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EXTENSIONS OF GRAPHS AND PROBLEMS OF HOMOGENEITY.

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Introduction. This part of our dissertation is devoted to extending some results of Gardiner [3], Enomoto [2] and Ronse [6].

Suppose that in a graph G, for every vertex subsets U, V of G such that $\langle U \rangle \cong \langle V \rangle$ and $\langle U \rangle$ belongs to some family $\mathcal{C}$ of graphs, the neighbourhoods $\langle \bigcap_{u \in U} G(u) \rangle$ and $\langle \bigcap_{v \in V} G(v) \rangle$ are relatively similar. What can then be said about G?

We will suppose simply that these neighbourhoods contain the same number of vertices. We show that for suitable families $\mathcal{C}$, the property is hereditary, in the sense that for any vertex v of G, $\langle G(v) \rangle$ satisfies the same property as G. This connects our question to the problem of finding the extensions of a given graph H, namely the graphs G such that for every vertex v of G, $\langle G(v) \rangle \cong H$. We can classify those extensions for some graphs H, and this allows us to solve our problem for several classes $\mathcal{C}$ of graphs, generalizing [2, 3, 6].

The author wishes to apologize for the rather unsmooth presentations and proofs of some results in this part of the dissertation, but this is due to lack of time, since this work had to be done, from inception to typing, in the space of three months.

Definitions. We use the notations of [8]. If G is a graph, then for $a \in V(G)$ we put $G(a) = \{ b \in V(G) \mid \{a,b\} \in E(G) \}$. If $U \subseteq V(G)$, then $\langle U \rangle_G$ is the graph with vertex-set U and edge-set $E(G)|_U\{2\}$. When no confusion is possible, we will simply write it $\langle U \rangle$.

We will define the graphs K_n, N_n and tK_n, $K_{t;n}$ as in [8] and [6] respectively.

We will consider as a graph the one with vertex-set and edge-set equal to $\emptyset$. We will write it K_0 or N_0.

If G is a graph and $U \subseteq V(G)$, then let $G(U) = \bigcap_{u \in U} G(u)$.

All graphs considered in this work will be finite.

§I Homogeneity conditions

We will call a family any collection of graphs given up to an isomorphism. Thus, if $\mathcal{C}$ is a family, if $A \in \mathcal{C}$ and $B \cong A$, then we may write $B \in \mathcal{C}$.

Definitions. Let G be a graph, let $\mathcal{C}$ be a family and let $A, B \subseteq V(G)$ such that $\langle B\rangle \cong \langle A\rangle \in \mathcal{C}$.

(1) If for any such A,B, every isomorphism $\langle A\rangle \to \langle B\rangle$ extends to an automorphism of G, then we say that G is $\mathcal{C}$-ultrahomogeneous, or G is a $\mathcal{C}$-UH-graph.

(2) If for any such A,B, there is an automorphism of G mapping A onto B, then we say that G is $\mathcal{C}$-homogeneous, or G is a $\mathcal{C}$-H-graph.

(3) If for any such A, every automorphism of $\langle A\rangle$ extends to an automorphism of G, then we say that G is locally $\mathcal{C}$-homogeneous, or G is a $\mathcal{C}$-LH-graph.

(4) If for any such A,B, $\langle G(A)\rangle \cong \langle G(B)\rangle$, then we say that G is combinatorially $\mathcal{C}$-homogeneous, or G is a $\mathcal{C}$-CH-graph.

(5) If for any such A, B, for any $\emptyset \neq X \subseteq A$, there is an isomorphism $f: \langle A\rangle \to \langle B\rangle$ with $\langle G(X)\rangle \cong \langle G(X^f)\rangle$, then we say that G is strongly combinatorially $\mathcal{C}$-homogeneous, or G is a $\mathcal{C}$-SCH-graph.

(6) If for any such A, B, $|G(A)| = |G(B)|$, then we say that G is $\mathcal{C}$-coherent, or G is a $\mathcal{C}$-C-graph.

(7) If for any such A,B, for any isomorphism $f: \langle A\rangle \to \langle B\rangle$ and $X \subseteq A$, $X \neq \emptyset$, $|G(X)| = |G(X^f)|$, then we say that G is strongly $\mathcal{T}$-coherent, or that G is a $\mathcal{T}$-SC-graph.

(8) If for any such A, for every automorphism f of $\langle A\rangle$ and $X \subseteq A$, $X \neq \emptyset$, $|G(X)| = |G(X^f)|$, then we say that G is locally $\mathcal{T}$-coherent, or G is a $\mathcal{T}$-LC-graph.

Notes. 1) If $\mathcal{T}$ is the family of all graphs, then we simply say that G is ultrahomogeneous or G is a UH-graph, etc...

2) A graph G is ultrahomogeneous if and only if it is both homogeneous and locally homogeneous.

We have the following implications:

$$\begin{tikzcd}
 & (2) \arrow[dr, Rightarrow] & & (4) \arrow[r, Rightarrow] & (6) \\
(1) \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] \arrow[rrrr, Rightarrow, bend right] & & (5) \arrow[ur, Rightarrow] & & (7) \arrow[u, Rightarrow] \\
 & (3) \arrow[rrr, Rightarrow] & & & (8)
\end{tikzcd}$$

If $\mathcal{T}$ is a family, the complement of $\mathcal{T}$ is the family $\overline{\mathcal{T}}$ consisting of all graphs $\overline{X}$, where $X \in \mathcal{T}$.

Gardiner classified UH-graphs [3], LH-graphs and even $\mathcal{T}$-LH-graphs, where $\mathcal{T}$ is the family of connected graphs [4]. Ronse classified H-graphs [6]. Enomoto [2] classified CH-graphs and $\mathcal{T}$-SCH-graphs. In this work, we will study $\mathcal{T}$-C-, $\mathcal{T}$-SC or $\mathcal{T}$-LC-graphs for several families $\mathcal{T}$.

In this section, we study some elementary properties of $\mathcal{T}$-C-graphs.

Proposition 1.1: Let $\mathcal{T}$ be a family and G be a $\mathcal{T}$-SC-graph. If V, W $\subseteq$ V(G), if $T \in \mathcal{T}$, if f_1 is an isomorphism $T \to \langle V \rangle$ and f_2 an isomorphism $T \to \langle W \rangle$, then for any A, $B \subseteq V(T)$ such that $A \cap B = \emptyset$ and $A \cup B \neq \emptyset$, $|G(A^{f_1}) \cap \bar{G}(B^{f_1})| = |G(A^{f_2}) \cap \bar{G}(B^{f_2})|$.

Proof. We use induction on $|B|$. If $|B| = 0$, then the result holds by hypothesis. Suppose that $|B| > 0$ and that the result holds for any B' such that $|B'| = |B| - 1$. Then take $b \in B$ and put $B' = B \setminus \{b\}$ and $A' = A \cup \{b\}$. Then for i = 1, 2, $|G(A^{f_i}) \cap \bar{G}(B'^{f_i})|$

$= |G(A^{f_i}) \cap G(b^{f_i}) \cap \bar{G}(B'^{f_i})| + |G(A^{f_i}) \cap \bar{G}(b^{f_i}) \cap \bar{G}(B'^{f_i})|$

$= |G(A'^{f_i}) \cap \bar{G}(B'^{f_i})| + |G(A^{f_i}) \cap \bar{G}(B^{f_i})|$.

By induction hypothesis, $|G(A^{f_i}) \cap \bar{G}(B'^{f_i})|$ and $|G(A'^{f_i}) \cap \bar{G}(B'^{f_i})|$ do not depend on i, and the result follows then.

If G is a $\mathcal{T}$-C-graph and $T \in \mathcal{T}$, then we write G(T) for $|G(T^f)|$, where f is any isomorphism $T \to \langle U \rangle$, where $\emptyset \neq U \subseteq V(G)$.

If G is a $\mathcal{T}$-SC-graph, if $T \in \mathcal{T}$, $A, B \subseteq V(G)$, $A \cup B \neq \emptyset$ and $A \cap B = \emptyset$, then we write G(T,A,B) for $|G(A^f) \cap \bar{G}(B^f)|$, where f is any isomorphism $T \to \langle U \rangle$, where $U \subseteq V(\bar{G})$. We have $G(T,V(T),\emptyset) = G(T)$. Let then $G'(T) = G(T,\emptyset,V(T))$.

Note that if $b \in B$, then $G(T,A,B\setminus\{b\}) = G(T,A\cup\{b\}, B\setminus\{b\}) + G(T,A,B)$. Clearly, if $f \in Aut(T)$, then $G(T,A,B) = G(T,A^f,B^f)$.

<u>Proposition 1.2.</u> Let G be a graph and $\mathcal{T}$ a family. Then G is a $\mathcal{T}$-SC-graph if and only if $\bar{G}$ is a $\bar{\mathcal{T}}$-SC-graph. Then we have for any $T \in \mathcal{T}$:

(i) $G'(T) = \bar{G}(\bar{T})$

(ii) If $A,B \subseteq V(G)$, $A\cup B \neq \emptyset$ and $A\cap B = \emptyset$, then $\bar{G}(\bar{T},A,B) = G(T,B,A)$.

The proof is elementary and is left to the reader.

