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Report R485

Some Topics on Digital Convexity

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Abstract: *We study various definitions of convexity on 2D digital images and examine their relations with 4- and 8-connectedness. We give a new treatment of the analysis by Rosenfeld and Kim of the chord property, digital straight lines, and the link between them.*

I. Introduction

The convexity of two-dimensional digital images has been the subject of several papers. An accurate description of the state of the art can be found in the works of Kim and Rosenfeld [15–17,20,21,23,26,27].

The goal of this Report is to complement the work done on this field, mainly on two aspects:

— An analysis of the relations between the various definitions of convexity and connectedness, including new proofs of some equivalence theorems.

— A new treatment of digital and cellular straight line segments, including a simpler proof of their characterization theorems, and the definition of a new concept, the "strong chord property".

We have not included the analysis made mainly by Sklansky and Kim [15,23,29–31] of the quantization relations between convex digital images and convex Euclidean images, although they are related to digital and cellular straight line segments. We have also left aside the various algorithms for testing convexity, building convex hulls or digital straight line segments, etc. found in the literature.

Our work is organized as follows:

— In Chapter II we review some important combinatorial properties of convexity in the n-dimensional real space: the theorems of Carathéodory, Helly and Santaló.

— In Chapter III we review several definitions of convexity and convex hulls in two-dimensional rectangular grid images and analyze their relations with 4- and 8-connectedness.

— In Chapter IV, we study the *chord property* and a specialization of it to the 4-connected case, which we call the *strong chord property*. Using Santaló's Theorem, we give a simpler proof of the characterization by Rosenfeld and Kim of digital and cellular straight line segments.

Throughout this Report, we adopt a geometrical point of view, which is in some way original. Such a mathematical treatment of the topic improves the theoretical understanding of digital geometry and should be useful for the generalization of results of 2-dimensional digital geometry to the 3- or n-dimensional case.

II. Convexity in the n-dimensional real space

The study of convexity in digital figures on a two-dimensional square grid is done with the aim of finding digital equivalents of concepts and properties found in the Euclidean case. Moreover, as we will see later on, some theorems on convexity in the two-dimensional real plane can find applications in the study of digital convexity. It is thus worthwile to recall some fundamental facts on convexity in the Euclidean plane. This is the purpose of this Chapter. Moreover, we will generalize this analysis to the n-dimensional case, since this generalization does not lead to a real complication.

Points in the real n-dimensional space $\mathbf{R}^n$ can be considered as vectors w.r.t. some origin. Given m points $p_1, \dots, p_m$ and m real numbers $\lambda_1, \dots, \lambda_m$ such that $\lambda_1 + \cdots + \lambda_m = 1$, the point

$$\sum_{i=1}^{m} \lambda_i p_i = p_1 + \sum_{i=2}^{m} \lambda_i (p_i - p_1) \tag{1}$$

is independent of the choice of the origin. When these m points do not belong to a common proper Euclidean subspace of $\mathbf{R}^n$ (this requires that $m > n$), the vectors $p_i - p_1$ $(i \geq 2)$ generate the vector space $\mathbf{R}^n$, and so every point takes the form (1). If moreover $m = n+1$, then the vectors $p_i - p_1$ are linearly independent, and so the form (1) of a point is unique.

Given a nonvoid set P of points in $\mathbf{R}^n$, we write $[P]$ for the set of all points of the form $\lambda_1 p_1 + \cdots + \lambda_m p_m$, where $m > 0$, $\lambda_i \geq 0$ and $p_i \in P$ for each i, and $\lambda_1 + \cdots + \lambda_m = 1$. Readily, if P is infinite, then $[P]$ is the union of all $[Q]$, with Q being any finite nonvoid subset of P.

When P is finite and $P = \{p_1, \dots, p_m\}$, we can write $[p_1, \dots, p_m]$ for $[P]$. It represents the topologically closed polytope spanned by $p_1, \dots, p_m$.

The operator [ ] has the following properties; for any nonvoids subsets P and Q of $\mathbf{R}^n$ we have:

(1°) $P \subseteq [P]$.

(2°) $[[P]] = [P]$.

(3°) If $P \subseteq Q$, then $[P] \subseteq [Q]$.

(4°) $[P \cup Q] = \bigcup [p, q]$ for $p \in [P]$ and $q \in [Q]$.

One says that a set C is *convex* if and only if for every $p, q \in C$, the closed segment $[p, q]$ is a subset of C. By iterating (4°), it is easily shown that this is equivalent to $[C] = C$.

The intersection of any number of convex sets is itself convex. One can thus define the *convex hull* of a set P as the intersection of all convex sets containing it; it is in fact equal to $[P]$.

The following result is called *Carathéodory's Theorem* for convex sets (see [33], page 35). For $n = 2$, it simply means that a convex polygon is the union of the triangles spanned by its vertices.

Proposition 1. *Let P be a subset of $\mathbb{R}^n$ and let $\mathcal{P}(n+1)$ be the set of all subsets of size $n+1$ of P. Then $[P] = \bigcup_{Q \in \mathcal{P}(n+1)}[Q]$.*

Proof. Suppose false. Then there is some $p \in [P]$ such that $p \notin \bigcup_{Q\in\mathcal{P}(n+1)}[Q]$. Now $p \in [R]$ for a finite subset R of P, and so there is a minimal subset M of R such that $p \in [M]$. Clearly $|M| > n+1$. We write $M = \{p_0, \dots p_m\}$ $(m \geq n+1)$, and so $p = \sum_{i=0}^m \lambda_i p_i$, where $\sum_{i=0}^m \lambda_i = 1$ and each $\lambda_i \geq 0$. In fact, by the minimality of M, each $\lambda_i > 0$.

As $m > n$, the vectors $p_1 - p_0, \dots, p_m - p_0$ are linearly dependent. There exist thus $\mu_1, \dots, \mu_m \in R$, not all zero, such that $\sum_{i=1}^m \mu_i(p_i - p_0) = 0$. Setting $\mu_0 = -\sum_{i=1}^m \mu_i$, we get $\sum_{i=0}^m \mu_i p_i = 0$ and $\sum_{i=0}^m \mu_i = 0$.

Choose k such that μ_k/λ_k is maximum; as $\lambda_k > 0$ and $\sum_{i=0}^m \mu_i = 0$, $\mu_k > 0$. We have:

$$p = \sum_{i=0}^m \lambda_i p_i = \sum_{i=0}^m \lambda_i p_i - \lambda_k/\mu_k \sum_{i=0}^m \mu_i p_i = \sum_{i=0}^m \nu_i p_i,$$

where

$$\nu_i = \lambda_i - \frac{\lambda_k \mu_i}{\mu_k} \quad \text{for } i = 0, \dots, m. \tag{2}$$

Clearly $\sum_{i=0}^m \nu_i = \sum_{i=0}^m \lambda_i - \lambda_k/\mu_k \sum_{i=0}^m \mu_i = 1 - 0 = 1$. By (2) and the fact that λ_k and μ_k are both $\neq 0$, we have:

$$\nu_i = \frac{\lambda_i \lambda_k}{\mu_k}\left(\frac{\mu_k}{\lambda_k} - \frac{\mu_i}{\lambda_i}\right). \tag{3}$$

As $\lambda_i > 0$, $\lambda_k > 0$, $\mu_k > 0$ and $\mu_k/\lambda_k \geq \mu_i/\lambda_i$, this means that $\nu_i \geq 0$. Moreover $\nu_k = 0$. Thus $p \in [M - \{p_k\}]$, which contradicts the minimality of M. ∎

Note that if the dimension of the space spanned by P is equal to some $d < n$, then $[P] = \bigcup_{Q\in\mathcal{P}(d+1)}[Q]$.

Another important result is *Helly's First Theorem* (see [33], page 117, and [36], page 16):

Proposition 2. *Let $m \geq n+2$. Given m convex sets such that the intersection of any $n+1$ among them is nonvoid, the intersection of these m sets is nonvoid.*

Proof. Let $P_1, \dots, P_m$ be these sets. We suppose first that $m = n+2$. For $i = 1, \dots, m$, we can choose some $p_i \in \bigcap_{j\neq i} P_j$, since this intersection is nonvoid by hypothesis.

The $m - 1 = n + 1$ vectors $p_2 - p_1, \dots, p_m - p_1$ are linearly dependent. There exist thus $\mu_2, \dots, \mu_m \in R$, not all zero, such that $\sum_{i=2}^m \mu_i(p_i - p_0) = 0$. Setting $\mu_1 = -\sum_{i=2}^m \mu_i$, we get $\sum_{i=1}^m \mu_i p_i = 0$ and $\sum_{i=1}^m \mu_i = 0$.

Let I_0 be the set of all i such that $\mu_i \geq 0$, and I_1 the set of all j such that $\mu_j < 0$. Let $\theta = \sum_{i\in I_0} \mu_i$; then $\theta > 0$ and $\sum_{j\in I_1} \mu_j = -\theta$. Moreover we have:

$$\sum_{i\in I_0} \frac{\mu_i}{\theta} p_i = \sum_{j\in I_1} \frac{-\mu_j}{\theta} p_i. \tag{4}$$

As $\mu_i/\theta \geq 0$ for $i \in I_0$, $-\mu_j/\theta \geq 0$ for $j \in I_1$, and $\sum_{i\in I_0} \mu_i/\theta = \sum_{j\in I_1} -\mu_j/\theta = \theta/\theta = 1$, (4) means that there is a point

$$p \in [\{p_i \mid i \in I_0\}] \cap [\{p_j \mid j \in I_1\}]. \tag{5}$$

But for each $i \in I_0$, $p_i \in \bigcap_{j\neq i} P_j \subseteq \bigcap_{j\in I_1} P_j$. As $\bigcap_{j\in I_1} P_j$ is convex, we get:

$$[\{p_i \mid i \in I_0\}] \subseteq \bigcap_{j\in I_1} P_j. \tag{6}$$

Similarly we have:

$$[\{p_j \mid j \in I_1\}] \subseteq \bigcap_{i\in I_0} P_i. \tag{7}$$

Combining (5), (6) and (7), we get $p \in \bigcap_{i=1}^m P_i$.

Suppose now that $m > n+2$. We use induction on m. Assume that the result is true for $m-1$. As $m-1 \geq n+2$, the intersection of any $n+2$ among $P_1, \ldots, P_m$ is nonvoid. Consider now the sets $P'_2 = P_1 \cap P_2, \ldots, P'_m = P_1 \cap P_m$. Then the intersection of any $n+1$ among them is nonvoid (since it is an intersection of $n+2$ among $P_1, \ldots, P_m$). Aplying again the result for $m-1$, we get $P_1 \cap \cdots \cap P_m = P'_2 \cap \cdots \cap P'_m \neq \emptyset$, and so the result holds for m. ∎

For $n = 2$, Helly's First Theorem has an interesting consequence, the *Tranversal Theorem of Santaló* (see [33], page 122, and [36], page 20), of which we give a slightly modified formulation:

Proposition 3. *Let $m \geq 4$ and consider m pairwise noncollinear parallel straight line segments in the plane (they may be open, closed, half-open, or even reduced to a single point). If for every 3 of them there is a straight line intersecting the 3 segments, then there is a line intersecting all m segments.*

Proof. We can choose the Y-axis parallel to the segments. If we write $S_1, \ldots, S_m$ for these segments, then for each $i = 1, \ldots, m$ there is some $x_i \in \mathbf{R}$ and some convex interval $Y_i \subseteq \mathbf{R}$ (which can be closed, open, half-open, or reduced to a single point) such that

$$S_i = \{(x_i, y) \mid y \in Y_i\}.$$

A line L intersecting more than one segment is not parallel to the Y-axis (since the segments S_i are pairwise not collinear). Thus it has an equation of the form

$$y = ax + b,$$

and we write it $L(a,b)$. Now we can associate to each S_i the set

$$C_i = \{(u,v) \mid x_i u + v \in Y_i\}.$$

Clearly the convexity of Y_i implies that C_i is a convex strip of the plane. We associate to $L(a,b)$ the point (a,b). Then $L(a,b)$ intersects S_i iff there is some $y_i \in Y_i$ such that $y_i = ax_i + b$, in other words iff $(a,b) \in C_i$.

The result follows then by applying Helly's First Theorem to the sets C_i. ∎

III. Convexity and convex hulls on two-dimensional digital rectangular grid images

III.1. Definitions of digital convexity

We will examine several definitions of convexity and convex hulls, analyze their relations with 4- and 8-connectedness, and see under which conditions they are equivalent.

These definitions are related to their real plane counterparts. We will thus use some of the properties of convex sets in the Euclidean plane stated in the previous section.

