$, we set $A' = \alpha(A^c) = \big(\varphi(A)\big)^c$, then the map $A \mapsto A'$ is a dual automorphism of $\mathrm{Inv}(\alpha\varphi)$, and it forms a *complementation* in the sense that $A \wedge A' = \emptyset$ (the least element of $\mathrm{Inv}(\alpha\varphi)$) and $A \vee A' = \alpha\varphi(A \cup A') = \alpha(\mathcal{E})$ (the greatest element of $\mathrm{Inv}(\alpha\varphi)$). Note however that these two lattice are generally not Boolean.

We illustrate in Figure 2 an element A of $\mathrm{Inv}(\alpha\varphi)$, where α is the structural $\mathbf{T}$-opening by a 3×3-square B, and the corresponding element $\varphi(A)$ of $\mathrm{Inv}(\varphi\alpha)$.

The set $\mathcal{R}$ of pairs (A, F) such that $A \in \mathrm{Inv}(\alpha\varphi)$ and $F = \varphi(A)$ is a complete lattice isomorphic to $\mathrm{Inv}(\alpha\varphi)$ and $\mathrm{Inv}(\varphi\alpha)$. As in the preceding subsection, we can now build models of objects by equivalence classes corresponding to such pairs. For example if we assume that we consider only sets X such that *the difference between* $\alpha(X)$ *and* $\varphi(X)$ *does not matter*, then to each $(A, F) \in \mathcal{R}$ corresponds the equivalence class of sets X such that $A \subseteq X \subseteq F$, and we restrict objects to members of these classes. It is also possible to associate to $(A, F) \in \mathcal{R}$ the class of sets X such that $\alpha\varphi(X) = A$ and $\varphi\alpha(X) = F$, but then the meaning of such equivalence classes is less clear.

What we said in this subsection for digital sets is also valid for subsets of the continuous space $\mathbb{R}^d$, or objects in more general types of spaces (e.g., grey-level functions), provided that we still have the structure of a complete lattice.

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References

[1] G. Birkhoff (1984). *Lattice Theory*, American Mathematical Society Colloquium Publications, Vol. 25, 3rd edition, Providence, RI.

[2] A. Bogomolny (1988). Digital geometry may not be discrete, *Computer Vision, Graphics, and Image Processing*, Vol. 43, no. 2, pp. 205–220.

[3] G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M. Mislove, D.S. Scott (1980). *A Compendium of Continuous Lattices*, Springer-Verlag, Berlin.

[4] A.M. Gleason (1958). Projective topological spaces, *Illinois J. Math.*, Vol. 2, pp. 482–489.

[5] H.J.A.M. Heijmans, C. Ronse (1988). The algebraic basis of mathematical morphology; part I: dilations and erosions, *CWI Report AM-R8807, PRLB Manuscript M248.*

[6] G. Matheron, *Théorie des ensembles aléatoires*, Ecole des Mines, Paris, 1969.

[7] G. Matheron, *Random Sets and Integral Geometry*, J. Wiley & Sons, New York, NY, 1975.

[8] C. Ronse (1985). An isomorphism for digital images, *J. Combinatorial Theory, Ser. A*, Vol. 39, pp. 132–159.

[9] C. Ronse, H.J.A.M. Heijmans (1989). The algebraic basis of mathematical morphology; part II: openings and closings, *CWI Report AM-R8904, PRLB Manuscript M291.*

[10] J. Serra (1982). *Image Analysis and Mathematical Morphology*, Academic Press, London.

[11] J. Serra, ed. (1988). *Image Analysis and Mathematical Morphology, Vol. 2: Theoretical Advances*, Academic Press, London.

[12] L. Vincent (1989). Graphs and mathematical morphology, *Signal Processing*, Vol. 16, no. 4, pp. 365–388.