<u>Definition.</u> Let $\mathcal{T}$ be a family. If for any $T \in \mathcal{T}$ and $\emptyset \neq A \subseteq V(G)$, $\langle A\rangle_{\mathcal{T}} \in \mathcal{T}$, then we say that $\mathcal{T}$ is a strong family.

<u>Proposition 1.3.</u> Let G be a graph and $\mathcal{T}$ a strong family. If G is $\mathcal{T}$-coherent, then G is strongly $\mathcal{T}$-coherent.

<u>Proof.</u> Let $A,B \subseteq V(G)$ such that $\langle A\rangle \cong \langle B\rangle \in \mathcal{T}$. If f is an isomorphism $\langle A\rangle \to \langle B\rangle$ and if $\emptyset \neq U \subseteq V(A)$, then $\langle U\rangle \cong \langle U^f\rangle$. As $\mathcal{T}$ is strong, $\langle U\rangle \in \mathcal{T}$ and so $|G(U)| = |G(U^f)|$. Hence G is a $\mathcal{T}$-SC-graph.

If G is a graph and $A \subseteq V(G)$, then we write $G - \langle A \rangle$ for $\langle V(G) \setminus A \rangle$.

If G and H are two graphs such that $V(G) \cap V(H) = \emptyset$, then we define the following graphs:

(i) $G \cup H$ is the graph with vertex-set $V(G) \cup V(H)$ and edge-set $E(G) \cup E(H)$.

(ii) $G + H$ is the graph with vertext-set $V(G) \cup V(H)$ and edge-set $E(G) \cup E(H) \cup \{\{a,b\} \mid a \in V(G) \text{ and } b \in V(H)\}$.

Then we have the following properties: If G, H, K are three graphs such that $V(G) \cap V(H) = V(H) \cap V(K) = V(G) \cap V(K)$, then we have:

(1) $\overline{G \cup H} = \overline{G} + \overline{H}$

(2) $\overline{G + H} = \overline{G} \cup \overline{H}$

(3) $(G \cup H) \cup K = G \cup (H \cup K)$

(4) $(G + H) + K = G + (H) + K$

(5) If $A \subseteq V(G)$, $B \subseteq V(H)$, then $\langle A \cup B \rangle_{G \cup H} = \langle A \rangle_G \cup \langle B \rangle_H$ and $\langle A \cup B \rangle_{G+H} = \langle A \rangle_G + \langle B \rangle_H$.

In particular, $(G - \langle A \rangle) \cup H = (G \cup H) - \langle A \rangle$ and $(G - \langle A \rangle) + H = (G + H) - \langle A \rangle$.

<u>Definition</u>. Let $\mathcal{C}$ be a family.

(1) If for any $T \in \mathcal{C}$, $T + K_1 \in \mathcal{C}$, then we say that $\mathcal{C}$ is closed.

(2) If for any $T \in \mathcal{C}$, $T \cup K_1 \in \mathcal{C}$, then we say that $\mathcal{C}$ is open.

(3) If $\mathcal{T}$ is open and closed, then we say that $\mathcal{T}$ is extensible.

We give here an example. Let $\mathcal{K}$ be the family of graphs of the form K_1, $K_1 \cup K_1$, $K_1 + K_1$, ..., $(\ldots((K_1 * K_1) * K_1) \ldots * K_1)$, where each $*$ is either a $+$ of a $\cup$. It is the extensible family generated by K_1. We will show here that $\mathcal{K}$ is strong. Indeed, if $T \varepsilon \mathcal{K}$ and $\emptyset \neq Y \subset V(T)$, then there is some $X \subset V(T)$ such that $Y \subseteq X$ and $|X| = |V(T)| - 1$ Let $Z = V(T) \setminus X$. As $T \varepsilon \mathcal{K}$, we can write $T = V * W$, where $|V(V)| = |V(T)| - 1$, $V \varepsilon \mathcal{K}$, $|V(W)| = 1$ and $*$ is either a $+$ or a $\cup$. Now $\langle X \rangle = T - \langle Z \rangle = (V * W) - \langle Z \rangle$. If $Z = V(W)$, then $\langle X \rangle = V * (W - \langle Z \rangle) = V \varepsilon \mathcal{K}$, while if $Z \neq V(W)$, then $Z \subseteq V(V)$ and so $\langle X \rangle = (V - \langle Z \rangle) * W$. Using induction we can see that $V - \langle Z \rangle \; \varepsilon \; \mathcal{K}$ and so $\langle X \rangle \varepsilon \mathcal{K}$ in this case. Using induction again, we get $\langle Y \rangle \; \varepsilon \mathcal{K}$.

In §III we will study $\mathcal{K}$-C-graphs.

<u>Proposition 1.4</u>: Let $\mathcal{T}$ be a family. Then $\mathcal{T}$ is open if and only if $\overline{\mathcal{T}}$ is closed. In particular, $\mathcal{T}$ is extensible if and only $\overline{\mathcal{T}}$ is extensible.

This result follows immediately from the properties (1) and (2) of $\cup$ and $+$.

<u>Proposition 1.5</u>: Let $\mathcal{T}$ be a closed family, let G be a graph and a $\varepsilon V(G)$

(i) If G is $\mathcal{T}$-coherent, then $\langle G(a) \rangle$ is $\overline{\mathcal{T}}$-coherent.

(ii) If G is strongly $\mathcal{T}$-coherent, then $\langle G(a)\rangle$ is strongly $\mathcal{T}$-coherent.

Proof. Let $H = \langle G(a)\rangle$. Let $U, V \subseteq G(a)$ such that $\langle U\rangle \cong \langle V\rangle \in \mathcal{T}$. Let $X \subseteq U$ and let f be an isomorphism $\langle U\rangle \to \langle V\rangle$. Then $\langle \{a\} \cup U\rangle \cong \langle \{a\} \cup V\rangle \cong K_1 + \langle V\rangle \in \mathcal{T}$. If g is the map $\langle \{a\} \cup U\rangle \to \langle \{a\} \cup V\rangle$ defined by $a^g = a$ and $x^g = x^f$ for $x \in U$, then g is an isomorphism. If G is $\mathcal{T}$-coherent, then $|H(U)| = |G(\{a\} \cup U)| = |G(\{a\} \cup V)| = |H(V)|$ and so H is $\mathcal{T}$-coherent. If G is strongly $\mathcal{T}$-coherent, then $|H(X)| = |G(\{a\} \cup X)| = |G((\{a\} \cup X)^g)| = |G(\{a\} \cup X^f)| = |H(X^f)|$ and so H is strongly $\mathcal{T}$- coherent.

Proposition 1.6: Let $\mathcal{T}$ be an open family, let G be a strongly $\mathcal{T}$- coherent graph and let $a \in V(G)$. Then $\langle \bar{G}(a)\rangle_G$ is strongly $\mathcal{T}$-coherent.

Proof: As $\mathcal{T}$ is open, $\overline{\mathcal{T}}$ is closed by Proposition 1.4. Then $\bar{G}$ is strongly $\overline{\mathcal{T}}$-coherent by Proposition 1.2. By Proposition 1.5, $\langle \bar{G}(a)\rangle_{\bar{G}}$ is strongly $\overline{\mathcal{T}}$- coherent and so $\langle \bar{G}(a)\rangle_G$ is strongly $\mathcal{T}$-coherent by Proposition 1.2.

Corollary 1.7. Let $\mathcal{T}$ be an extensible family, let G be a strongly $\mathcal{T}$- coherent graph and let $a \in V(G)$. Then $\langle G(a)\rangle$ and $\langle \bar{G}(a)\rangle_G$ are strongly $\mathcal{T}$-coherent.

We will now characterize some classes of graphs as $\mathcal{T}$-coherent graphs for some families $\mathcal{T}$.

<u>Proposition 1.8.</u> Let $\mathcal{T} = \{K_n \mid n \geq 1\}$ and let G be a $\mathcal{T}$-coherent graph. Then :

(i) If $|V(G)|=n$ and $G(K_1) = 0$, then $G \cong N_n$.

(ii) If $|V(G)|=tn$ $(t \geq 1,\ n \geq 1)$ and $G(K_i) = n-i$ for $i = 1, \ldots, n$, then $G \cong t\, K_n$.

(iii) If $G(K_1) = 2$ and $G(K_2) = 0$, then G is isomorphic to a union of cycles, each of length at least 4.

(iv) If $|V(G)| = 5$, $G(K_1) = 2$ and $G(K_2) = 0$, then $G \cong C_5$.

(v) If $|V(G)| = tn$ (where $t \geq 2$), if $G'(K_1) = n - 1$ and $G'(K_2) = 0$, then $G \cong K_{t;n}$.

The proof of this result is left to the reader. From this we get the following:

<u>Corollary 1.9:</u> Let $\mathcal{T} = \{N_n \mid n \geq 1\}$ and let G be a $\mathcal{T}$-coherent-graph. Then:

(i) If $|V(G)| = n$ and $G'(N_1) = 0$, then $G \cong K_n$

(ii) If $|V(G)| = n > 1$ and $G'(N_i) = n - i$ for $i = 1, \ldots n$, then $G \cong N_n$.