Let us introduce some notation. Let G be a rectangular grid in two dimensions. It can be viewed as a subset of $\mathbf{Z}^2$, where $\mathbf{Z}$ is the set of rational integers, and so the pixels can be represented with integer coordinates (i, j). We choose the orientation of the coordinate axes on $\mathbf{Z}^2$ in such a way that the labelling of pixels is consistent with matrix notation: the pixel (i, j) corresponds to the intersection of row i and column j, with rows counted from top to bottom and columns counted from left to right. In other words, the X- and Y-axes point downwards and rightwards respectively. The natural embedding of $\mathbf{Z}^2$ into $\mathbf{R}^2$ leads to the identification of pixels of G with integer-coordinated points of the plane.

The grid G corresponds also to a decomposition of the plane into square surfaces; thus to every pixel p corresponds the closed square surface $\overline{p}$, which is defined as follows:

$$\overline{p} = \{x \in \mathbf{R}^2 \mid d_8(x, p) \leq 1/2\}. \tag{8}$$

Given $S \subseteq G$, one defines $\overline{S}$ as the union of all $\overline{p}$ for $p \in S$.

For a subset P of G, we write $\langle P \rangle$ for the set of all pixels of G which belong to $[P]$; if P is finite and $P = \{p_1, \ldots, p_m\}$, then we can write $\langle p_1, \ldots, p_m \rangle$ for $\langle P \rangle$. By Proposition 1,

$$\langle P \rangle = \bigcup_{p,q,r \in P} \langle p, q, r \rangle. \tag{9}$$

We illustrate this definition in Figure 1.

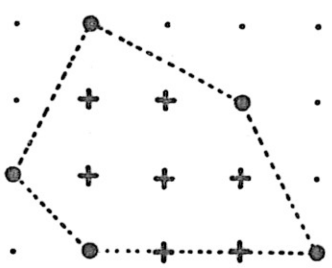


Figure 1.

We can now state several definitions of digital convexity. Let $P \subseteq G$.

First definition [24]. P is convex iff for every $p, q, \in P$, $\langle p, q \rangle \subseteq P$.

Second definition [15]. P is convex iff $P = \langle P \rangle$. By (9), this holds iff for every $p, q, r \in P$, $\langle p, q, r \rangle \subseteq P$.

Third definition [15]. Assume that P is simply 8-connected. Then the border $\partial(\overline{P})$ of $\overline{P}$ is a simple closed curve. For every $p, q \in P$, let $\mathcal{P}(P; p, q)$ be the plane area consisting of all polygons whose boundary is a subset of $[p, q] \cup \partial(\overline{P})$ and whose interior lies outside $\overline{P}$ (see Figure 2). Then P is convex iff for every $p, q \in P$, the area $P \cap \mathcal{P}(P; p, q)$ is empty.

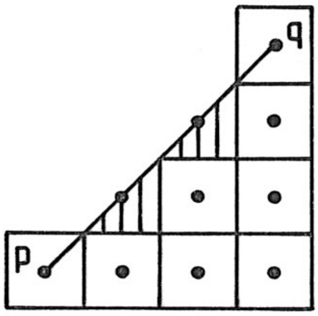


Figure 2. $\mathcal{P}(P; p, q)$, *non-convex case*

Fourth definition [29,30]. P is convex iff it is the digitization of a convex image in the real plane.

It is easy to see that when P is simply 8-connected, the second and third definitions are equivalent (see [15], Lemma 10). This is because for $p, q \in P$, $\mathcal{P}(P; p, q) \subseteq \bigcup_{r \in P} [p, q, r]$. Kim showed the non-trivial result that when P is simply 8-connected, the first and third definitions are equivalent (see [15], Theorem 5). It is not hard to see that when P satisfies to the first or the second definition, the assumption that P is simply 8-connected is a consequence of the weaker assumption that P is 8-connected (see Section III.3). Thus, when P is 8-connected the first two definitions of convexity are equivalent. We will give another proof of this fact in Theorem 4.

The fourth definition depends upon the definition of the digitization of an image in the real plane (see [15,23]). Generally, for a plane figure F, its digitization is the set of $p \in G$ such that $\overline{p} \cap F \neq \emptyset$, or rather the set of $q \in G$ such that $\overline{q}^\circ \cap F \neq \emptyset$, where $\overline{q}^\circ$ is the interior of $\overline{q}$. Kim [15] showed that the fourth definition implies the second one, and that the second one implies the fourth one when no pixel of P meets the border of P on more than 2 sides and P is 8-connected. When a pixel p of P meets the border of P in at least 3 sides, then one can have a counter-example, as shown in Figure 3.

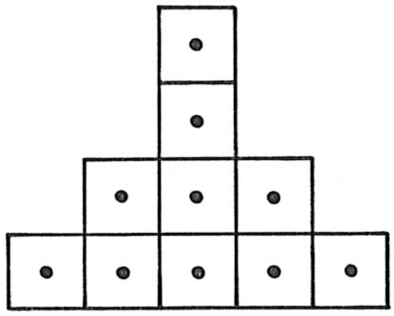


Figure 3.

It is clear that the first two definitions are the more practical. We make then the following:

Definition 1. Let $P \subseteq G$. Then:

(i) P is *line-convex* (or briefly *L-convex*) iff for every $p, q \in P$, $\langle p, q\rangle \subseteq P$

(ii) P is *triangle-convex* (or briefly *T-convex*) iff for every $p, q, r \in P$, $\langle p, q, r\rangle \subseteq P$.

When P is not (line-, triangle-) convex, then we say that it is (line-, triangle-) *concave*. Note that P is T-convex iff $P = \langle P\rangle$.

As announced above, we have the following result, due essentially to Kim [15]:

Theorem 4. *If P is 8-connected and L-convex, then P is T-convex.*

For its proof we need the following:

Lemma 5. *Let $H(\alpha, \beta, n)$ be the set of all pixels $(i, j) \in G$ such that $\alpha i + \beta j \leq n$, where $\alpha, \beta \in \{0, 1, -1\}$, $(\alpha, \beta) \neq (0, 0)$ and n is any rational integer. (It is easy to see that $H(\alpha, \beta, n)$ is one half of the plane determined by a line along the horizontal, vertical or diagonal directions). If $C \subseteq G$ is 8-connected and L-convex, then $C \cap H(\alpha, \beta, n)$ is empty or 8-connected and L-convex.*

Proof. It is obvious that $H(\alpha, \beta, n)$ is L-convex. Thus $C \cap H(\alpha, \beta, n)$ is L-convex. Suppose that the result is false, that $C \cap H(\alpha, \beta, n)$ is not 8-connected. Take two connected components D and E of C and $d \in D$ and $e \in E$ such that $d_4(d, e)$ (the 4-distance between them) is minimum. We have two cases:

(a) d and e lie both on the border of $H(\alpha, \beta, n)$. Now this border is a horizontal, vertical or diagonal line of G, and so this means that $\langle d, e\rangle$ is 8-connected. By convexity, $\langle d, e\rangle \subseteq C$, and as $\langle d, e\rangle \subseteq H(\alpha, \beta, n)$, d is 8-connected to e in $C \cap H(\alpha, \beta, n)$, a contradiction.

(b) One of d and e, say d, does not lie on the border of $H(\alpha, \beta, n)$. Then all pixels 8-adjacent to d lie in $H(\alpha, \beta, n)$. There are 3 of them, p_1, p_2 and p_3, whose 4-distance to e is less than $d_4(d, e)$ (see Figure 4). If one of them belongs to C, then it belongs to D and so $d_4(d, e)$ is not minimum. Thus $p_1, p_2, p_3 \notin C$. We draw from p the half-lines L_1, L_2, L_3

of G passing through p_1, p_2 and p_3 respectively (see Figure 4 again). By the convexity of C, they lie all 3 outside C. But then $L_1 \cup L_2 \cup L_3$ disconnects G, and d and e cannot be 8-connected in C, a contradiction. ∎

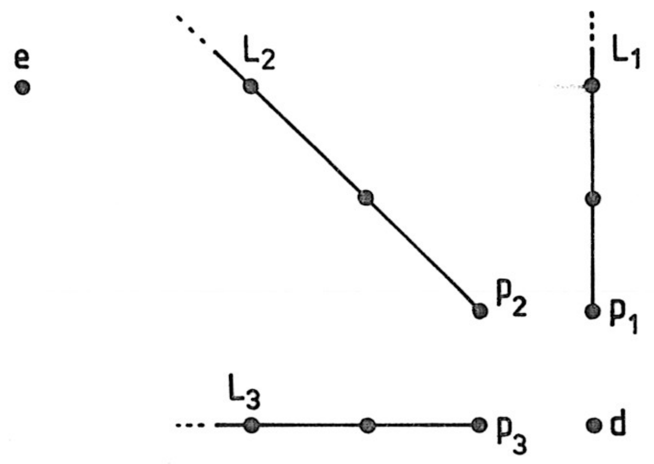


Figure 4.

Proof of Theorem 4. Given a finite set $S = \{p_1, \ldots, p_m\}$ of pixels, we write $R(S)$ or $R(p_1, \ldots, p_m)$ for the smallest rectangle with horizontal and vertical sides containing S. If the coordinates of the elements of S have minima $imin$ and $jmin$, and maxima $imax$ and $jmax$, then $R(S)$ is the intersection of $H(-1, 0, -imin)$, $H(0, -1, -jmin)$, $H(1, 0, imax)$ and $H(0, 1, jmax)$. Thus by Lemma 5, if $S \subseteq P$, then $R(S) \cap P$ is 8-connected and L-convex.

Suppose that the result is false. There is thus a triple $\{p_1, p_2, p_3\} \subseteq P$ such that $\langle p_1, p_2, p_3 \rangle$ contains a pixel $q \in G - P$. Such a triple will be called a *C-triple*. A C-triple T such that $R(T)$ has minimum possible area is called a *CC-triple*.

Now the proof is divided into 4 steps:

Step 1. In a CC-triple T, two pixels are opposite corners of $R(T)$. In other words there is a pair $T' \subseteq T$ such that $R(T) = R(T')$.

Suppose indeed that it is false. Then either T contains two adjacent corners of $R(T)$ or only one corner of $R(T)$. The two cases are illustrated in Figures 5 (a) and (b) respectively. In both we construct the pixel p as shown on the figure.

It is clear that $p \in P$ in (a) (by L-convexity). Suppose that in (b) $p \notin P$. Then by L-convexity, the two half-lines L and L' built by extending the lines joining p from the two pixels x and y (see Figure 5 (b) again) contain no pixel in P. But then $L \cup L'$ disconnects G and X cannot be 8-connected to z.

Thus $p \in P$ in both cases. Now $\langle T \rangle = \bigcup_{q,r \in T} \langle p, q, r \rangle$ and so one of the triples $\{p, q, r\}$ is a C-triple, with $R(p, q, r)$ smaller than $R(T)$. Thus T is not a CC-triple, a contradiction.

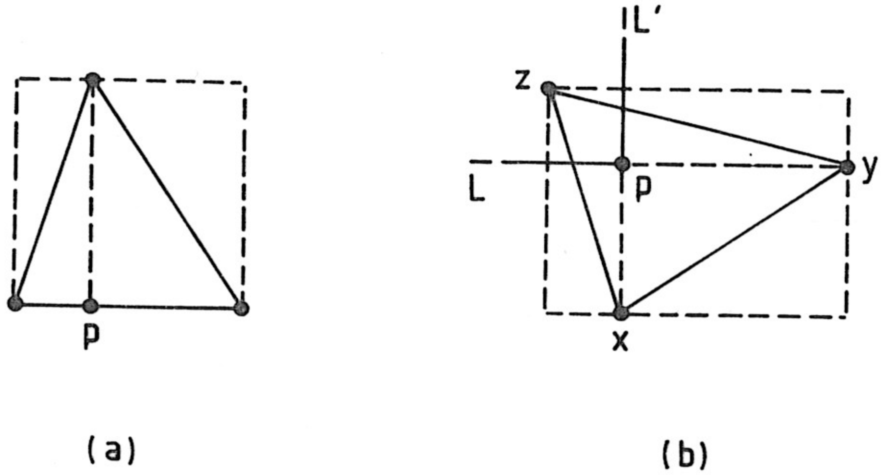


Figure 5.

We take a CC-triple $\{x, y, z\}$, where y and z are opposite corners of $R = R(x, y, z)$. We can assume (by symmetry) that y is the top left corner, z the bottom right corner, that x lies below the diagonal $[y, z]$, and that the height h of R is not larger than its width w. Let u and v be the two other corner pixels of R, lying respectively above and below $[y, z]$. The situation is illustrated in Figure 6.

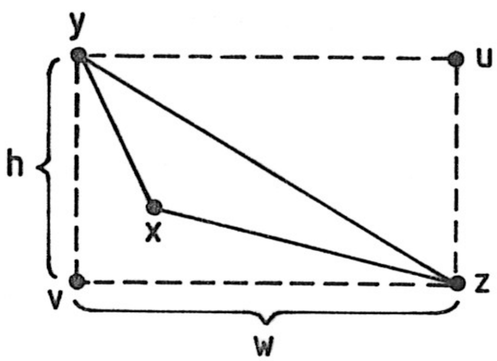


Figure 6.

Step 2. $P \cap \langle y, z, u\rangle = \{y, z\}$.

Suppose indeed that $P \cap \langle y, z, u\rangle$ contains another pixel t; then we have:

$$\langle x, y, z\rangle \subseteq \langle x, y, t\rangle \cup \langle x, z, t\rangle$$

As $\{x, y, z\}$ is a C-triple, one of $\{x, y, t\}$ and $\{x, z, t\}$ is a C-triple, say $\{x, y, t\}$. But then either $R(x, y, t)$ is smaller than R, which contradicts the minimality of R, or $R = R(x, y, t)$ and Step 1 implies that x and t are opposite corners of R, in other words $x = v$ and $t = u$. Thus P contains the 4 corner pixels of R, and L-convexity implies that $R \subseteq P$, which contradicts the fact that $\{x, y, z\}$ is a C-triple.

Thus $P \cap \langle y, z, u \rangle = \{y, z\}$. It follows that $h \neq w$ and so $h < w$.

By Lemma 5, $P \cap R$ is 8-connected, as we explained above. Thus $P \cap \langle y, z, v \rangle$ is 8-connected.

Step 3. Let t be the left neighbor of z (see Figure 7). Then $t \in P$ and $\langle y, z, t \rangle$ contains a pixel of $G - P$.

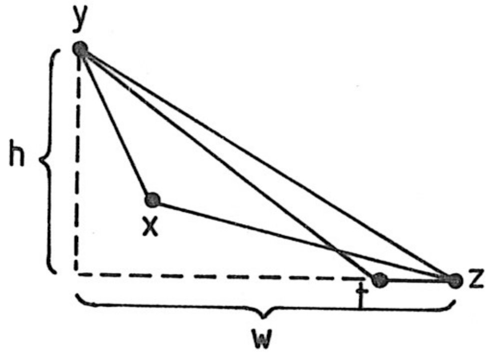


Figure 7.

As $P \cap \langle y, z, v \rangle$ is 8-connected, one of the 8-neighbors of z belongs to it. As $h < w$, this can only be t. If $x \in \langle y, z, t \rangle$, then it is obvious that $\langle y, z, t \rangle$ contains a pixel of $G - P$. If $x \notin \langle x, z, t \rangle$, then $\langle x, y, z \rangle \subseteq \langle y, z, t \rangle \cup \langle x, y, t \rangle$ (see Figure 7 again). If $\langle y, z, t \rangle$ contains no pixel of $G - P$, then $\langle x, y, t \rangle$ is a C-triangle, and $R(x, y, t)$ is smaller than R, which contradicts the minimality of R. Thus $\langle y, z, t \rangle \cap (G - P) \neq \emptyset$.

Step 4. Final contradiction.

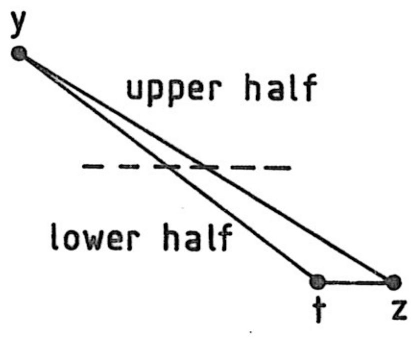


Figure 8.

Let p be the pixel in $\langle y, z, t \rangle \cap (G - P)$ such that the distance between p and $[y, z]$ is maximum. If p is in the upper half of $\langle y, z, t \rangle$ (see Figure 8), then we take the pixel q such that p is the middle of $[y, q]$. As $y \in P$ and $p \notin P$, $q \notin P$ by convexity. But $q \in \langle y, z, t \rangle$ (since p is in the upper half of it) and q lies at double distance from $[y, z]$ than p, a contradiction. Thus p is in the lower half of $\langle y, z, t \rangle$.

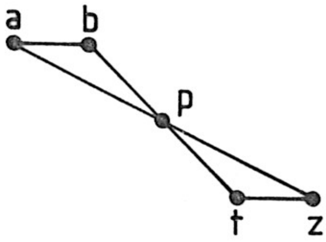


Figure 9.

Now take $a, b \in G$ such that p is the middle of $[b, t]$ and of $[a, z]$ (see Figure 9).

As $t, z \in P$ and $p \notin P$, L-convexity implies that $a, b \notin P$. As t is the left neighbor of z, a is the left neighbor of b (since $abzt$ is a parallelogram). Now a lies at double distance from $[y, z]$ than p; thus $a \notin \langle y, z, t\rangle$. As p is in the lower half of $\langle y, z, t\rangle$, a and b lie below the line yu. As p lies to the right of $[y, t]$, b lies also to the right of it, and as $a \notin \langle y, z, t\rangle$, a lies to the left of $[y, t]$. Thus $[a, b]$ crosses $[y, t]$, and it is clear that $a \in \langle y, z, v\rangle$.

As we said above, $P \cap \langle y, z, v\rangle$ is 8-connected. There is thus a pixel $c \in P \cap \langle y, z, v\rangle$ such that c lies on the horizontal line L passing through a and b. As $d(t, z) = 1$, $L \cap [y, z, t]$ has width at most 1, and as b lies to the right of $[y, t], L \cap \langle y, z, v\rangle$ contains no pixel to the right of b. Thus c lies to the left of a. The situation is illustrated in Figure 10.

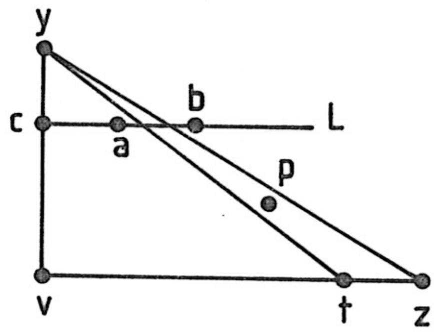


Figure 10.

Hence $a \in \langle c, y, t\rangle$, and so $\{c, y, t\}$ is a C-triple and $R(c, y, t)$ is smaller than R, a contradiction. ∎

In the real plane, convex figures are connected. This is not the case with L- or T-convexity. A L- or T-convex subset of G can be 8-disconnected, or even 8-connected but 4-disconnected. However we have some relation between L-convexity and connectivity:

Proposition 6. *Let P be a L-convex subset of G, and $k = 4$ or 8. Then any k-connected component of P is L-convex (and even T-convex, by Theorem 4).*

Proof. Let X be a k-connected component of X, let $p, q \in X$ and $r \in \langle p, q\rangle - \{p, q\}$.

Let L be the vertical line passing through r. Then L disconnects G, and as p and q lie on opposite sides of L, p and q are not k-connected in $G - L$. As p and q are k-connected in X, this means that there is a pixel $t \in X \cap L$. As P is L-convex and $r \in P$, $\langle t, r \rangle \subseteq P$. As $\langle t, r \rangle$ is a 4-connected vertical run, t and r lie in the same k-connected component, in other words $t \in X$. Hence X is L-convex. ∎

Taking into account Theorem 4 and Proposition 6, we can state the following:

Definition 2. Let $P \subseteq G$, and take $k = 4$ or 8. Then we say that P is k-convex iff P is k-connected and L-convex.

Clearly, 4-convexity implies 8-convexity, which implies T-convexity, which implies L-convexity.

III.2. Definitions of digital convex hull

The properties of L- and T-convexity are preserved by intersection; this leads thus to the definition of the *L- or T-convex hull* of a set P of pixels as the intersection of all L- or T-convex sets of pixels containing it.

The T-convex hull is the easiest to characterize. For $P, Q \subseteq G$ we have trivially:

$$P \subseteq \langle P \rangle. \tag{10}$$

$$\text{If } P \subseteq Q, \text{ then } \langle P \rangle \subseteq \langle Q \rangle. \tag{11}$$

We can also show the following:

$$\langle \langle P \rangle \rangle = \langle P \rangle. \tag{12}$$

Indeed, $\langle P \rangle \subseteq \langle \langle P \rangle \rangle$ by (10). By the definition of $\langle \rangle$, $\langle P \rangle \subseteq [P]$ and similarly $\langle \langle P \rangle \rangle \subseteq [\langle P \rangle]$. But then

$$\langle \langle P \rangle \rangle \subseteq [\langle P \rangle] \subseteq [[P]] \quad (\text{since } \langle P \rangle \subseteq [P]),$$

$$\langle \langle P \rangle \rangle \subseteq [P] \quad (\text{since } [[P]] = [P]).$$

Thus $\langle \langle P \rangle \rangle \subseteq [P] \cap G = \langle P \rangle$, and so (12) holds.

This means that $\langle P \rangle$ is T-convex, and so $\langle P \rangle$ is the T-convex hull of P.

The L-convex hull is more difficult to build. We define $\mathcal{L}(P)$ as follows:

$$\mathcal{L}(P) = \bigcup_{p,q \in P} \langle p, q \rangle. \tag{13}$$

Then for $m \geq 1$ we can define $\mathcal{L}^m(P)$ iteratively as follows:

$$\begin{aligned} \mathcal{L}^m(P) &= \mathcal{L}(P) \quad \text{if } m = 1; \\ &= \mathcal{L}\left(\mathcal{L}^{m-1}(P)\right) \quad \text{if } m > 1. \end{aligned} \tag{14}$$

Then the L-convex hull of P is the set

$$\langle P\rangle_L = \bigcup_{m=1}^{\infty} \mathcal{L}^m(P). \tag{15}$$

When P is finite, then there is some m such that $\mathcal{L}^{m+1}(P) = \mathcal{L}^m(P)$ and so $\langle P\rangle_L = \mathcal{L}^m(P)$.

Clearly $\langle P\rangle_L \subseteq \langle P\rangle$. We have equality in some circumstances. For example:

Proposition 7. *If P is 8-connected, then $\langle P\rangle_L = \langle P\rangle$ and $\langle P\rangle$ is 8-connected.*

Proof. As P is 8-connected, there is an 8-connected component Y of $\langle P\rangle_L$ such that $P \subseteq Y$. By Proposition 6, Y is L-convex. By Theorem 4, Y is T-convex, and so $\langle P\rangle \subseteq Y \subseteq \langle P\rangle_L$. As $\langle P\rangle_L \subseteq \langle P\rangle$, $\langle P\rangle_L = \langle P\rangle = Y$, and so $\langle P\rangle$ is 8-connected. ∎

One should note also that when $\langle P\rangle_L$ is 8-connected, $\langle P\rangle = \langle P\rangle_L$ by Theorem 4.

One can wonder whether the intersection of all 8- (or 4-) convex sets of pixels containing P would give another type of convex hull. This is not the case:

Proposition 8. *Let $P \subseteq G$ and $k = 4$ or 8. Then $\langle P\rangle$ is the intersection of all k-convex sets containing P.*

Proof. For any k-convex set X such that $P \subseteq X$, $\langle P\rangle \subseteq X$. Let P^* be the intersection of all such sets X. Then $\langle P\rangle \subseteq P^*$.

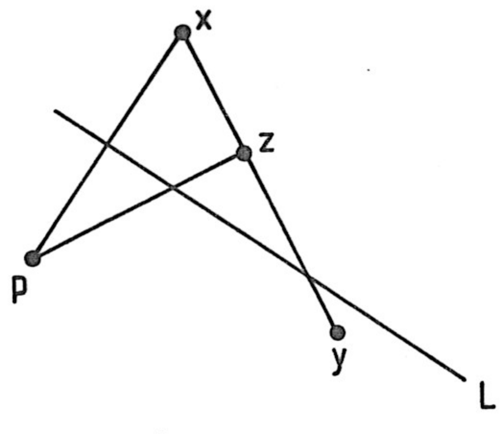


Figure 11.

Consider now a pixel $p \notin \langle P\rangle$. As $[P]$ is topologically closed, there is a point $x \in [P]$ for which the Euclidean distance $d(p,x)$ is minimum. As $p \notin [P]$, $p \neq x$. Let L be the mediatrix of $[p,x]$ (i.e., the line perpendicular to $[p,x]$ and passing through the middle of it). Then every $y \in [P]$ lies on the same side of L as x. Suppose indeed that y lies on the same side of L as p. Then the angle $\angle pxy$ is acute, and as $d(p,y) \geq d(p,x)$, the angle $\angle pyx$ is not larger than $\angle pxy$, and so it is also acute. Hence the line perpendicular to the line xy drawn from p intersects $[x, y]$ in a point z (see Figure 11). But then $z \in [P]$ (since $x, y \in [P]$) and $d(p, z) < d(p, x)$ a contradiction.