(iii) If $|V(G)| = tn$ (where $t > 1$ and $n > 1$) and $G'(N_i) = n - i$ for $i=1, \ldots n$, then $G \cong K_{t;n}$.

(iv) If $|V(G)| = 5$, $G'(N_1) = 2$ and $G'(N_2) = 0$, then $G \cong C_5$.

(v) If $|V(G)| = tn$ (where $t \geq 2$), if $G(N_1) = n-1$ and $G(N_2) = 0$, then $G \cong tK_n$.

This result follows from the fact that $\mathcal{T}$ is strong and from Propositions 1.2 and 1.3, since $\overline{K_n} = N_n$, $\overline{tK_n} = K_{t;n}$ (for $t \geq 2$) and $\overline{C_5} = C_5$.

Proposition 1.10: Let $\mathcal{T} = \{K_m | m \geq 1\}$ and let G be a graph such that $|V(G)| = n^2$, where $n \geq 3$. Then the following are equivalent:

(i) $G \cong L_{n,n}$.

(ii) G is $\mathcal{T}$-coherent, $G(K_1) = 2(n-1)$ and $G(K_i) = n-i$ for $i = 2, \ldots, n$.

(iii) $\langle G(a) \rangle \cong 2K_{n-1}$ for any $a \in V(G)$.

Proof: It is clear that (i) implies (ii). Suppose that (ii) holds and let $a \in V(G)$. Then $H = \langle G(a) \rangle$ is $\mathcal{T}$-coherent with $|V(H)| = 2(n-1)$ and $H(K_i) = n-1-i$ for $i = 1, \ldots, n-1$. By Proposition 1.8(ii), $H \cong 2K_{n-1}$ and so (iii) holds.

Suppose now that (iii) holds. Let N be the set of subsets X of G such that $\langle X \rangle \cong K_n$. Then $|N|.n$ = the number of pairs (K,x) such that $K \in N$ and $x \in K = n^2.2$.

Thus $|N| = 2n$. If $K, K' \in N$ and $K \cap K' \neq \emptyset$, then there is some $x \in K \cap K'$ and so $\langle K \cup K' \rangle = \langle G(x) \cup x \rangle \cong K_1 + 2K_{n-1}$.

Let $a \in V(G)$. Then $\{a\} \cup G(a) = X_1 \cup Y_1$, where $\langle X_1 \rangle = \langle Y_1 \rangle \cong K_n$ and $\langle (X_1 \cup Y_1) \setminus \{a\} \rangle = \langle G(a) \rangle \cong 2K_{n-1}$. We can write $X_1 = \{a, x_2, \ldots, x_n\}$ and $Y_1 = \{a, y_2, \ldots, y_n\}$. Then for $i = 2, \ldots, n$, $\{x_i\} \cup G(x_i) = X_1 \cup Y_i$ where $\langle Y_i \rangle \cong K_n$ and $X_1 \cap Y_i = \{x_i\}$, while $\{y_i\} \cup G(y_i) = Y_1 \cup X_i$, where $\langle X_i \rangle \cong K_n$ and $Y_1 \cap X_i = \{y_i\}$. For $1 \leq i < j \leq n$, $X_i \cap X_j = \emptyset$. Indeed, if $x \in X_i \cap X_j$, then $x \notin Y_1$ and so $y_i, y_j \in G(x)$; but $y_i \in X_i \setminus \{x\}$ and $y_j \in X_j \setminus \{x\}$ and $y_i \sim y_j$, which contradicts the fact that $\langle G(x) \rangle \cong 2K_{n-1}$. Similarly, $Y_i \cap Y_j = \emptyset$. Therefore $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ are both partitions of $V(G)$, since $|V(G)| = n^2$. Now, for any $i, j \in \{1, \ldots, n\}$ $X_i \neq Y_j$, otherwise $\{y_i, x_j\} \subseteq X_i$ and so $y_i \sim x_j$ which contradicts the fact that $\langle G(a) \rangle \cong 2K_{n-1}$. For any $z \in V(G)$, there is some X_i and some Y_j such that $z \in X_i \cap Y_j$, and they are unique. Conversely, if $z' \in V(G)$ and $z' \neq z$, then $z' \notin X_i \cap Y_j$, since $X_i \neq Y_j$. Thus we can label the elements of $V(G)$ as ordered pairs (i,j), where $i,j \in \{1, \ldots n\}$, and we have $x = (i,j)$ if and only if $x \in X_i \cap Y_j$. It follows that $G \cong L_{n,n}$ and (i) holds.

<u>Corollary 1.11</u>. Let $\mathcal{T} = \{N_m \mid m \geq 1\}$ and let G be a graph such that $|V(G)| = n^2$, where $n \geq 3$. Then the following are equivalent:

(i) $G \cong \overline{L_{n,n}}$.

(ii) G is $\mathcal{T}$-coherent, $G'(N_1) = 2(n-1)$ and $G'(N_i) = n-i$ for $i = 2, \ldots, n$.

(iii) $\langle \bar{G}(a) \rangle_G \cong K_{2;n}$ for any $a \in V(G)$.

Let us end the section on a property of $\mathcal{T}$-coherent graphs, where $\mathcal{T}$ is an open or closed family. The proof is left to the reader.

<u>Proposition 1.12</u>: Let G be a $\mathcal{T}$-coherent graph, let $a \in V(G)$ $H = \langle G(a) \rangle$ and $L = \langle \bar{G}(a) \rangle_G$. Then:

(i) If $\mathcal{T}$ is closed, then for any $T \in \mathcal{T}$, $H(T) = G\ (T+K_1)$

(ii) If $\mathcal{T}$ is closed and G is strongly $\mathcal{T}$-coherent, then for any $T \in \mathcal{T}$ and $A, B \subseteq V(T)$ such that $A \cap B = \emptyset \neq A \cup B$, $H(T,A,B) = G(T + K_1,\ A \cup V(K_1), B)$.

(iii) If $\mathcal{T}$ is open and G is strongly $\mathcal{T}$-coherent, then for any $T \in \mathcal{T}$ and $A, B \subseteq V(T)$ such that $A \cup B \neq \emptyset = A \cap B$, $L(T, A, B) = G(T \cup K_1,\ A,\ B \cup V(K_1))$.

II. Extensions of graphs

Definition. The graph G is an extension of the graph H if for any $x \in V(G)$, $\langle G(x) \rangle \cong H$. If G is connected then we say that G is a connected extension of H.

It is clear that if G is an extension of H, then G is the disjoint union of one or several connected extensions of H.

Proposition 2.1. For any $n \geq 0$, K_{n+1} is the only connected extension of K_n. Any extension of K_n is of the form tK_{n+1}, where $t \geq 1$.

The proof of this result is elementary and is left to the reader.

Proposition 2.2: Let G be a connected graph such that for any $z \in V(G)$, $\langle G(z) \rangle$ is a null graph or a complete multipartite graph. If there is some $x \in V(G)$ such that $\langle G(x) \rangle$ is not a null graph, then G is a complete multipartite graph.

Proof. For any $y \in V(G)$, the relation $a = b$ or $a \not\sim b$ is an equivalence on $G(y)$. (*) Take x as in the statement of the proposition. We have $G(x) = P_1 \cup \ldots \cup P_k$, where $k \geq 2$, and for $a \in P_i$ and $b \in P_j$, $a \sim b$ if and only $i \neq j$. Take $y \in G(x)$ such that $|G(y) \setminus G(x)|$ is minimal. If $G(y) \setminus G(x) = \{x\}$, then $G(z) \setminus G(x) = \{x\}$ for any $z \in G(x)$, and it follows that $V(G) = \{x\} \cup G(x)$, in other words G is a complete multipartite graph. If $G(y) \setminus G(x) \supset \{x\}$, then put $y \in P_i$

(where $1 \leq i \leq k$), $D = P_i \setminus \{y\}$, $E = G(y) \setminus (G(x) \cup \{x\})$ and $F = G(x) \setminus P_i$. Then $E \neq \emptyset \neq F$. Take any $u \in E$; if $E \neq \{u\}$, let $u' \in E \setminus \{u\}$; then $\{x, u, u'\} \subseteq G(y)$ and $u \not\sim x \not\sim u'$. This implies that $u \not\sim u'$ by (*) and so $\langle G(y) \setminus G(x) \rangle$ is a null graph. Take any $z \in F$; then $\{x, u, z\} \subseteq G(y)$ and $u \not\sim x \sim z$. Thus (*) implies that $u \sim z$ and so $G(y) \setminus G(x) \subseteq G(z)$. If $D \neq \emptyset$, then take any $t \in D$. As $\{t, x, u\} \subseteq G(z)$ and $t \sim x \not\sim u$, (*) implies that $t \sim u$ and so $G(y) \setminus G(x) \subseteq G(t)$. We have thus proved that $G(y) \setminus G(x) \subseteq G(a)$ for any $a \in G(x)$. As $|G(a) \setminus G(x)| \leq |G(y) \setminus G(x)|$ by definition of y, it follows that $G(y) \setminus G(x) = G(a) \setminus G(x)$ and $G(a) \subseteq G(y) \cup G(x)$. If $b \in G(y) \setminus G(x)$, then $G(b) \subseteq G(x) \cup G(y)$; indeed, if $c \in G(b) \setminus G(y)$, then for any $z \in F$, we have $\{c, y, z\} \subseteq G(b)$ and $z \sim y \not\sim c$; but then (*) implies that $z \sim c$, in other words $c \in G(z) \subseteq G(x) \cup G(y)$. We conclude that $V(G) = G(x) \cup G(y) = P_1 \cup \ldots \cup P_{k+1}$ where $P_{k+1} = G(y) \setminus G(x)$, and for $a \in P_i$ and $b \in P_j$, $a \sim b$ if and only if $i \neq j$. Therefore G is a complete multipartitie graph.