Thus every pixel of P lies on the other side of L than p. Let H be the set of all pixels of G lying on the same side of L as P. Then H is k-convex and contains P. Thus $P^* \subseteq H$. Moreover $p \notin H$ and so $p \notin P^*$. As p was chosen arbitrarily, $P^* \subseteq \langle P \rangle$. Thus $P^* = \langle P \rangle$. ∎

Note. There are other ways to prove the existence of a line L separating p from P, notably with the use of Helly's First Theorem (Proposition 2).

III.3. Other relations between digital convexity and connectedness

The previous two sections gave us already some relations between 4- and 8-connectivity on the one hand, and L- and T-convexity on the other hand.

In Lemma 5, we mentioned the sets $H(\alpha, \beta, n)$ containing all pixels located on one side of a line along the horizontal, vertical, or one of the two diagonal directions. These directions can be called *fundamental*, and so we will call such set $H(\alpha, \beta, n)$ a *fundamental half-grid*. Then Lemma 5 can be stated as follows:

Lemma 5. *Let P be an 8-convex subset of G and H a fundamental hald-grid. Then $P \cap H$ is empty or 8-convex.*

Now the horizontal and vertical directions can be called *axial*, and they determine the fundamental half-grids $H(\alpha, \beta, n)$ for which $\alpha\beta = 0$. Such a half-grid is called an *axial half-grid*. The following result can be proved in essentially the same way as Lemma 5:

Lemma 9. *Let P be a 4-convex subset of G and H an axial half-grid. Then $P \cap H$ is empty or 4-convex.*

It seems that one can prove a converse for these two results, namely that for $H \subseteq G$:

(*i*) H is an intersection of fundamental half-planes iff for every 8-convex $P \subseteq G$, $P \cap H$ is empty or 8-convex.

(*ii*) H is an intersection of axial half-planes iff for every 4-convex $P \subseteq G$, $P \cap H$ is empty or 4-convex.

Now L-convexity is related to *simple k-connectedness* ($k = 4$ or 8). We say that a subset S of G is *unbounded* if it intersects the border of G (when G is finite) or if it is infinite (when G is infinite). Otherwise it is *bounded*.

For $k = 4, 8$ and $k' = 12 - k$, one says that a subset X of G is *simply k-connected* if X is k-connected and $G - X$ has no bounded k'-connected component. Note that for every bounded subset Y of G, $G - Y$ has exactly one unbounded k'-connected component.

The following result is trivial.

Lemma 10. *Let P be an L-convex subset of G. Then every 4-connected component* *of $G - P$ is unbounded. Moreover, if P is bounded, then $G - P$ is 4-connected.*

This is due to the fact that for every $p \in G - P$, if L is the set of all pixels along the horizontal line through p, then $L \cap (G - P)$ is unbounded on at least one side of p.

Thus an 8-convex set P is simply 8-connected.

IV. Digital convexity and straight line segments

In the Euclidean space, the relation between straight line segments and convexity is evident. In the case of digital images, this relation exists also, but it is more complicated. Many papers have studied digital straight line segments independently from convexity [4,5,8–11,28,34,35].

Rosenfeld and Kim [16,17,20,21,26] have investigated the relations between digital convexity and straight line segments, showing in particular that a simple k-connected path ($k = 4$ or 8) is L-convex iff it is a quantization of a straight line segment of the plane (we will give more details on the type of quantization corresponding to $k = 4$ and $k = 8$ respectively).

In this Chapter, we will recall the concept of the *chord property* introduced by Rosenfeld [26] and introduce a particularization of it for the 4-connected case, the *strong chord property*. We will also prove the main results obtained by Kim and Rosenfeld on the relations between the chord property, k-convexity, and straight line segments. Some of our our proofs will be simpler than the original ones, thanks to the use of Santaló's Theorem (Proposition 3).

IV.1. The chord property and the strong chord property

The *chord property* was defined in the context of a study of digital straight line segments [26]; it can in fact be considered as another definition of digital convexity, as we will see later. We could thus have introduced it in Section III.1, but we have preferred to introduce it only here, because it is mainly used in relation with straight line segments.

Definition 3 [26]. Let $S \subseteq G$. Then we say that S has the *chord property* iff for every pixels $p, q \in S$ and every point $x \in [p, q]$, there is some pixel $r \in S$ such that $d_8(x, r) < 1$.

Kim [16] showed that when S is 8-connected, it is L-convex iff it has the chord property. In fact, we will prove that S has the chord property iff S is 8-convex. This equivalence led us to search for a specialization of the chord property which would be equivalent to 4-convexity. We found the following:

Definition 4. Let $S \subseteq G$. Then we say that S has the *strong chord property* iff for every two distinct pixels $p, q \in S$ and every point $x \in [p, q]$, there exist two distinct pixels $r, s \in S$ such that $d_4(r, s) = 1$ and $d_8(x, r) + d_8(x, s) < 2$.

In this definition we require that p and q are distinct in order to ensure that a set S of size 1 will have the strong chord property. We will show that S has the strong chord property iff it is 4-convex. This definition and this result are original.

From Definitions 3 and 4 it seems that in order to verify if a set satisfies the chord property or the strong chord property, one must make an infinite number of verifications, namely measuring for each $x \in [p, q]$ the distance $d_8(x, r)$ for pixels $r \in S$ in the vicinity of x. This is not the case, we will show that one can restrict the verification to a finite number of points x lying in the intersection of $[p,q]$ with certain lines drawn between the pixels of the grid. Let us describe here these lines; they belong to two classes:

First, the *axial grid lines*, which consist in the *horizontal* and *vertical grid lines* $\mathcal{H}_i$ and $\mathcal{V}_i$ respectively (where i is an integer), defined as follows:

$$\begin{aligned} \mathcal{H}_i &= \{(x,y) \in \mathbf{R}^2 \mid x = i\}; \\ \mathcal{V}_i &= \{(x,y) \in \mathbf{R}^2 \mid y = i\}. \end{aligned} \tag{16}$$

Second, the *diagonal grid lines*, which consist in the *principal* and *secondary diagonal grid lines* $\mathcal{P}_i$ and $\mathcal{S}_i$ respectively (where i is an integer), defined as follows:

$$\begin{aligned} \mathcal{P}_i &= \{(x,y) \in \mathbf{R}^2 \mid x - y = i\}; \\ \mathcal{S}_i &= \{(x,y) \in \mathbf{R}^2 \mid x + y = i\}. \end{aligned} \tag{17}$$

These lines are the lines along the fundamental directions (horizontal, vertical, principal and secondary diagonal) drawn between the pixels of G. We will show that in order to verify the chord property (*resp.* strong chord property) for a set S, we will only have to look at the intersections of the segments $[p,q]$ $(p,q \in S)$ with axial (*resp.* diagonal) grid lines.

We need first the following lemma. But let us introduce beforehand some further notation. Given two distinct points x and y, we write $]x,y[$ for the open line segment joining them; in other words $]x,y[= [x,y] - \{x,y\}$.

Lemma 11. *Let $S \subseteq G$, $k = 4$ or 8 and $k' = 12 - k$. If S is k-convex, $p,q \in S$, a and b are two k'-adjacent pixels of G and $]p,q[$ intersects $]a,b[$ in a single point x, then a or $b \in S$.*

Proof. Let $\mathcal{L}$ be the line drawn through a and b and let $M = \mathcal{L} \cap G$. Then p and q lie on opposite sides of M. As M is a k'-connected line, it is easily seen that the two sides of M are not k-connected in $G - M$. But as S is k-connected and S intersects both sides of M, $S \cap M \neq \emptyset$. Let $z \in S \cap M$. If $z \in [a,b]$, then $z = a$ or b and the result holds. If $z \notin [a,b]$, then $[x,z]$ contains a or b. Suppose that it is a. Then $a \in [x,z] \subseteq [p,q,z]$ (since $x \in [p,q]$), and so Theorem 4 implies that $a \in S$. ∎

We can now give a characterization of the chord property and the strong chord property, which also indicates for which points x in each $[p,q]$ $(p,q \in S)$ one must find some $r \in S$ with $d_8(x,r) < 1$.

Theorem 12. *Let $S \subseteq G$. Then the following three statements are equivalent:*

(i) *S has the chord property.*

(ii) *S is 8-convex.*

(iii) *For every two distinct $p,q \in S$ and every axial grid line $\mathcal{L}$ such that $]p,q[\cap\mathcal{L}$ contains exactly one point x, there exists some pixel $r \in S \cap \mathcal{L}$ such that $d_8(x,r) < 1$.*

Proof. The result is trivial if $|S| = 1$, and so we can assume that $|S| \geq 2$.

(1°) (i) implies (ii).

Let $p, q \in S$ and $r \in \langle p, q\rangle$. As for any pixel s of G we have $d_8(r, s) \geq 1$, we must have $r \in S$. Thus S is L-convex. It remains to show that S is 8-connected. Suppose false. Then S is the disjoint union of two sets U and V such that no pixel of U is 8-adjacent to a pixel of V. Take $u \in U$, $v \in V$, and define the two sets:

$$U^* = \{x \in [u, v] \mid d_8(x, U) < 1\},$$
$$V^* = \{x \in [u, v] \mid d_8(x, V) < 1\}.$$

Then clearly $U^* \neq \emptyset \neq V^*$ and the chord property implies that $[u, v] = U^* \cup V^*$. Now U^* must have a limit point y. This means that for every $\epsilon > 0$, there exist $c \in U^*$ and $d \in V^*$ such that $d_8(y, c) < \epsilon$ and $d_8(y, d) < \epsilon$. We can assume that $y \in U^*$. Then $d_8(y, U) < 1$. Let $\eta = 1 - d_8(y, U)$. There is some $z \in V^*$ such that $d_8(y, z) < \eta$. Then $d_8(z, V) < 1$ and so we have

$$d_8(U, V) \leq d_8(U, y) + d_8(y, z) + d_8(z, V) < 1 - \eta + \eta + 1 = 2,$$

which means that $d_8(U, V) = 1$, in other words that U is 8-adjacent to V, a contradiction.

(2°) (ii) implies (iii).

If $]p, q[\cap \mathcal{L} = \{x\}$, then there are two 8-adjacent pixels a and b of $\mathcal{L}$ such that $x \in [a, b]$. If $x = a$, then $a \in S$ by L-convexity, and obviously $d_8(x, a) < 1$. The same holds for $x = b$. If $x \in]a, b[$, then Lemma 11 implies that a or $b \in S$, and clearly $d_8(x, a) < 1$ and $d_8(x, b) < 1$.

(3°) (iii) implies (i).

Let $p, q \in S$ and $x \in [p, q]$. We must show that there is some $r \in S$ such that $d_8(x, r) < 1$. This is obviously true if $x = p$ or q, since $p, q \in S$ and $d_8(p, p) = d_8(q, q) = 0$. This is also true by hypothesis if $x \in]p, q[$ and $\mathcal{L} \cap]p, q[= \{x\}$ for some axial grid line $\mathcal{L}$. We can thus assume that $x \in]p, q[$ and that one of the following two statements is true:

(a) There is an axial grid line $\mathcal{L}$ such that $]p, q[\cap \mathcal{L}$ contains x, but also another point y.

(b) $x \notin]p, q[\cap \mathcal{L}$ for any axial grid line $\mathcal{L}$.

In case (a) it is clear that $[p, q] \subset \mathcal{L}$. There is some pixel $r \in [p, q]$ such that $d_8(x, r) < 1$. Now there is another axial grid line $\mathcal{L}'$, perpendicular to $\mathcal{L}$, intersecting it in r only. Then (by hypothesis) there is some pixel $r' \in S$ such that $d_8(r, r') < 1$, in other words $r \in S$.

In case (b) there are four 8-adjacent pixels $a, b, c, d \in G$ forming a 2×2 square such that x lies in the interior of $[a, b, c, d]$. Then $d_8(x, y) < 1$ for every point $y \in [a, b, c, d]$. We can assume that $d_8(x, p) \geq 1$, otherwise we can choose $r = p$, and similarly $d_8(x, q) \geq 1$. Thus p and q lie outside $[a, b, c, d]$ and so $]p, q[$ intersects one of the border segments of $[a, b, c, d]$, say $[a, b]$, in a single point y. Applying the hypothesis to y, given the axial grid line $\mathcal{L}$ containing $[a, b]$, there is some $r \in \mathcal{L} \cap S$ such that $d_8(y, r) < 1$. This means that $r = a$ or b, but then then $d_8(x, r) < 1$.

Thus in each case there is some $r \in S$ such that $d_8(x, r) < 1$, and so S has the chord property. ∎

Theorem 13. *Let $S \subseteq G$. Then the following three statements are equivalent:*

(i) *S has the strong chord property.*

(ii) *S is 4-convex.*

(iii) *For every two distinct $p, q \in S$ and every diagonal grid line $\mathcal{L}$ such that $]p,q[\cap\mathcal{L}$ contains exactly one point x, there exist some pixel $r \in S \cap \mathcal{L}$ such that $d_8(x,r) < 1$.*

Proof. The result is trivial if $|S| = 1$, and so we can assume that $|S| \geq 2$.