<u>Corollary 2.3</u>: (i) $K_{m_1, \ldots, m_r}$ $(r \geq 2)$ has an extension if and only if $m_1 = m_2 = \ldots = m_r$.

(ii) The only connected extension of $K_{t;m}$ is $K_{t+1;m}$ (for $t \geq 2$).

<u>Proposition 2.4</u> If G is a nonconnected $\{N_2\}$-coherent graph, then $G = K_{m_1} \cup \ldots \cup K_{m_r}$ for some $r \geq 2$.

<u>Proof</u>: Let L_1 and L_2 be two connected components of G.

If $\langle L_1 \rangle \cong K_{|L_1|}$ then there are some $x, y, z \in L_1$ such that $x \sim y \sim z \not\sim x$. Take $u \in L_2$. Then $\langle x, u \rangle \cong \langle x, z \rangle \cong N_2$ and $|G(x) \cap G(z)| \geq 1 > 0 = |G(x) \cap G(u)|$, which contradicts our hypothesis. Therefore $\langle L_1 \rangle = K_{|L_1|}$ and the result holds.

Corollary 2.5. If $t \geq 2$ and $m \geq 2$, then the only $\{N_2\}$-coherent extension of $K_{t;m}$ is $K_{t+1;m}$.

Proposition 2.6. The only connected extension of C_5 is the graph of the icosahedron, which is not $\{N_2\}$-coherent.

Proof. Let G be such an extension and let $x \in V(G)$. Let $C = G(x)$; we can write $C = \{x_1, x_2, x_3, x_4, x_5\}$, where $x_i \sim x_{i+1}$ (i is taken modula 5). For any $i = 1, \ldots, 5$, $\{x, x_{i-1}, x_{i+1}\} \subseteq G(x_i)$, and as $x_{i-1} \sim x \sim x_{i+1}$, there exist $y_i, y'_i \in G(x_i)$ such that $x_{i-1} \sim y_i \sim y'_i \sim x_{i+1}$. As $x_i \sim x_{i+1}$, $|G(x_i) \cap G(x_{i+1})| = 2$, and it follows that $y'_i = y_{i+1}$. It is clear that $y_i \neq y_{i+1}$ for any i. Moreover, $y_i \neq y_{i+2}$; otherwise $\{x_i, y_{i+1}, x_{i+1}\} \subseteq G(y_i)$, with $\langle x_i, y_{i+1}, x_{i+1} \rangle = K_3$, which is impossible. Therefore the 5 points $y_1, \ldots, y_5$ are pairwise distinct. Now for $i = 1, \ldots, 5$, $y_i \not\sim y_{i+2}$, otherwise $\{x_i, y_i, y_{i+2}, x_{i+1}\} \subseteq G(y_{i+1})$, with $\langle x_i, y_i, y_{i+2}, x_{i+1} \rangle \cong C_4$, which is impossible. Therefore $\langle y_1, \ldots, y_5 \rangle \cong C_5$ and if $K = \{x\} \cup \{x_1, \ldots, x_5\} \cup \{y_1, \ldots, y_5\}$, then $\langle K \rangle$ is uniquely determined.

For $i = 1, \ldots, 5$, $|G(y_i) \cap K| = 4$ and so $|G(y_i) \setminus K| = 1$. Let $z \in G(y_2) \setminus K$. Then $G(y_2) = \{y_1, x_1, x_2, y_3, z\}$, and as $\langle G(y_2) \rangle = C_5$, we must have $y_1 \sim z \sim y_3$. Therefore $\{z\} = G(y_1) \setminus K = G(y_2) \setminus K = G(y_3) \setminus K$. Repeating that argument, we find that $\{z\} = G(y_i) \setminus K$ for $i = 1, \ldots, 5$. If $L = K \cup \{z\}$, then $\langle L \rangle$ is isomorphic to the graph of the isocohedron, and as G is connected, $L = V(G)$ and $G = \langle L \rangle$. As G has diameter 3, G is not $\{N_2\}$-coherent, and the result holds.

<u>Corollary 2.7</u>: C_5 has no $\{N_2\}$-coherent extension.

<u>Proposition 2.8</u> [1] Let G be a connected extension of $L_{3,3}$. Then one of the following holds:

(i) $G \cong \overline{L_{4,4}}$.

(ii) $G \cong X(3)$, where $X(3)$ is constructed as follows: Take a set V such that $|V| = 6$, take the 3-subset of V as vertices of $X(3)$ and write $A \sim B$ if $|A \cap B| = 2$.

<u>Proof</u>. If $H = L_{3,3}$, then $V(H)$ may be identified with $AG(2,3)$ and then $E(H)$ is the set of pairs of points parallel to one of two directions $\mathcal{D}_1$ and $\mathcal{D}_2$. Let $\mathcal{D}$ and $\mathcal{D}'$ be the two other directions. We define the oblique lines of H. These are the lines in the two directions $\mathcal{D}$ and $\mathcal{D}'$.

Let G be a connected extension of H. Let $x \in V(G)$, let $M = G(x)$ and $N = \{y \in V(G) \mid d(x,y) = 2\}$. As $\langle M \rangle \cong H$, every vertex of M is joined to 4 vertices of N.

There are thus 36 edges between M and N. Let $y \in N$ and $Y = G(x) \cap G(y)$. Then $Y \subseteq M$ and if $z \in Y$, then $\langle G(z) \cap Y \rangle \cong \langle G(a) \cap G(b) \rangle$ for $a, b \in V(H)$ such that $a \not\sim b$, since $\{x, y\} \subseteq G(z)$ and $x \not\sim y$. Thus $\langle G(z) \cap Y \rangle \cong N_2$. Therefore Y is a graph of valency 2 without triangles. It is easy to see that such a set is either a square (a set of the form $M \cap \tilde{G}(x)$, where $x \in M$) or a hexagon (a set of the form $M \setminus L$, where L is an oblique line of $\langle M \rangle$).

If $y, y' \in N$ and if there exist $z_1, z_2, z_3 \in G(x) \cap G(y) \cap G(y')$ such that $z_1 \sim z_2 \sim z_3$, then $\{x, y, y', z_2\} \subseteq G(z_1) \cap G(z_3)$ and $\{x, y, y'\} \subseteq G(z_2)$, which implies that $y = y'$, since $d(z_1, z_3) = 2$. It follows that for $y \neq y'$, $G(x) \cap G(y) \neq G(x) \cap G(y')$ and that if $G(x) \cap G(y)$ is a hexagon and $G(x) \cap G(y')$ is a square, then $|G(x) \cap G(y) \cap G(y')| \leq 2$. In fact, we have the equality, because a square does not contain more than 2 points of an oblique line.

Suppose that there exist $y, y' \in N$ such that $S = G(x) \cap G(y)$ is a square and $T = G(x) \cap G(z)$ is a hexagon. Let $L = M \setminus T$. As any other square of the form $G(x) \cap G(z)$ (where $z \in N$) contains 2 vertices of L and as 2 vertices of L are contained in only one square, there are at most 3 such squares. As there are only 2 oblique lines intersecting S in 2 points, there are at most 2 hexagons of the form $G(x) \cap G(z)$ (where $z \in N$). As any hexagon gives 6 edges between M and N and any square gives 4, we get $36 \leq 3 \times 4 + 2 \times 6 = 24$, which is impossible.

Therefore, for any $x \in V(G)$, one of the following holds:

(a) If $y \in V(G)$ and $d(x, y) = 2$, then $G(x) \cap G(y)$ is a square.

(b) If $y \in V(G)$ and $d(x,y) = 2$, then $G(x) \cap G(y)$ is a hexagon.

Now, if $a, b \in V(G)$ and $a \sim b$, then there exist $c, d \in V(G)$ such that $c \in G(a) \cap G(b)$ (in other words $\{a,b\} \subseteq G(c)$) and $d \in G(c) \setminus (G(a) \cap G(b))$. But then $d(a,d) = d(b,d) = 2$, and we deduce from it that for any two vertices x and y of G, there are some $a_1, \ldots, a_k \in V(G)$ such that $a_1 = x$, $d(a_i, a_{i+1}) = 2$ $(i = 1, \ldots, k-1)$ and $a_k = y$. It follows that $|G(a_1) \cap G(a_2)| = \ldots = |G(a_{k-1}) \cap G(a_k)|$ and we conclude that either (a) holds for any $x \in V(G)$ or (b) holds for any $x \in V(G)$.