(1°) (i) implies (ii).
As the inequality $d_8(x, r) + d_8(x, s) < 2$ implies that $d_8(x, r) < 1$ or $d_8(x, s) < 1$, the strong chord property implies the chord property. Thus S is 8-convex by Theorem 12. It remains thus to prove that S is 4-connected. Suppose false. Then S is the disjoint union of two sets sets U and V such that no pixel of U is 4-adjacent to a pixel of V. As S is 8-connected, there exist $u \in U$ and $v \in V$ which are 8-adjacent. Let c and d be the two pixels which are 4-adjacent to both u and v (thus u, v, c and d form a 2×2-square). Then $c, d \notin S$, otherwise U would be 4-connected to V. Let x be the middle of $[u, v]$. By the strong chord property, there exist $r, s \in S$ such that $d_4(r, s) = 1$ and $d_8(x, r) + d_8(x, s) < 2$. But $d_8(x, u) = d_8(x, v) = d_8(x, c) = d_8(x, d) = 1/2$, while $d_8(x, t) \geq 3/2$ for $t \in G - \{u, v, c, d\}$. Thus $r, s \in \{u, v, c, d\}$, and as r is 4-adjacent to s, one of r and s is equal to c or d, which contradicts the fact that $c, d \notin S$. Thus S is 4-connected, and so it is 4-convex.

(2°) (ii) implies (iii).
The proof is the same as for the point (2°) in the proof of Theorem 12.

(3°) (iii) implies (i).
For any $u, v \in S$ and two diagonally adjacent $a, b \in G$, if $[u, v]$ intersects $[a, b]$, then a or $b \in S$. Indeed, let y be a point in this intersection. If $y = u$, then $u = a$ or b, and similarly for v. If $y \in]u, v[$, then this follows from the hypothesis. Moreover, if $y = a$, then $a \in S$. Let us call this property the *intersection argument.*

Let p, q be two distinct pixels of S; we must show that for every $x \in [p, q]$ there exist two pixels $r, s \in S$ such that $d_8(x, r) + d_8(x, s) < 2$ and $d_4(r, s) = 1$. This is obvious if p and q are 4-adjacent, for we can take $r = p$ and $s = q$ in this case. We can thus assume that q is not 4-djacent to p.

Consider the decomposition of the plane in triangles by the grid lines. Then we have the situation shown in Figure 12, where $x \in [b, e, y]$.

As $x \in [b, e, y] \cap [p, q]$, $[p, q]$ must intersect at least two sides of $[b, e, y]$, and so it must intersect at least one of the two diagonals $[b, d]$ and $[a, e]$. We can assume that it intersects $[b, d]$. We have two cases:

(a) $[p, q]$ intersects $[a, e]$ too.

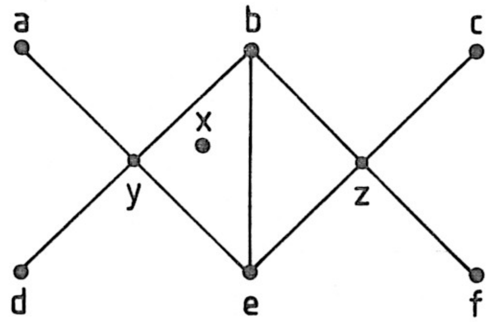


Figure 12.

Then the intersection argument implies that S contains two pixels $g \in \{a, e\}$ and $h \in \{b, d\}$. Clearly g and h are 4-adjacent. Now if $x \notin [b, e]$, then x lies in the interior of $[a, b, d, e]$ and so $d_8(x, g) < 1$ and $d_8(x, h) < 1$. In this case the result holds with $r = g$ and $s = h$.

We can thus assume that $x \in [b, e]$. If $b \in S$, then either $x = e$ and the result holds with $r = e$ and $s = b$, or $d_8(x, b) < 1$ and so the result holds with $r = b$ and $s = g \in \{a, e\}$. A similar argument holds if $e \in S$. We can thus assume that $b, e \notin S$. Now $x \in [b, e, z]$, and the same argument as the one in the preceding paragraph implies that $[p, q]$ must intersect one of the diagonals $[b, f]$ and $[c, e]$, say $[b, f]$. Again the intersection argument implies that b or $f \in S$. As $b \notin S$, $f \in S$. As $d, f \in S$ and $[d, f]$ intersects $[a, e]$ in e, $e \in S$, a contradiction.

(b) $[p, q]$ does not intersect $[a, e]$.
As $x \in [a, e, c]$ and $d_4(x, y) \geq 2$, $[p, q]$ must intersect two sides of $[a, e, c]$, and so it must intersect $[c, e]$. The intersection argument implies that S contains two pixels $g \in \{b, d\}$ and $h \in \{c, e\}$. We have four cases:

— $b, e \in S$. Then the result holds with $r = b$ and $s = e$.

— $b \notin S$ and $e \in S$. Then $d \in S$. The intersection argument applied to $[p, q]$ and $[b, d]$ implies that $x \neq b$. Thus the result holds with $r = d$ and $s = e$.

— $b \in S$ and $e \notin S$. Then $c \in S$. Let $\alpha = d(x, [a, b])$ and $\beta = d(x, [b, e])$, where d is the Euclidean distance. As $x \in [b, d, e]$,

$$\alpha \geq \beta. \tag{18}$$

As $x \in [a, b, e]$ and $x \notin [a, e]$ (since $[p, q]$ does not intersect $[a, e]$),

$$\alpha + \beta < 1. \tag{19}$$

Now $d_8(x, b) = max(\alpha, \beta)$ and $d_8(x, c) = max(\alpha, 1 + \beta)$. But then

$$\begin{aligned} d_8(x, b) + d_8(x, c) &= max(\alpha, \beta) + max(\alpha, 1 + \beta), \\ &= \alpha + max(\alpha, 1 + \beta) \quad \text{(by (18))}, \\ &= \alpha + 1 + \beta \quad \text{(by (19))}, \\ &< 2 \quad \text{(by (19))}. \end{aligned}$$

Hence the result holds with $r = b$ and $s = c$.

— $b, e \notin S$. Then $c, d \in S$. We can apply the intersection argument with $[d, c]$ and $[a, e]$, and so $a \in S$ (since $e \notin S$). But then we apply the intersection argument with $[a, c]$ and $[b, d]$, and so $b \in S$, a contradiction.

Thus we can find r and s with $d_8(x, r) + d_8(x, s) < 2$ in each possible case, and so S has the strong chord property. ∎

Note that the statements (iii) in Theorems 12 and 13 are very similar; in particular for the strong chord property, this form (iii) requires only a pixel r at 8-distance < 1 from x, and not two 4-adjacent pixels r and s with $d_8(x, r) + d_8(x, s) < 2$, as in Definition 4.

The next two sections will exhibit the relations between the digitization of straight line segments and the chord/strong chord property.

IV.2. Grid-intersect quantization of straight lines and the chord property

The *grid-intersect quantization* was introduced by Freeman [10] for the digitization of curves. It associates 8-connected sets of pixels to connected sets of points in the plane.

Rosenfeld [26] studied the grid-intersect quantization of straight line segments, calling them *digital straight line segments*. He showed that a simple 8-connected path having the chord property is a digital straight line segment and conversely. Then Kim and Rosenfeld [17,21] showed that for every set S of pixels having the chord property and for every two pixels $a, b \in S$, S contains a digital straight line segment D whose extremities are a and b.

We will give here a simplified proof of these two results, using the Tranversal Theorem of Santaló (Proposition 3). But let us first recall the definition of a simple k-connected path and of the grid-intersect quantization.

For $k = 4$ or 8, a simple k-connected path is a path $\{p_0, \ldots, p_n\}$ such that for $0 \leq i < j \leq n$, the pixels p_i and p_j are k-adjacent if and only if $j = i + 1$.

Given a continuous and orientable curve C in $\mathbf{R}^2$, its *grid-intersect quantization* is a set $Q(C)$ of pixels built as follows: whenever C intersects an axial grid line ($\mathcal{H}_i$ or $\mathcal{V}_i$, see (16)), the pixel on that grid line closest to this intersection is taken as a member of $Q(C)$. Thus the pixel (u, v) belongs to $Q(C)$ if $\mathcal{H}_u \cap C$ contains a point (u, y) with $|y - v| < 1/2$ or if $\mathcal{V}_v \cap C$ contains a point (x, v) with $|x - u| < 1/2$. When C intersects an axial grid line exactly in the middle of two 4-adjacent pixels p and q, one must choose one of them as quantization pixel; the best is to choose the one which is locally to the left of C w.r.t. its orientation.

When C is connected, its grid-intersect quantization $Q(C)$ is 8-connected, because two successive intersections of C with axial grid lines lie in the border of a square $[a, b, c, d]$ formed by pairwise 8-adjacent pixels a, b, c, d, and so the corresponding quantization points are in $\{a, b, c, d\}$. Moreover, when following C along its orientation, these quantization points form an 8-connected path. We illustrate this in Figure 13, where the quantization pixels are represented as black squares and numbered in their order of occurrence when following C.

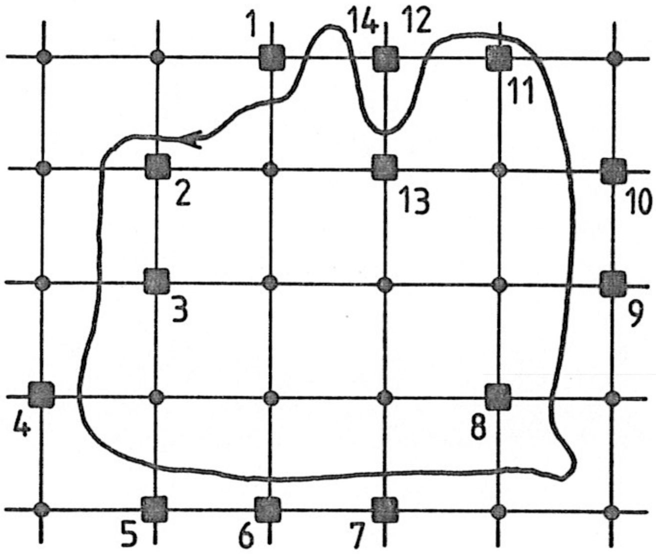
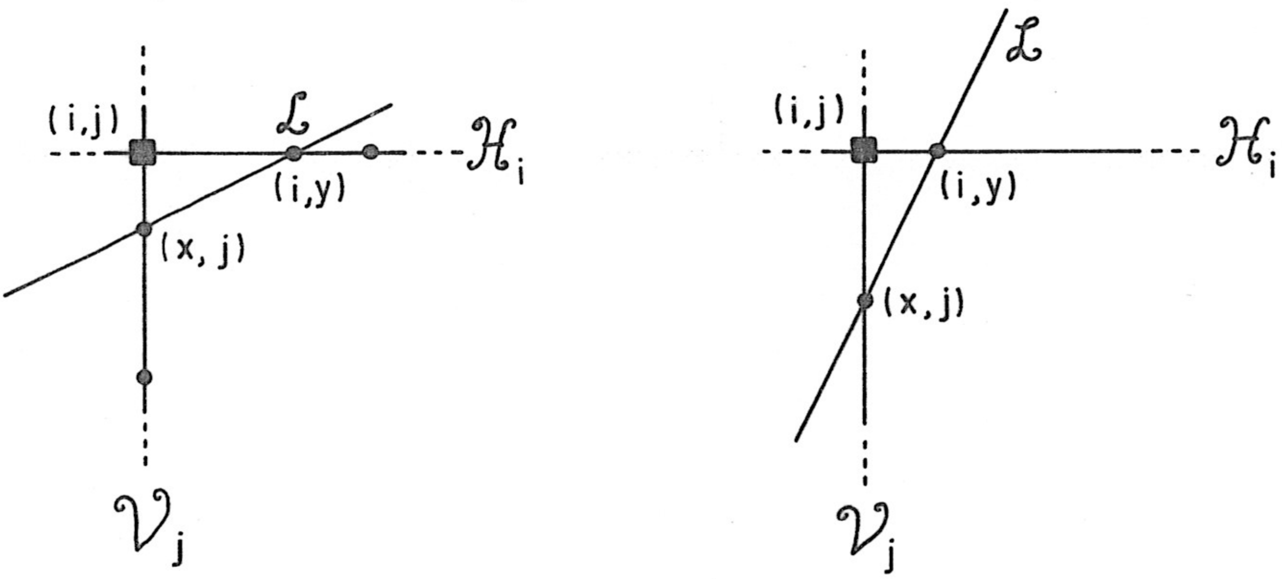


Figure 13. *Grid-intersect quantization of a curve*



(a) (b)

Figure 14.

Consider now a straight line $\mathcal{L}$. We will study $Q(\mathcal{L})$ and the sets $Q(C)$, where C is any segment of $\mathcal{L}$. As remarked by Rosenfeld [26], $Q(\mathcal{L})$ is determined only by the intersections $\mathcal{L} \cap \mathcal{V}_j$ with vertical grid lines when the orientation of $\mathcal{L}$ is closer to the horizontal than the vertical (*i.e.*, when it forms with the horizontal an angle smaller than 45°). On the other hand, it is determined only by the intersections $\mathcal{L} \cap \mathcal{H}_i$ with horizontal grid lines when the orientation of $\mathcal{L}$ is closer to the vertical than the horizontal (*i.e.*, when it forms with the vertical an angle smaller than 45°). This can be understood from Figure 14. When the orientation of $\mathcal{L}$ is closer to the horizontal, if $\mathcal{L} \cap \mathcal{H}_i$ induces the pixel (i, j) in $Q(\mathcal{L})$, then $\mathcal{L}$ contains a point (i, y) with $|y - j| \leq 1/2$; but then $\mathcal{L} \cap \mathcal{V}_j$ contains a point (x, j) with $|x - i| \leq |y - j|$, and so $\mathcal{L} \cap \mathcal{V}_j$ induces (i, j) in $Q(\mathcal{L})$ (see Figure 14 (a)). When the orientation of $\mathcal{L}$ is closer to the vertical, then a similar argument holds with Figure 14 (b). Note that when $\mathcal{L}$ is parallel to one of the two diagonal directions, then $Q(\mathcal{L})$ is determined by the intersections $\mathcal{L} \cap \mathcal{H}_i$ or $\mathcal{L} \cap \mathcal{V}_j$ indifferently.

The same argument holds for $Q(C)$ when C is a segment of $\mathcal{L}$, except that at the extremities of C the intersection with one axial grid line may be missing (namely $C \cap \mathcal{V}_j$ in Figure 14 (a) and $C \cap \mathcal{H}_i$ in Figure 14 (b)). But then we can extend C into a larger segment C' containing the missing intersections, and $Q(C) = Q(C')$. We can thus assume that C is a closed segment whose extremities lie on horizontal or vertical grid lines according to whether the orientation of C is closer to the vertical or to the horizontal.

This property of the grid-intersect quantization of straight line segments will be useful in the proof of the following two results characterizing digital straight line segments:

Proposition 14 [26]. *Given a straight line segment C, $Q(C)$ is a simple 8-connected path having the chord property.*

Proof. We can assume that the angle between the orientation of C and the horizontal is not larger than 45°, that C is oriented in such a way that its left side is above it, and that the extremities of C are two points (g, s) and (h, t), where s and t are integers and $s < t$. For every integer j such that $s \leq j \leq t$, C intersects $\mathcal{V}_j$ in a single point $q_j = (x_j, j)$ (with $g = x_s$ and $h = x_t$), and this intersection induces in $Q(C)$ a pixel $p_j = (i_j, j)$, where $x_j - 1/2 \leq i_j < x_j + 1/2$. Write $\Delta = \{w \in \mathrm{R} \mid -1/2 \leq w < 1/2\}$ and $\lambda_j = i_j - x_j$. Then $\lambda_j \in \Delta$. Note that for every $\lambda, \lambda' \in \Delta$, $|\lambda - \lambda'| < 1$.

Let us first show that for $s \leq a < b \leq t$, $|i_a - i_b| \leq b - a$. Indeed, $|\lambda_a - \lambda_b| < 1$ (since $\lambda_a, \lambda_b \in \Delta$), and $|x_a - x_b| \leq b - a$ (since D forms an angle of at most 45° with the horizontal). We get thus:

$$|i_a - i_b| = |(\lambda_a - \lambda_b) - (x_a - x_b)| \leq |(\lambda_a - \lambda_b)| + |(x_a - x_b)| < 1 + b - a.$$

As both $b - a$ and $|i_a - i_b|$ are integers, $|i_a - i_b| \leq b - a$.

$Q(C)$ is 8-connected by the properties of the grid-intersect quantization stated above. More precisely, for $s \leq j < t$, the preceding paragraph implies that $|i_{j+1} - i_j| \leq 1$. Thus $d_8(p_j, p_{j+1}) = max(|i_{j+1} - i_j|, 1) = 1$, and so p_j is an 8-neighbor of p_{j+1}. As $d_8(p_j, p_{j'}) \geq j' - j > 1$ for $j' > j + 1$, $Q(C) = \{p_s, \ldots, p_t\}$ is a simple 8-connected path.

Let us now show that $Q(C)$ has the chord property. It is sufficient to prove the statement (iii) of Theorem 12. Consider a, b such that $s \leq a < b \leq t$, and let x be a point in $[p_a, p_b] \cap \mathcal{A}$, where $\mathcal{A}$ is an axial grid line. We must show that $\mathcal{A}$ contains a pixel r such that $d_8(x, r) < 1$. We have two cases:

(a) $\mathcal{A} = \mathcal{V}_j$ for some j. Then $a \leq j \leq b$. As $\lambda_a, \lambda_b \in \Delta$, p_a and p_b belong to the strip consisting of all points which are either at most $1/2$ above C or less than $1/2$ below it. As this strip is convex, x belongs to it, and so $x = (x_j + \lambda, j)$, where $\lambda \in \Delta$. Now $p_j \in Q(C) \cap \mathcal{V}_j$, and as $\lambda, \lambda_j \in \Delta$, $d_8(x, p_j) = |\lambda - \lambda_j| < 1$. Thus we can take $r = p_j$.

(b) $\mathcal{A} = \mathcal{H}_i$ for some i, but $x \notin \mathcal{V}_j$ for any j. Then we have $x = (i, j + \theta)$, where $a \leq j < b$ and $0 < \theta < 1$. We can assume that $i_a \leq i_b$. As $|i_a - i_b| \leq b - a$ (see above), $[p_a, p_b]$ contains two points $x' = (i - \psi, j)$ and $x'' = (i + \phi, j + 1)$, where $0 \leq \psi = \theta(i_b - i_a)/(b - a) \leq \theta < 1$ and $0 \leq \phi = (1 - \theta)(i_b - i_a)/(b - a) \leq 1 - \theta < 1$. Let $r_1 = (i, j)$, $r_2 = (i - 1, j)$, $r_3 = (i, j + 1)$ and $r_4 = (i + 1, j + 1)$. As every point of $\mathcal{V}_i - \{r_1, r_2\}$ is at 8-distance at least 1 from x', (a) implies that $p_i = r_1$ or r_2. The same argument with x'' implies that $p_{i+1} = r_3$ or r_4. If $p_i = r_1$, then the result holds with $r = r_1$, since $r_1 \in \mathcal{H}_i$. If $p_{i+1} = r_3$, then it holds too. If $p_i \neq r_1$ and $p_{i+1} \neq r_3$, then $p_i = r_2$, $p_{i+1} = r_4$, and so $d_8(p_i, p_{i+1}) = 2$, a contradiction.

Thus we can find the appropriate pixel r in each case, and therefore $Q(C)$ has the chord property. ∎

Rosenfeld [26] showed the converse of this result, namely that a simple 8-connected path having the chord property is the grid-intersect quantization of a straight line segment. Rosenfeld's proof was rather long and intricate (much more than that of Proposition 14). This converse will be a consequence of the following result, due to Kim and Rosenfeld [17,21], of which we give here a simple proof based on Santaló's Theorem (Proposition 3):

Theorem 15 [17,21]. *Let S be a set of pixels having the chord property and let $p, q \in S$. Then there is a straight line segment C such that $Q(C) \subseteq S$ and the simple 8-connected path $Q(C)$ has p and q as its extremities.*

Proof. Write $p = (a, u)$ and $q = (b, v)$. By symmetry, we can assume that $u < v$ and $|b - a| \leq v - u$. If the equality holds, then p and q are along a diagonal, and then the chord property implies that $\langle p, q\rangle \subseteq S$. But then the result holds because $\langle p, q\rangle = Q([p, q])$. Thus $|b - a| < v - u$.

Define $T_u = \{p\}$, $T_v = \{q\}$ and for $u < j < v$, $T_j = S \cap \mathcal{V}_j$ (in other words the set of all pixels of S of the form (i, j)). The chord property implies that for $u \leq j \leq v$ there are two integers m_j and M_j such that $m_j \leq M_j$ and T_j is the set of all pixels (i, j) such that $m_j \leq i \leq M_j$. For every pixel $r = (c, d)$, let

$$D(r) = \{(z, d) \mid z \in \mathbf{R} \text{ and } c - 1/2 < z \leq c + 1/2\},$$

and for each U_j, let D_j be the union of all $D(r)$ for $r \in U_j$. Then

$$D_j = \{(z, j) \mid z \in \mathbf{R} \text{ and } m_j - 1/2 < z \leq M_j + 1/2\}.$$

Let us show that $D_u, \ldots, D_v$ satisfy the hypothesis of Proposition 3. Clearly they are parallel convex straight line segments. Consider the three segments D_j, $D_{j'}$ and $D_{j''}$, where

$u \leq j < j' < j'' \leq v$. Take two pixels $r \in D_j$ and $s \in D_{j''}$; then $[r, s]$ intersect $\mathcal{V}_{j'}$ in a point x. By the chord property, there is some pixel $t \in S$ such that $d_8(x, t) < 1$. But then $t \in \mathcal{V}_{j'}$, in other words $t \in T_{j'} = S \cap \mathcal{V}_{j'}$. Let t' be the middle of $[x, t]$ and let $\epsilon = d_8(x, t)/2$. Consider the straight line $\mathcal{L}$ parallel to $[r, s]$ and passing through t' (see Figure 15). Then $\mathcal{L}$ intersects $\mathcal{V}_j$ in a point r' such that $d_8(r, r') = \epsilon$, and similarly it intersects $\mathcal{V}_{j''}$ in a point s' such that $d_8(s, s') = \epsilon$. But as $\epsilon < 1/2$, $r' \in D(r)$, $t' \in D(t)$ and $s' \in D(s)$, in other words $\mathcal{L}$ intersects D_j, $D_{j'}$ and $D_{j''}$.

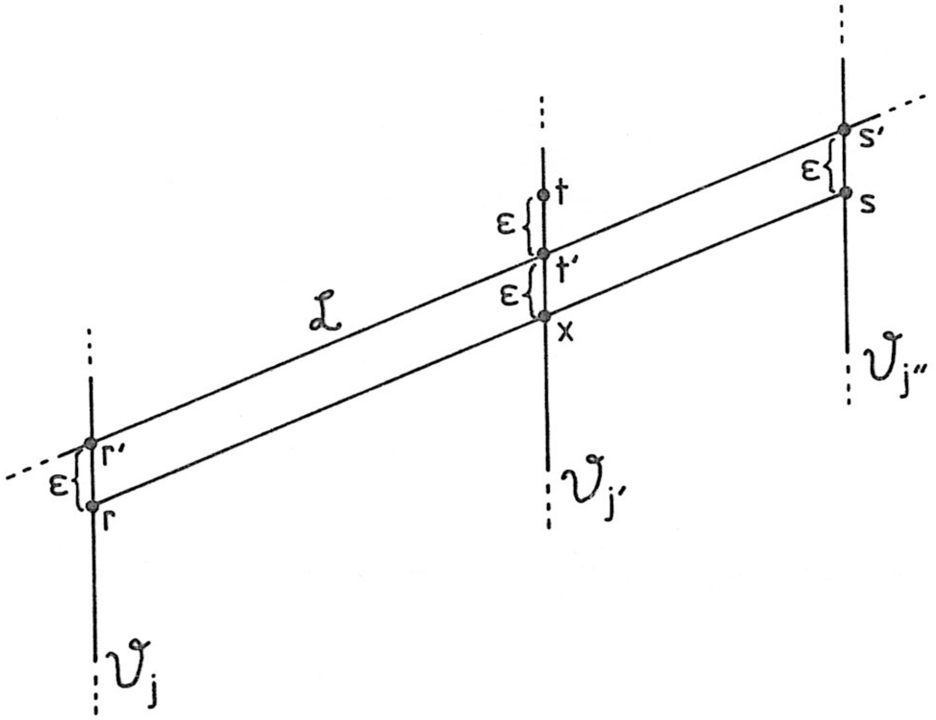


Figure 15.