Suppose that (a) holds. Every vertex of M can be written (i,j), where $i, j \in \{1,2,3\}$. A square S of M corresponds to a unique vertex (i,j) of M, where $S = M \cap \bar{G}((i,j))$. We write then $S = S_{ij}$. Now $S_{ij} = G(x) \cap G(y)$ for a unique $y \in N$, and we write $y = (i,j)'$. If $j \neq j'$, then $\langle S_{ij} \cap S_{ij'} \rangle \cong K_2$; suppose that $S_{ij} \cap S_{ij'} = \{a, b\}$. Then $\langle G(a) \cap G(b) \rangle \cong 2K_2$, $\langle G(a) \cap G(b) \cap (\{x\} \cup M) \rangle \cong K_2$ and so $K_2 \cong \langle G(a) \cap G(b) \cap N \rangle = \langle (i, j)', (i,j')' \rangle$, in other words $(i,j)' \sim (i,j')'$. Similarly, $(i',j)' \sim (i,j)'$ for $i \neq i'$. Now, if $i \neq i'$ and $j \neq j'$, then let $i'' \in \{1,2,3\} \setminus \{i,i'\}$ and $j'' \in \{1,2,3\} \setminus \{j,j'\}$. Then $(i'',j'') \in S_{ij} \cap S_{ij'} \cap S_{i'j} \cap S_{i'j'}$ and it follows that $\{(i,j)', (i,j')', (i',j), (i',j')\} \subseteq G((i'',j''))$. But $G((i'',j'')) \cap N \cong L_{2,2} \cong C_4$, and as $(i,j)' \sim (i,j')' \sim (i',j')' \sim (i',j)' \sim (i,j)'$, it follows that $(i,j)' \not\sim (i',j')'$. Therefore $\langle N \rangle \cong H$. But every vertex of N is joined to 4 vertices of N and 4 of M.

Therefore, if $(i,j)' \in N$, then $|G((i,j)') \setminus (\{x\} \cup M \cup N)| = 1$ and let $z \in G((i,j)') \setminus (\{x\} \cup M \cup N)$. As $\langle G(z) \cap G((i,j)') \rangle \cong 2K_2$ and $G((i,j)') \cap G(z) \subseteq N$, it follows that for $i \neq i'$ and $j \neq j'$, $(i,j')' \sim z \sim (i',j)'$. Repeating the argument for $(i,j')'$, we see that $(i',j')' \sim z$, and so $N \subseteq G(z)$. Let $L = \{x,z\} \cup M \cup N$. Then $\langle L \rangle$ has valency 9, and so $L = V(G)$ and $G = \langle L \rangle$. Consider the set $V = \{a_1, a_2, a_3, b_1, b_2, b_3\}$. If we make correspond x to $\{a_1, a_2, a_3\}$, z to $\{b_1, b_2, b_3\}$, (i,j) to $(\{a_1, a_2, a_3\} \setminus \{a_i\}) \cup \{b_j\}$ and $(i,j)'$ to $\{a_i\} \cup (\{b_1, b_2, b_3\} \setminus \{b_j\})$, then we see that $G \cong X(3)$ and (ii) holds.

Suppose now that (b) holds. Write $N = \{a_1, a_2, a_3, b_1, b_2, b_3\}$, where $G(x) \cap G(a_i) = M \setminus L_i$ $(L_i \in \mathcal{D})$ and $G(x) \cap G(b_i) = M \setminus L'_i$ $(L'_i \in \mathcal{D}')$ for $i = 1, 2, 3$.

If $y \in M$ and $\{y\} = L_i \cap L'_j$, then $G(y) \cap N = N \setminus \{a_i, b_j\}$. As $\langle G(y) \rangle \cong H$, $\langle N \setminus \{a_i, b_j\} \rangle$ is a square. But this is true for any $i,j \in \{1,2,3\}$. Suppose that $a_i \not\sim b_j$. Then for $k \neq i$ and $\ell \neq j$, $\langle a_i, a_k, b_j, b_\ell \rangle$ is a square and it follows that $a_i \sim a_k$ and $a_i \sim b_\ell$. But then $|G(a_i) \cap N| \geq 4$, and as $|G(a_i) \cap M| = 6$, $|G(a_i)| \geq 10$, which is impossible. Thus $a_i \sim b_j$ and for $k \neq i$, $a_i \not\sim a_k$, since $|G(a_i) \cap N| \leq 9 - |G(a_i) \cap M| = 3$. If $L = \{x\} \cup M \cup N$, then $\langle L \rangle$ is regular of degree 9 and so $L = V(G)$ and $G = \langle L \rangle$. Now $|V(G)| = 16$ and $\langle \bar{G}(x) \rangle_{\bar{G}} = \overline{\langle N \rangle} \cong 2K_3$. By proposition 1.10, $\bar{G} \cong L_{4,4}$ and so $G = \overline{L_{4,4}}$ and (i) holds.

Corollary 2.9: $\overline{L_{4,4}}$ is the only $\{N_2\}$-coherent extension of $L_{3,3}$.

This is due to the fact that X(3) has diameter 3.

Definition: The graph G is an antiextension of the graph H if for any $x \in V(G)$, $\langle \overline{G}(x) \rangle_G \cong H$. In other words, G is an antiextension of H if and only if $\overline{G}$ is an extension of $\overline{H}$.

From Proposition 2.1 and Corollaries 2.5, 2.7 and 2.9, one derives the 4 following results :

Corollary 2.10 (i) The antiextensions of N_o are the graphs K_n, where $n \geq 1$.

(ii) If $n \geq 1$, then any antiextension of N_n is either N_{n+1} or $K_{t;n+1}$, where $t \geq 2$.

Corollary 2.11. If $t \geq 2$ and $m \geq 2$, then the only $\{K_2\}$-coherent antiextension of tK_m is $(t+1)K_m$.

Corollary 2.12. C_5 has no $\{K_2\}$-coherent antiextension.

Corollary 2.13: $L_{4,4}$ is the only $\{K_2\}$-coherent antiextension of $L_{3,3}$.

<u>Proposition 2.14:</u> Let G be a $\{K_2\}$-coherent anti-extension of $L_{n,n}$, where $n \geq 4$. Then $G \cong L_{n+1,\, n+1}$.

<u>Proof.</u> Take $a \in V(G)$ and let $X = G(a)$ and $Y = \bar{G}(a)$. For $y \in Y$, $\langle \bar{G}(y) \rangle_G \cong L_{n,n}$ and $\langle \bar{G}(y) \cap \bar{G}(a) \rangle_G = L_{n-1,\, n-1}$. So there exist, $A, B \subseteq X$ such that $\langle A \rangle \cong \langle B \rangle \cong K_{n-1}$ and $A \cup B = \bar{G}(y) \cap G(a)$. Now $\langle \{a\} \cup A \cup B \cup (\bar{G}(y) \cap \bar{G}(a)) \rangle \cong L_{n,n}$. We can write $y = (1,1)$ and $\bar{G}(y) \cap \bar{G}(a) = \{(a,b) \mid 1 < a \leq n,\ 1 < b \leq n\}$. Take $x \in A$, we may suppose that x is joined to the points $(a,2)$ such that $1 < a \leq n$, and not to the points (a,b), where $1 < a$ and $2 < b$.

For any $z = (c,d)$ (where $1 < c \leq n$ and $2 < d \leq n$), $x \in \bar{G}(z)$ and $\bar{G}(z) \cap \bar{G}(a) = \{(e,f) \mid e \neq c \text{ and } f \neq d\}$. Now x must be joined to a line of $\bar{G}(z) \cap \bar{G}(a)$. As x is joined to the points $(e,2)$ such that $1 \neq e \neq c$ and as there are at least 2 such points (since $n \geq 4$), x is joined to the points $(e, 2)$ such that $e \neq c$, and not to the points (e,f) such that $e \neq c$ and $2 \neq f \neq d$. As this is true for any (c,d) such that $1 < c \leq n$ and $2 < d \leq n$ and as $n \geq 4$, it follows that $G(x) \cap \bar{G}(a) = \{(e,2) \mid 1 \leq e \leq n\}$ and so $|G(x) \cap \bar{G}(a)| = n$.

Let $k = |X|$; then for any $u \in V(G)$, $|G(u)| = |V(G)| - 1 - n^2 = k$. If $y \in G(u)$ and $z \in \bar{G}(u)$, then $|G(u) \cap G(z)| = k - |\bar{G}(u) \cap G(z)| = k - 2(n-1)$, while $|G(u) \cap G(y)|$ is a constant which does not depend on the choice of u and y, since G is $\{K_2\}$-coherent: as $|G(a) \cap G(x)| = k-1 - |\bar{G}(a) \cap G(x)| = k-1-n$ for $x \in A$ (see above), $|G(u) \cap G(y)| = k-1-n$. Therefore G is strongly regular and we have

$|\{(x,y) \mid x\in X,\ y\in Y,\ x\sim y\}| = kn = n^2\ (k-2(n-1))$.
Hence $k= 2n$. Now if $x,\ z\in V(G)$ and $x\not\sim z$, then $|G(x)\cap G(z)| = k-2\ (n-1) = 2$; but as $\langle \bar{G}(x)\rangle_G \cong L_{n,n}$, there is some $u\in\bar{G}(x)\cap\bar{G}(z)$, and as $\{x,\ z\}\subseteq \bar{G}(u)$ and $\langle\bar{G}(u)\rangle_G \cong L_{n,n}$, $\langle G(x)\cap G(z)\cap \bar{G}(u)\rangle \cong N_2$. As $|G(x)\cap G(z)| = 2$, it follows that $\langle G(x)\cap G(z)\rangle \cong N_2$.