By Proposition 3, there is a straight line $\mathcal{L}$ intersecting each D_j $(u \leq j \leq v)$. In other words, each T_j contains a pixel p_j such that $\mathcal{L}$ intersects $D(p_j)$. We have of course $p_u = p$ and $p_v = q$. Let p' and q' be the two points of $\mathcal{L}$ in $D(p)$ and $D(q)$ respectively. We have $p' = (a + \lambda, u)$ and $q' = (b + \mu, v)$, where $1/2 < \lambda, \mu \leq 1/2$. Let $C = [p', q']$. The horizontal distance between p' and q' is $v - u$, while the vertical distance between them is $|b + \mu - a - \lambda| \leq |b - a| + |\mu - \lambda| < (u - v - 1) + 1 = u - v$. Thus C forms an angle smaller than 45° with the horizontal, and so $Q(C)$ is determined by the intersections $C \cap \mathcal{V}_j$, $u \leq j \leq v$. Now for $z_j = (x_j, j) \in C \cap \mathcal{V}_j$, the point z_j induces the pixel $t = (i, j) \in G$ iff $x_j - 1/2 \leq i < x_j + 1/2$, in other words iff $i - 1/2 < x_j \leq i + 1/2$, that is $z_j \in D(t)$. Now this is possible only for $t = p_j$ (since the sets $D(t)$ are pairwise disjoint), and so $Q(C)$ is the path $p_u, \ldots, p_v$, which lies in S. As $p_u \in T_u = \{p\}$, $p_u = p$, and similarly $p_v = q$. ∎

As we said above, Theorem 15 implies the converse of Proposition 14, and so we have the following:

Corollary 16. *Let $S \subseteq G$; then the following two statements are equivalent:*

(i) *S is a simple 8-connected path having the chord property.*

(ii) *There is a straight line segment C such that $S = Q(C)$.*

Proof. (ii) implies (i) by Proposition 14. Let us show that (i) implies (ii). Indeed, if S is a simple 8-connected path having the chord property, we take p and q as its extremities. Then Theorem 15 implies that there is a straight line C such that $p, q \in Q(C) \subseteq S$. As $Q(C)$ is 8-connected (by Proposition 14), this means that $Q(C) = S$. ∎

Following Rosenfeld [26], such a set S will be called a *digital straight line segment.*

It is easy to prove the converse of Theorem 15. We get thus the following:

Corollary 17. *Let $S \subseteq G$; then the following two statements are equivalent:*

(i) *S has the chord property.*

(ii) *For every $p, q \in S$, there is a straight line segment C such that $Q(C) \subseteq S$ and the simple 8-connected path $Q(C)$ has p and q as its extremities.*

Proof. (i) implies (ii) by Theorem 15. Let us show that (ii) implies (i). Take any two $p, q \in S$, and let C be the straight line segment such that $p, q \in Q(C) \subseteq S$. Proposition 14 implies that $Q(C)$ has the chord property, and so for every $x \in [p, q]$, there is some $r \in C(Q)$ such that $d_8(x, r) < 1$. But $r \in S$, and as p and q were arbitrarily chosen, S has the chord property. ∎

IV.3. Square box quantization of straight lines and the strong chord property

The *square box quantization* is a digitization scheme used for surfaces. It associates 4-connected sets of pixels to sets of points whose interior is connected.

Kim [16] studied —in essentially the same way as Rosenfeld [26] did for the grid-intersect quantization— the square box quantization of straight line segments. He called such sets *cellular straight line segments.* He showed that a set of pixels is a cellular straight line segment iff it is a simple 4-connected path having the chord property, and that for every 4-connected set S of pixels having the chord property and every two pixels $a, b \in S$, S contains a cellular straight line segment D whose extremities are a and b.

In this section we will link these results to the strong chord property (which is equivalent to 4-connectedness plus the chord property, see Theorems 12 and 13), and prove them with the same methods as in the preceding section.

Let us recall first the definition of the square box quantization. To each pixel p one associates the square box $\overline{p}$ consisting of all points of $\mathbf{R}^2$ whose 8-distance to p is at most $1/2$ (see (8)). The interior $\overline{p}^\circ$ of $\overline{p}$ is the set of points whose 8-distance to p is less than $1/2$. Given a continuous surface $\mathcal{S}$ in $\mathbf{R}^2$, its *square box quantization* is the set $Q^\circ(\mathcal{S})$ of all pixels $p \in G$ such that $\mathcal{C}$ intersects $\overline{p}^\circ$.

For a continuous and orientable curve C, this definition of the square box quantization is incomplete, because a portion of C may traverse the edge between $\overline{p}$ and $\overline{q}$ (where p and q are two 8-adjacent pixels), without intersecting $\overline{p}^{\circ}$ and $\overline{q}^{\circ}$. In this case, we adopt the same remedy as for the grid-intersect quantization, and we choose as member of $Q^{\circ}(S)$ the one of p and q which is locally to the left of C. Thus $Q^{\circ}(C)$ is the set of all pixels $p \in G$ such that either:

(i) C intersects $\overline{p}^{\circ}$; or

(ii) C intersects $\overline{p}$, but not $\overline{p}^{\circ}$, and p lies locally to the left of C w.r.t. its orientation.

When C is connected, $Q^{\circ}(C)$ is 4-connected. Moreover, when following C along its orientation, the quantization points of $Q^{\circ}(C)$ form a 4-connected path, because the successive square boxes $\overline{p}$ traversed by C are 4-adjacent. We illustrate this property in Figure 16, with quantization pixels marked as black squares and numbered in their order of occurrence when following C.

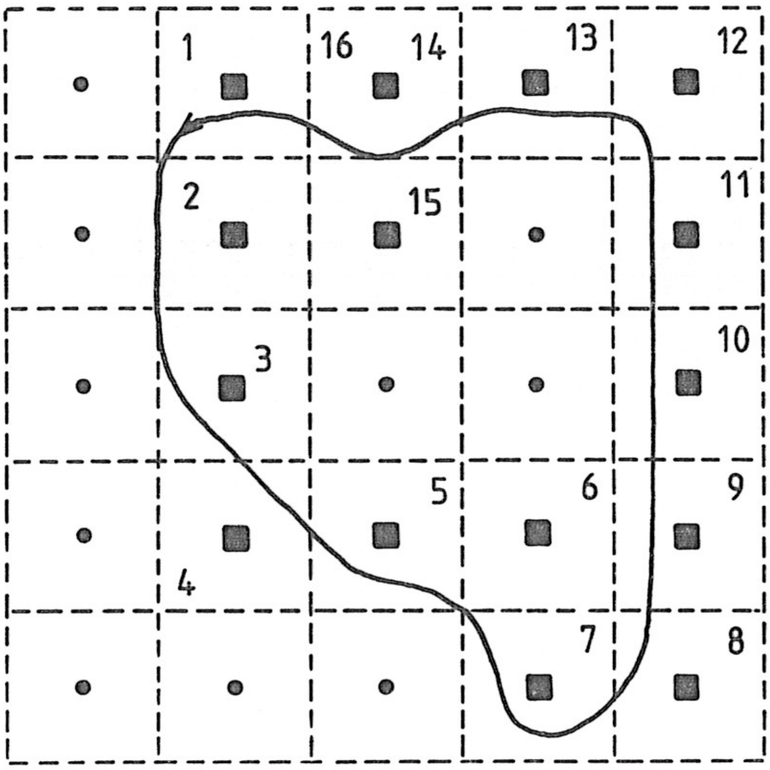


Figure 16. *Square box quantization of a curve*

In the same way as the grid-intersect quantization of straight lines was related to 8-convexity (or equivalently the chord property), the square box quantization of straight lines will be related to 4-convexity (or equivalently the strong chord property). In order to exhibit this relation, we will need to introduce another quantization scheme similar to the grid-intersect quantization, except that it is based on diagonal grid lines instead of axial ones. We will call it the *diagonal-intersect quantization.* We will see that the square box quantization of a straight line is equal to its diagonal-intersect quantization. It will then be possible to adapt Proposition 14 and Theorem 15 to the diagonal-intersect quantization.

The *diagonal-intersect quantization* $Q^*(C)$ of the curve C is a set of pixels built as follows: whenever C intersects a diagonal grid line (P_i or S_i, see (17)) in a point x which is not the middle of two diagonally adjacent pixels, the pixel p on that grid line closest to this intersection is taken as a member of $Q^*(C)$. Note that $d_8(x, p) < 1/2$ and $d_4(x, p) < 1$, but $d_8(x, q) \geq 1/2$ and $d_4(x, q) \geq 1$ for any other pixel q.

When C intersects a diagonal grid line in the middle x of two diagonally adjacent pixels, $x = (i + 1/2, j + 1/2)$ for two integers i and j. Then the pixels in $Q^*(C)$ determined by x will be chosen among the 4 pixels $a = (i, j)$, $b = (i, j+1)$, $c = (i+1, j)$ and $d = (i+1, j+1)$. We assume that x is not an isolated point of C. We have two cases:

(*i*) Only one of the four pixels a, b, c and d lies locally to the left of C w.r.t. x. This happens when the local orientation of C on x is parallel to a diagonal orientation. Then this unique pixel is taken as member of $Q^*(C)$. We illustrate this in Figure 17 (a).

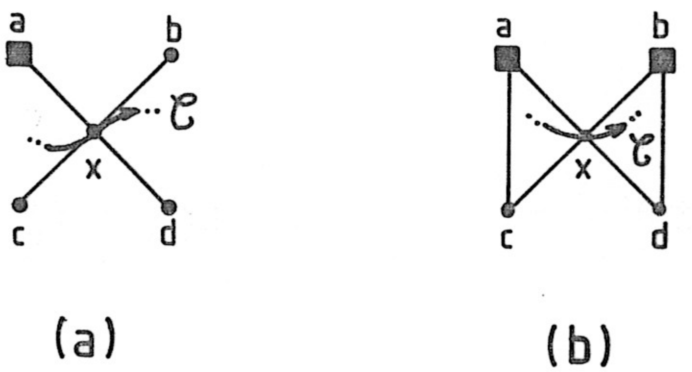


Figure 17.

(*ii*) Two adjacent pixels among the four a, b, c and d lies locally to the left of C w.r.t. x. We can assume that these pixels are a and b. Now the local orientation of C on x is not parallel to a diagonal orientation, and so the neighborhood of x in C must intersect the interior of $[x, a, c]$ or the interior of $[x, b, d]$ (and both if x is not an endpoint of C). Then we take $a \in Q^*(C)$ if this neighborhood intersects the interior of $[x, a, c]$ and $b \in Q^*(C)$ if it intersects the interior of $[x, b, d]$. We illustrate this in Figure 17 (b).

Now if $\mathcal{L}$ is a straight line, then $Q^*(\mathcal{L}) = Q^\circ(\mathcal{L})$. Indeed, consider a pixel $p \in P_m \cap S_n$; then:

— $\mathcal{L}$ intersects $\overline{p}^\circ$ iff it intersects P_m or S_n at 8-distance less than $1/2$ from x (see Figure 18 (a)).

— $\mathcal{L}$ intersects $\overline{p}$, but not $\overline{p}^\circ$, and leaves p to the left iff it intersects P_m or S_n at 8-distance $1/2$ from p, leaving p on the left, in such a way that this induces p in $Q^*(\mathcal{L})$ (see Figures 18 (b) and 18 (c)).

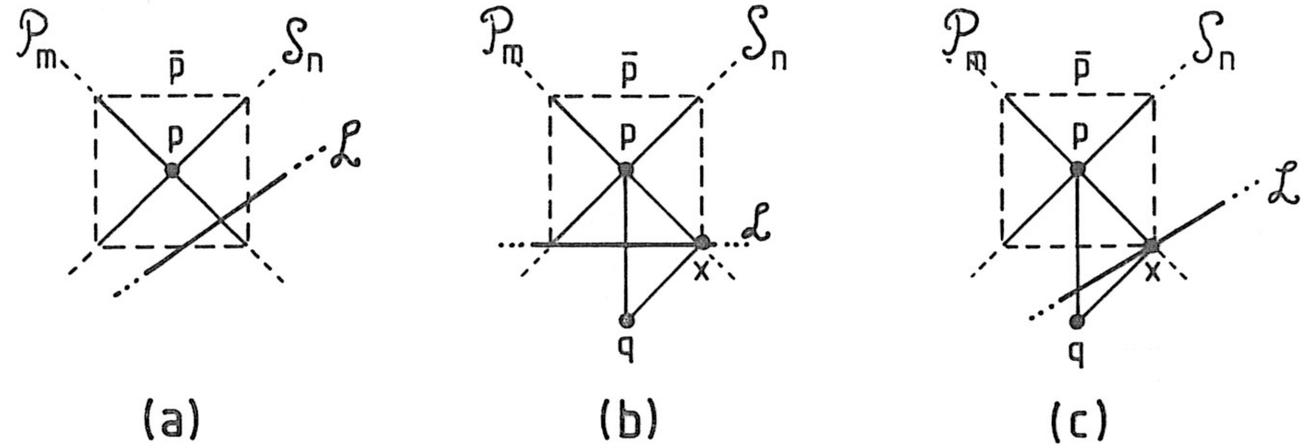


Figure 18.