Now $\langle X\rangle$ has valency $k-1-n = n-1$, and if $x, x'\in X$ and $x\not\sim x'$, then $\langle G(x)\cap G(x')\rangle \cong N_2$. As $a\in G(x)\cap G(x')$, $G(x)\cap G(x') = \{a,\ y\}$, where $y\in Y$. Therefore all connected components of X are complete graphs and so $\langle X\rangle \cong 2K_n$ By Propositon 1.10, $G\cong L_{n+1,\ n+1}$, since $|V(G)| = 1 + 2n + n^2 = (n+1)^2$.

Corollary 2.15. Let G be a $\{N_2\}$-coherent extension of $\overline{L_{n,n}}$, where $n \geq 4$. Then $G\cong \overline{L_{n+1,n+1}}$.

Proposition 2.16. Let $\mathcal{C} = \{N_m \mid m\geq 1\}$, let $t\geq 2$ and $n\geq 1$. If G is a $\mathcal{C}$-coherent extension of tK_n, then one of the following holds:

(i) $t = 2$, $n\geq 2$ and $G\cong L_{n+1,n+1}$.

(ii) $n = 1$, and $G\cong K_{2;t}$

(iii) $n = 1$, $t = 2$ and $G\cong C_5$.

Proof. By Proposition 2.4, G is connected. Therefore G has diameter 2. Let $a\in V(G)$, let $X = G(a)$ and $Y = \bar{G}(a)$, let $U_1, \ldots, U_t$ be the connected components of $\langle X\rangle$.

If $y, x \in V(G)$ and $x \not\sim y$, then $\langle G(x) \cap G(y) \rangle$ is a null graph. Indeed, if we had $z_1, z_2 \in G(x) \cap G(y)$ such that $z_1 \sim z_2$, then we would have: $\{x, z_2, y\} \subseteq G(z_1)$ and $x \sim z_2 \sim y \not\sim x$, which contradicts the fact that $\langle G(z_1) \rangle \cong tK_n$.

Therefore if $y \in Y$, then $G(y) \cap X$ containts at most one element of each U_i.

Let $g_1 = t$ and for $i \geqslant 2$, let $g_i = G(N_i)$. Then $g_i \geqslant g_j$ for $1 \leqslant i \leqslant j$. If $i \geqslant 2$ and $\langle x_1, \ldots, x_i \rangle \cong N_i$, then $\langle G(x_1) \cap \ldots . \cap G(x_i) \rangle \cong N_{g_i}$, since $G(x_1) \cap G(x_2)$ does not contain two adjacent vertices. Therefore, for $j \geqslant 1$, g_j is the size of a maximal null graph contained in the neighbourhood of a null subgraph of size j. From this we deduce the following:

(1) If $i, j \geqslant 1$, then $g_i \geqslant j$ if and only if $g_j \geqslant i$.

Let $\mu = g_2$, $\theta = g_3$, $s = g_\mu - 1$ and $r = |Y|$. For all $i = 1, \ldots, t$, for every $b \in U_i$, $G(b) \setminus Y$ = $(U_i \setminus \{b\}) \cup \{a\}$ and so $|G(b) \cap Y| = tn - n = (t-1)n$.

The number of pairs (x,y) such that $x \in X$, $y \in Y$ and $x \sim y$ is equal to $r\mu$ and also to $nt.n(t-1) = t(t-1)n^2$. Therefore:

(2) $r\mu = t(t-1)\, n^2$.

Let us choose μ vertices $p_1, \ldots, p_\mu$ of X, each in a distinct U_i. Then $|Y \cap G(p_1) \cap \ldots \cap G(p_\mu)| = g_\mu - 1 = s$. By (1) we have $s \geq 1$ since $g_2 \geq \mu$.

The number of sets $\{y\} \cup (G(y) \cap X)$ such that $y \in Y$ is equal to r and also to $s.\binom{t}{\mu} n^\mu$. Hence:

$$(3) \qquad r = s\binom{t}{\mu} n^\mu.$$

By eliminating r from (2) and (3), we get $s\binom{t}{\mu} n^\mu = t(t-1)n^2/\mu$. As $t \geq 2$ and $\mu \geq 1$, we get:

$$(4) \qquad s\binom{t-1}{\mu-1} n^{\mu-2} = t-1$$

It follows that if $\mu \geq 2$, then $s n^{\mu-2} \geq 1$ and so $\binom{t-1}{\mu-1} \leq t-1$. But this implies that $\mu - 1 = 0, 1, t-2$ or $t-1$. Thus we have:

$$(5) \quad \mu = 1, 2, t-1 \text{ or } t.$$

If $g_1 = t \geq 3$, then (1) implies that $\theta = g_3 \geq 1$. If $y \in Y$, then $|Y \cap \bar{G}(y)| = |Y| - |\{y\}| - |Y \cap G(y)| = r - 1 - (tn-\mu) = \dfrac{t(t-1)n^2}{\mu} - 1 - tn + \mu$ by (2).

The number of couples (x,z) such that $x \in X \cap G(y)$ and $z \in G(x) \cap Y \cap \bar{G}(y)$ is equal firstly to $|Y \cap \bar{G}(y)|.g_3 = \theta\left(\dfrac{t(t-1)n^2}{\mu} - tn + \mu - 1\right)$, secondly to $|X \cap G(y)|.|G(x) \cap \bar{G}(y) \cap Y|$ (with $x \in X \cap G(y)$), that is to $\mu(t-2)n$.

Therefore:

(6) If $t \geq 3$, then $\theta\left(\frac{t(t-1)n^2}{\mu} - tn + \mu - 1\right) = \mu(t-2)n$.

In other words $\frac{\theta t(t-1)n^2}{\mu} - ((\theta+\mu)t-2\mu)\, n + \theta \cdot (\mu-1) = 0$.

Let $A = \theta t(t-1)/\mu$, $B = -((\theta+\mu)t - 2\mu)$ and $C = \theta(\mu-1)$. Then n is a solution of the equation $Ax^2 + Bx + C = 0$. Let m be the other solution. As $A > 0$, $B < 0$ and $C \geq 0$, we have $m \geq 0$. Now $nm = \frac{C}{A} = \frac{\mu(\mu-1)}{t(t-1)} \leq 1$. Therefore $m \leq 1$ (since $n \geq 1$), and the equality holds if and only if $\mu = t$ and $n = 1$. As $\frac{B}{A} = -(m+n)$, we get: $A + B + C = A\,(1-(m+n)+mn) = A\,(n-1)(m-1)$. As $A > 0$, $A + B + C \leq 0$, and the equality holds if and only if $n = 1$ and $\mu = t$. Hence:

(7) If $t \geq 3$, then $\frac{\theta}{\mu}t(t-1) - ((\theta+\mu)t-2\mu) + \theta(\mu-1) \leq 0$, and the equality holds if and only if $t = \mu$ and $n = 1$.

We have four cases:

(1°) $1 = \mu < t-1$

(2°) $2 = \mu < t-1$

(3°) $\mu = t-1$.

(4°) $\mu = t$.

Suppose that (1°) holds. Then $t \geq 3$, and (1) implies that $1 \leq \theta < 2$, in other words $\theta = 1$. By (7), we get $(t-1)(t-2) = t(t-1) - (2t-2) \leq 0$, which is impossible, since $t \geq 3$.

If (2^{o}) holds, then $t \geq 4$ and $\theta = 1$ by (1). Applying (7), we get:

$\frac{1}{2}t(t-1) - (3t-4) + 1 \leq 0$, or simply

$(t-2)(t-5) = t^2 - 7t + 10 \leq 0$, which implies that $3 \leq t \leq 4$. As $t \geq 4$, we have $t = 4$ and (6) gives $0 = \frac{1}{2}t(t-1)n^2 - (3t - 4)n+1 = 6n^2 - 8n+1$.

But this equation has discriminant $8^2 - 4 \times 6 = 40$, which is not a square. Hence n is not an integer, and so we have a contradiction.

Suppose now that (3^{o}) holds. If $n = 1$, then for $y \in Y$, there exists some $z \in G(y) \cap Y$. Now $G(y) \cap G(z) = \emptyset$ and so $t \geq |(X \cap G(y) \cup (X \cap G(z)| = 2\mu = 2(t-1)$ and we conclude that $t = 2$, $r = 2$ and so (iii) holds.

Therefore we may suppose that $n > 1$. By (4) we have $\varsigma n^{\mu-2} = 1$ and so $\mu-2 \leq 0$, in other words $t \leq 3$. If $t = 3$, then (1) implies that $\theta = 1$ and applying (6), we get $3n^2 - 5n + 1 = 0$. But this equation has discriminant $5^2 - 4 \times 3 = 13$, which is not a square. Hence n is not an integer, and we have a contradiction.

If $t = 2$, then let K be the set of all subsets M of V(G) such that $\langle M \rangle \cong K_{n+1}$. Then the number of pairs (x,M) such that $x \in M \in K$ is equal to $|K|(n+1)$ and also to $|V(G)|$. $t = 2(1+tn+r) = 2(1+tn + tn^2)$. Therefore $n + 1 \mid 2(1+tn+tn^2)$ and so $n + 1 \mid 2$, which contradicts the fact that $n > 1$.

Suppose lastly that (4^{o}) holds. If $n = 1$, then $r = t-1$ and every point of Y is joined to every point of X. Thus (ii) holds. If $n > 1$, then (4) implies that $\varsigma n^{t-2} = t-1$. If $t \geq 4$, then $n^{t-2} \geq 2^{t-2} > t-1$, which is impossible. Therefore $t = 2$ or $t = 3$, and we see that $\varsigma = 1$. If $t = 3$, then $\theta = \varsigma + 1 = 2$ and $n = 2$. But we see that (6) is not verified in this case. Hence $t = 2$ $r = n^2$ (by (2)) and so $|V(G)| = (1+n)^2$. By Proposition 1.10, $G \cong L_{n+1,\, n+1}$ and so (i) holds.

Corollary 2.17. Let $\mathcal{C} = \{K_m \mid m \geq 1\}$ and let $t \geq 2$.

(i) If G is a $\mathcal{C}$-coherent antiextension of K_t, then $G \cong 2\,K_t$ or $t = 2$ and $G \cong C_5$;

(ii) If $n \geq 2$ and G is a $\mathcal{C}$-coherent antiextension of $K_{t;n}$, then $t = 2$ and $G \cong \overline{L_{n+1,n+1}}$.

Proposition 2.18. If $n \geq 4$, then $L_{n,n}$ has no $\{N_2, N_3\}$-coherent extension.

Proof. Suppose that G is a $\{N_2, N_3\}$-coherent extension of $L_{n,n}$. By Proposition 2.4, G is connected. Moreover, G has diameter 2.

Let $a \in V(G)$, let $X = G(a)$ and $Y = \overline{G}(a)$. Let $\mu = G(N_2)$ and $\theta = G(N_3)$. If $r = |Y|$, then

$$r\mu = |\{(x,y) \mid x \in X, y \in Y, x \sim y\}| = n^2(n-1)^2.$$

As $n \geq 4$, $\theta \geq 1$. If $y \in Y$, then $|\overline{G}(y) \cap Y| = r-1-(n^2-\mu) = \frac{n^2(n-1)^2}{\mu} - n^2 + \mu - 1$. For each such y, the number of ordered pairs (x,z) such that $x \in G(y) \cap X$ and $z \in G(x) \cap \overline{G}(y) \cap Y$ is equal to $|G(y) \cap X| . |G(x) \cap \overline{G}(y) \cap \overline{G}(a)|$

$(x \in G(y) \cap G(a)$, in other words to $\mu.(n-2)^2$. But it is also equal to $|\bar{G}(y) \cap Y|$. $G(N_3) =$
$\theta\left(\frac{n^2(n-1)^2}{\mu} - n^2 + \mu - 1\right)$.

Therefore $\theta\left(\frac{n^2(n-1)^2}{\mu} - n^2 + \mu - 1\right) = \mu(n-2)^2$. $\quad (*)$

As in the beginning of the proof of Proposition 2.8, we can show that for $y \in Y$, $\langle G(y) \cap X\rangle$ is a union of cycles of length at least 4 of $\langle X\rangle$. Thus $\mu = |G(y) \cap X|$ is even and $4 \leq \mu \leq 2n$.

Let K be the set of subsets M of V(G) such that $\langle M\rangle \cong K_{n+1}$. Then every vertex of G is contained in $2n$ elements of K. It follows that $(1 + n^2 + n^2(n-1)^2/\mu).2n = |V(G)|.2n = |\{(x,M) \mid x \in M \in K\}| = |K|.(n+1)$. Therefore $n + 1$ divides $(1 + n^2 + n^2.(n-1)^2/\mu)\,2n = (\mu + \mu n^2 + n^2(n-1)^2)n/\frac{\mu}{2}$. As $(n + 1, n) = 1$, it follows that $n + 1$ divides $\mu + \mu n^2 + n^2(n-1)^2$. But then $n + 1 \mid 2\mu + 4$, and as $\mu \leq 2n$, $2\mu + 4 \leq 4(n+1)$. Therefore $2\mu + 4 = n + 1$, $2(n+1)$, $3(n+1)$ or $4(n+1)$. Hence $\mu = \frac{n-3}{2}$, $n - 1$, $\frac{3n-1}{2}$ or $2n$.

<u>Case 1:</u> If $\mu = \frac{n-3}{2}$, then $\frac{n-3}{2}$ is even and $\frac{n-3}{2} \mid n^2(n-1)^2$. We verify then that $\frac{n-3}{2} \mid 36$. If $4 \mid \mu$, then $8 \mid 2\mu = n - 3$ and $4 \mid n+1$. But then $4 \nmid n-1$, $8 \nmid (n-1)^2$ and so $n^2(n-1)^2/\mu$ is odd. As n^2 is odd, it follows that $1 + n^2 + \frac{n^2(n-1)^2}{\mu}$ is odd and so may not be divisible by $n + 1$ (which is even) Therefore $4 \nmid \mu$ and so $\mu = 6$ or 18. In both cases we verify that the equation $(*)$ is not satisfied,

and so we have a contradiction.

Case 2: If $\mu = n-1$, then (*) implies that

$\theta\,(n^2(n-1)-n^2 + n-2) = (n-1)(n-2)^2$, or

$\theta\,(n-2)\,(n^2 + 1) = (n-1)\,(n-2)^2$, or

$\theta\,(n^2 + 1) = (n-1)(n-2) = n^2 - 3n + 2 < n^2 + 1$.

But then $\theta < 1$, which is impossible.

Case 3: If $\mu = \frac{3n-1}{2}$, then $\mu \mid n^2(n-1)^2$. Now $(\mu, n) \mid (3n-1, n) = 1$ and so $\mu \mid (n-1)^2$. As $(\mu, n-1) \mid (3n-1, n-1)/2$, we have $\mu \mid 4$. As $\mu \geq 4$, we get $\mu = 4$ and so $n = \frac{2\mu + 1}{3} = 3$ which contradicts our hypothesis.

Case 4: If $\mu = 2n$, then (*) implies that

$\theta(\frac{1}{2}\, n(n-1)^2 - n^2 + 2n-1) = 2n(n-2)^2$, or

$\theta\,(\frac{1}{2}\, n(n-1)^2 - (n-1)^2) = 2n\,(n-2)^2$, or

$\theta\,(n-1)^2\,(\frac{n-2}{2}) = 2n(n-2)^2$, or

$\theta\,(n-1)^2 = 4\,n(n-2)$

But $(n-1, n) = (n-1, n-2) = 1$ and so $(n-1)^2 \mid 4$, in other words $n \leq 3$, which contradicts the hypothesis.

Therefore each case leads to a contradiction and so G does not exist.

Corollary 2.19: If $n \geq 4$, then $\overline{L_{n,n}}$ has no $\{K_2, K_3\}$-coherent antiextension.

Proposition 2.20 [5]. Let $m, n \geqslant 1$ and $s, t \geqslant 2$. If the graph G is an extension of tK_n and an antiextension of $K_{s;m}$ then one of the following holds:

(i) $m = n = s = t = 2$ and $G \cong L_{3,3}$

(ii) $m = n = 2$, $s = t = 1$ and $G \cong C_5$.

Proof. Let $a \in V(G)$, let $X = G(a)$ and $Y = \tilde{G}(a)$. If $n=1$, then G contains no triangles and it follows that $s=2$. As $\langle Y \rangle$ is of valency m, we have $m \leqslant t$. If $b \in X$, then $\langle G(b) \cap Y \rangle \cong (t-1)K_1$ and so $t-1 \leqslant m$. Therefore $m=t$ or $m=t-1$. Every vertex of X is joined to $t-1$ vertices of Y and every vertex of Y is joined to $t-m$ vertices of X. Thus $t(t-1)=2m(t-m)=$ the number of pairs (x,y) such that $x \in X$, $y \in Y$ and $x \sim y$. It follows that $t-1=m$ and $t=2$. Hence $n=m=1$ and $t=s=2$. Thus $|V(G)|=5$ and (ii) holds.

If $m=1$, then the same argument applied to $\overline{G}$ shows that (ii) holds also in this case. Therefore we may suppose that $m > 1 < n$. If $b \in X$, then $\langle G(b) \setminus Y \rangle \cong K_n$. As $\langle G(b) \rangle \cong K_n$, it follows that $\langle G(b) \cap Y \rangle \cong (t-1)K_n$. As $n > 1$, we must have $n \leqslant s$ and $t-1=1$ (since $\langle Y \rangle$ does not contain any $2K_2$). Applying the same argument to $\overline{G}$, we find that $m \leqslant t$ and $s-1=1$. Therefore $s=t=m=n=2$. Now $|V(G)|=9$ and so (i) holds by Proposition 1.10.

III. Characterizations of $\mathcal{T}$-C-graphs

In this section, we will classify $\mathcal{T}$-C-graphs for several families of graphs. Let us define first a few families:

(1) $\mathcal{H} = \{N_n \mid n \geq 0\} \cup \{tK_m \mid t \geq 1 \text{ and } m \geq 2\}$

$\cup \{K_{r;s} \mid r \geq 2 \text{ and } s \geq 2\} \cup \{C_5, L_{3,3}\}$

$\mathcal{H}$ is the family of all UH-graphs [3].

(2) $\mathcal{M} = \{N_n \mid n \geq 0\} \cup \{tK_m \mid t \geq 1 \text{ and } m \geq 2\}$

$\cup \{K_{r;s} \mid r \geq 2 \text{ and } s \geq 2\} \cup \{L_{n,n} \mid n \geq 3\}$

$\cup \{\overline{L_{n,n}} \mid n \geq 4\} \cup \{C_5\}$

$= \mathcal{H} \cup \{L_{n,n} \mid n \geq 4\} \cup \{\overline{L_{n,n}} \mid n \geq 4\}$

(3) $\mathcal{K}$ is the family defined in page 6.

(4) $\mathcal{L} = \{N_m \mid m \geq 1\} \cup \{K_n + N_m \mid n \geq 1 \text{ and } m \geq 1\}$.

<u>Note</u>: We have $\mathcal{H} = \overline{\mathcal{H}}$, $\mathcal{M} = \overline{\mathcal{M}}$, $\mathcal{K} = \overline{\mathcal{K}}$, but $\mathcal{L} \neq \overline{\mathcal{L}}$. Also $\mathcal{L} \cup \overline{\mathcal{L}} \subseteq \mathcal{K}$.

We will show that a $\mathcal{T}$-C-graph belongs to $\mathcal{H}$ or $\mathcal{M}$ when $\mathcal{T}$ is suitably chosen.

<u>Lemma 3.1</u> (i) If $n \geqslant 4$, then $L_{n,n}$ is not $\{2K_2\}$-coherent.

(ii) If $n \geqslant 4$, then $\overline{L_{n,n}}$ is not $\{K_{2,2}\}$coherent.

Proof. Take $a = (1,1)$, $b = (1,2)$, $c = (2,3)$, $d = (3,3)$ and $e = (2,4)$ in $V(G)$, where $G = L_{n,n} (n \geqslant 4)$. Then $\langle a, b, c, d \rangle \cong \langle a, b, c, e \rangle \cong 2K_2$, but $|G(\{a, b, c, e\})| = 0$ and $1 = |G(\{a, b, c, d\})|$. Hence (i) holds.

For $G = \overline{L_{n,n}}$ ($n \quad 4$), take the same points a,b,c,d,e. Then $\langle a, b, c, d \rangle \cong \langle a, b, c, e \rangle \cong K_{2,2}$ but $|G(\{a, b, c, e\})| = (n-4)(n-2)$ and $(n-3)^2 = |G(\{a, b, c, d\})|$. Hence (ii) holds.

<u>Theorem 3.2.</u> If G is a C-graph, then $G \in \mathcal{H}$.

Proof: Suppose that the result is false and let G be a counter-example with $V(G)$ minimal. Let $a, b \in V(G)$, let $H = \langle G(a) \rangle$, $L = \langle \bar{G}(a) \rangle_G$, $H_1 = \langle G(b) \rangle$ and $L_1 = \langle \bar{G}(b) \rangle_G$. By Corollary 1.7 and Proposition 1.12, H, H_1, L and L_1 are C-graphs and we have $H(X) = H_1(X)$ and $L(X) = L_1(X)$ for any graph X. Thus $H, H_1, L, L_1 \in \mathcal{H}$ by induction hypothesis, and Propositions 1.8 and 1.10 and Corollaries 1.9 and 1.11 imply that $H \cong H_1$ and $L \cong L_1$. Thus G is an extension of H and an antiextension of L. By Proposition 2.1, Corollaries 2.5, 2.7 and 2.9 and Lemma 3.1, we may not have $H \cong K_m$ $(m \geqslant 0)$, $K_{r;s}$ $(r \geqslant 2, s \geqslant 2)$, C_5 or $L_{3,3}$. By Corollaries 2.10, 2.11, 2.12 and 2.13 and Lemma 3.1, we may not have $L \cong N_n$ $(n \geqslant 0)$, tKm $(t \geqslant 2, m \geqslant)$, C_5 or $L_{3,3}$. Therefore $H \cong tK_m$ $(t \geqslant 2, m \geqslant 1)$ and

$L \cong K_{r,s}$ ($r \geq 2$, $s \geq 1$), which contradicts Proposition 2.20.

Therefore our supposition was false and the result must be true.

Theorem 3.3. If G is a $\mathcal{K}$-C-graph, then $G \in \mathcal{M}$

Proof. Suppose that the result is false and that G is counterexample with V(G) minimal. As in the preceding theorem, G is an extension of $H \in \mathcal{M}$ and an anti-extension of $L \in \mathcal{M}$, since $\mathcal{K}$ is strong and extensible. By Propositions 2.14 and 2.18 and Corollaries 2.15 and 2.19, H and L may not be isomorphic to $L_{n,n}$ or $\overline{L_{n,n}}$ with $n \geq 4$. Therefore $H, L \in \mathcal{K}$ and we get the same contradiction as in Theorem 3.2.

Corollary 3.4. If G is a $\mathcal{K} \cup \{2K_2, K_{2,2}\}$-C-graph, then $G \in \mathcal{H}$.

Our proof of Theorem 3.2 is similar to the proof that we gave in [5] of Gardiner's classification of UH-graphs [3]. The next result has been inspired by the method we used to classify H-graphs [6]:

Theorem 3.5 If G is a $\mathcal{L}$-C-graph, then $G \in \mathcal{M}$.

<u>Proof</u>. Suppose that the result is false and that G is a counterexample with $|V(G)|$ minimal. As $\mathcal{L}$ is closed and contains the graphs K_n and N_n ($n \geq 1$), we can show again that G is an extension of a graph $H \in \mathcal{M}$ By Propositions 2.1, 2.16 and 2.18 and Corollaries 2.5, 2.7, 2.9 and 2.15, $H \notin \mathcal{M}$, and so we have a contradiction Therefore the result holds.

<u>Corollary 3.6.</u> If G is a $\mathcal{L} \cup \{2K_2, K_{2,2}\}$ -$\mathcal{C}$-graph, then $G \in \mathcal{H}$.

As L is strong, we deduce the following:

<u>Corollary 3.7.</u> If G is a $\overline{\mathcal{L}}$ -$\mathcal{C}$-graph, then $G \in \mathcal{M}$

<u>Corollary 3.8.</u> If G is a $\overline{\mathcal{L}} \cup \{2K_2, K_{2,2}\}$-$\mathcal{C}$-graph, then $G \in \mathcal{H}$.

It is easy to verify that $\mathcal{M}$ is the family of all $\mathcal{H}$ -UH-graphs.

Enomoto [2] classified $\mathcal{C}$ -SCH-graphs, where $\mathcal{C}$ is the family of connected graphs. In fact, we can prove his result with a weaker hypothesis (the proof is identical, apart from a few details):

<u>Theorem 3.9</u>. Let G be a $\mathcal{C}$ -coherent graph. Suppose that for every U, $V \subseteq V(G)$ such that $\langle U \rangle \cong \langle V \rangle \in \mathcal{C}$ and $X \subseteq U$ such that X is stabilized by Aut($\langle U \rangle$), $|G(X)| = |G(X^f)|$ for any isomorphism $f: U \to V$ (X^f does not depend on the choice of f). Then G is the disjoint union of

for any isomorphism $f: \langle U \rangle \to \langle V \rangle$ (X^f does not depend on the choice of f); then G is the union of isomorphic graphs, which are isomorphic to one of the following graphs:

(i) K_n ($n \geq 1$).

(ii) C_n ($n \geq 4$).

(iii) $K_{t;r}$ ($t \geq 2, r \geq 2$).

(iv) $L_{n,n}$ ($n \geq 3$).

(v) $\overline{L_{2,n}}$ ($n \geq 3$).

(vi) Petersen's graph.

(vii) Clebsch's graph (the graph obtained by identifying antipodal points in the 5-dimensional cube.).

In particular, this classifies all $\mathcal{C}$-SC-graphs.

What about $\mathcal{C}$-LC-graphs? It is clear that if G is a $\mathcal{C}$-LC-graph, then its connected components are $\mathcal{C}$-LC-graphs. Therefore we may suppose that G is connected. If $a \in V(G)$, the $H = \langle G(a) \rangle$ is a C-graph. Indeed, if U, $V \subseteq G(a)$ and $\langle U \rangle \cong \langle V \rangle$, then $\langle \{a\} \cup U \rangle \cong \langle \{a\} \cup V \rangle$ and as $\{a\} \cup U$ and $\{a\} \cup V \subseteq \{a\} \cup G(a) \in \mathcal{C}$, we must have $|H(U)| = |G(\{a\} \cup U)| = |G(\{a\} \cup V)| = |H(V)|$. Thus $H \in \mathcal{H}$.

It is not hard to show that for any $b \in V(G)$, $\langle G(b) \rangle \cong H$ (by induction on $d(b,a)$). Thus G is a connected extension of H.

It seems that the methods of Enomoto [2] could enable us to classify $\mathcal{C}$-LC-graphs, but the author had not the time to do it.

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