The same holds for a straight line segment C, except that in some cases we have to extend its extremities; in other words there is another segment C' containing C, such that $Q^*(C) = Q^*(C') = Q^\circ(C')$.

Now we can proceed as we did in the previous section. When the orientation of $\mathcal{L}$ is closer to the orientation of the secondary diagonals than to the orientation of the principal diagonals, $Q^*(\mathcal{L})$ is determined by the intersections $\mathcal{L} \cap \mathcal{P}_m$ only. On the other hand, when this orientation is closer to orientation of the principal diagonals, $Q^*(\mathcal{L})$ is determined by the intersections $\mathcal{L} \cap \mathcal{S}_n$ only. Finally, when $\mathcal{L}$ is vertical or horizontal, then we can restrict ourselves to the intersections $\mathcal{L} \cap \mathcal{P}_m$ or $\mathcal{L} \cap \mathcal{S}_n$ indifferently. The same holds for $Q^*(C)$ for a segment C, but here again we might have to extend the extremities of C.

Let us now introduce the *diagonal coordinates* of points in the plane, *i.e.*, coordinates along diagonal axes. A point p with diagonal coordinates u and v will be labelled $(u, v)^*$, where the $*$ serves to avoid confusion with ordinary coordinates (x, y). These diagonal coordinates must be such that the point at the intersection of $\mathcal{P}_m$ and $\mathcal{S}_n$ will have m and n as diagonal coordinates. Thus by (17), a point $p = (x, y)$ will be equal to $(u, v)^*$ iff

$$u = x - y \qquad \text{and} \qquad v = x + y, \tag{20}$$

or equivalently

$$x = (u + v)/2 \qquad \text{and} \qquad y = (v - u)/2. \tag{21}$$

Then for every integer i,

$$\begin{aligned} \mathcal{P}_i &= \{(u, v)^* \in \mathbf{R}^2 \mid u = i\}; \\ \mathcal{S}_i &= \{(u, v)^* \in \mathbf{R}^2 \mid v = i\}. \end{aligned} \tag{22}$$

Note that x and y are both integers iff u and v are both integers and congruent modulo 2.

Given two points $p = (x, y) = (u, v)^*$ and $p' = (x', y') = (u', v')^*$, (21, 22) imply the following formulas for the 4- and 8-distance between them:

$$d_4(p, p') = |x' - x| + |y' - y| = max(|(x' - x) + (y' - y)|, |(x' - x) - (y' - y)|)$$

$$= max(|u' - u|, |v' - v|). \tag{23}$$

$$d_8(p, p') = max(|x' - x|, |y' - y|) = max(|(u' - u) + (v' - v)|, |(u' - u) - (v' - v)|)/2$$
$$= (|u' - u| + |v' - v|)/2. \tag{24}$$

This is due to the formula $|a| + |b| = max(|a + b|, |a - b|)$.

We can now get the equivalents of Proposition 14 and Theorem 15. These two results were proven first by Kim [16] (with Q° instead of Q^*, but this does of course not matter):

Proposition 18 [16]. *Given a straight line segment C, $Q^*(C)$ is a simple 4-connected path having the strong chord property.*

Proof. We can proceed in the same way as in the Proof of Proposition 14, but we substitute the diagonal coordinates to the ordinary ones, $\mathcal{P}_m$ to $\mathcal{H}_i$, and $\mathcal{S}_n$ to $\mathcal{V}_j$. We assume that the orientation of C forms an angle of less than 45° with the orientation of principal diagonals. The extremities of C are two points $(g, s)^*$ and $(h, t)^*$, where s and t are integers and $s < t$. For every integer n such that $s \leq n \leq t$, C intersects $\mathcal{S}_n$ in a single point $q_n = (u_n, n)^*$ (with $g = u_s$ and $h = u_t$), and this intersection induces in $Q^*(C)$ a pixel $p_n = (m_n, n)^*$, where $u_n - 1 \leq m_n < u_n + 1$ (note that we have 1 instead of 1/2 in Proposition 14). Write $\Delta = \{w \in \mathbf{R} \mid -1 \leq w < 1\}$ and $\lambda_n = m_n - u_n$. Then $\lambda_n \in \Delta$. Note that for every $\lambda, \lambda' \in \Delta$, $|\lambda - \lambda'| < 2$.

To show that for $s \leq a < b \leq t$, $|m_a - m_b| \leq b - a$, we do the same as in the second paragraph of the proof of Proposition 14, but here we have $|\lambda_a - \lambda_b| < 2$, and so we get $|m_a - m_b| =< 2 + b - a$. Now both $b - a$ and $|m_a - m_b|$ are integers, but they must also be congruent modulo 2 (since a and b are congruent to m_a and m_b respectively). Thus $|m_a - m_b| \leq b - a$.

An argument similar to the one in the third paragraph of the proof of Proposition 14 shows that $Q^*(C) = \{p_s, \ldots, p_t\}$ is a simple 4-connected path.

To show that $Q^*(C)$ has the strong chord property, it is sufficient to prove the statement (iii) of Theorem 13. Consider a, b such that $s \leq a < b \leq t$, and let x be a point in $[p_a, p_b] \cap \mathcal{D}$, where $\mathcal{D}$ is an diagonal grid line. We must show that $\mathcal{D}$ contains a pixel r such that $d_8(x, r) < 1$. We have two cases:

(a) $\mathcal{D} = \mathcal{S}_n$ for some n. An argument similar to the one in the corresponding point (a) in the proof of Proposition 14 shows that $x = (u_n + \lambda, n)^*$, where $\lambda \in \Delta$. Now $p_n \in Q^*(C) \cap \mathcal{S}_n$, and as $\lambda, \lambda_n \in \Delta$, $d_8(x, p_n) = |\lambda - \lambda_n|/2 < 1$. Thus we can take $r = p_n$.

(b) $\mathcal{D} = \mathcal{P}_m$ for some m, but $x \notin \mathcal{S}_n$ for any n. This case and the following argument are illustrated in Figure 19. We have $x = (m, n + \theta)^*$, where $a \leq n < b$, n is congruent to m modulo 2, and $0 < \theta < 2$. We can assume that $m_a \leq m_b$. Suppose that $p_a \in \mathcal{S}_{n+1}$. Then $p_a = (m + 2t + 1, n + 1)^*$ for some integer t. As p_a, p_b and x are collinear, we have

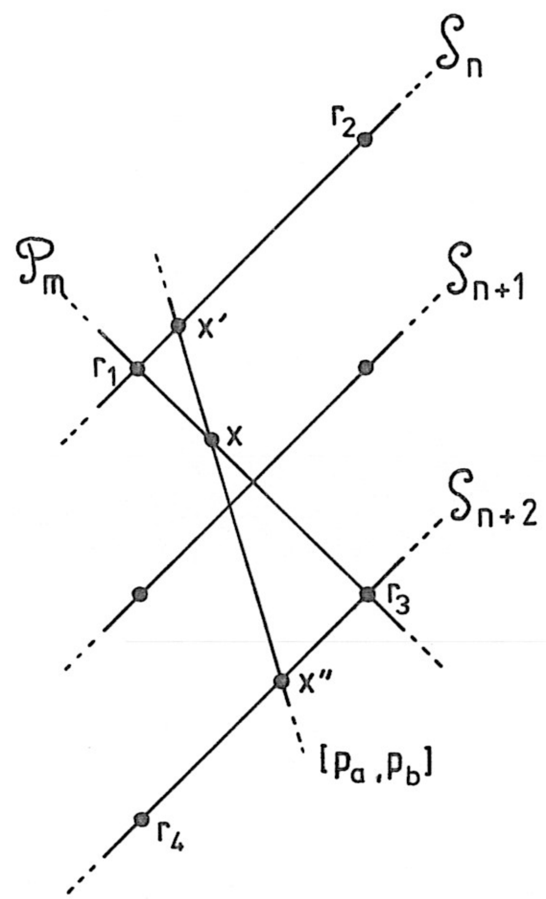


Figure 19.

$(m_b - m_a)/(b-a) = (2t+1)/(1-\theta)$. As $|m_a - m_b| \le b-a$ (see above), this implies that

$$1 \ge \left|\frac{m_b - m_a}{b-a}\right| = \left|\frac{2t+1}{1-\theta}\right| \ge \frac{1}{|\theta - 1|} > 1,$$

a contradiction. Thus $p_a \notin S_{n+1}$, and a similar argument shows that $p_b \notin S_{n+1}$. Thus $[p_a, p_b]$ crosses S_n and S_{n+2}, and so it contains two points $x' = (m-\psi, n)^*$ and $x'' = (m+\phi, n+2)^*$, where $0 \le \psi = \theta(m_b - m_a)/(b-a) \le \theta < 2$ and $0 \le \phi = (2-\theta)(m_b-m_a)/(b-a) \le 2-\theta < 2$. Let $r_1 = (m,n)^*$, $r_2 = (m-2,n)^*$, $r_3 = (m, n+2)^*$ and $r_4 = (m+2, n+2)^*$. As every pixel of $S_n - \{r_1, r_2\}$ is at 8-distance at least 1 from x', (a) implies that r_1 or $r_2 = p_m$. The same argument with x'' implies that r_3 or $r_4 = p_{m+2}$. If $r_1 = p_m$, then the result holds with $r = r_1$, since $r_1 \in P_m$ and $d_8(x, r_1) = \theta/2 < 1$). If $r_3 = p_{m+2}$, then it holds with $r = r_3$. If $r_1 \ne p_m$ and $r_3 \ne p_{m+2}$, then $p_m = r_2$, $p_{m+2} = r_4$, and so $d_4(p_m, p_{m+2}) = 4$, which contradicts the fact that $\{p_s, \ldots, p_t\}$ is a simple 4-connected path.

Thus we can find the appropriate pixel r in each case, and therefore $Q^*(C)$ has the strong chord property. ∎

Theorem 19 [16]. *Let S be a set of pixels having the strong chord property and let $p, q \in S$. Then there is a straight line segment C such that $Q^*(C) \subseteq S$ and the simple 4-connected path $Q^*(C)$ has p and q as its extremities.*

The proof is similar to the one of Theorem 15, if we substitute the diagonal coordinates to the ordinary ones, $\mathcal{P}_m$ to $\mathcal{H}_i$, and $\mathcal{S}_n$ to $\mathcal{V}_j$, and we make a few other small changes:

(1°) In the first paragraph, we can apply a parity argument similar to the one in the second paragraph in the proof of Proposition 18: $|b-a|$ and $v-u$ must be congruent modulo 2, and as $|b-a| < v-u$, this implies that $|b-a| \leq v-u-2$.

(2°) We must replace $1/2$ by 1 in the definition of $D(r)$ and of D_j and in the inequalities satisfied by x_i, λ and μ in the last paragraph. By (1°), the distance between p' and q' along the secondary diagonal direction is then $< (u-v-2)+2 = u-v$.

It is finally possible to derive the following two results in the same way as Corollaries 16 and 17:

Corollary 20. *Let $S \subseteq G$; then the following two statements are equivalent:*

(*i*) *S is a simple 4-connected path having the strong chord property.*

(*ii*) *There is a straight line segment C such that $S = Q^*(C)$.*

Corollary 21. *Let $S \subseteq G$; then the following two statements are equivalent:*

(*i*) *S has the strong chord property.*

(*ii*) *For every $p, q \in S$, there is a straight line segment C such that $Q^*(C) \subseteq S$ and the simple 4-connected path $Q^*(C)$ has p and q as its extremities.*

Note that in the 4 results stated above, one can replace $Q^*(C)$ by $Q^\circ(C)$.

V. Further topics in digital convexity

Chapter IV introduced relations between digital convexity and quantization of a particular class of convex subsets of the Euclidean space, namely straight line segments. One can go further and analyze the relations between digital convexity and quantization of Euclidean convex sets through various digitization schemes. This is done in particular in [15,23].

Digital convexity and straightness can also be defined in 3 or more dimensions. A study of the 3-dimensional case has begun [18,19,22]. Some properties studied in this Report can be generalized to n-dimensional digital images. For example Proposition 1 leads to the following generalization of L- and T-convexity: a n-dimensional digital image S is said to be (n, x)-convex (where $x = 1, \ldots, n$) iff for every $U \subseteq S$ such that $|U| \leq x+1$, $\langle U \rangle \subseteq S$. Thus L- and T-convexity are $(2,1)$ and $(2,2)$-convexity respectively. One can then investigate whether there exists an n-dimensional correspondent of Theorem 5. Another example of possible study is the generalization of the method used in Chapter IV in order to derive from n-dimensional theorems of Santaló type characterizations of the digitization of u-dimensional subspaces $(u = 1, \ldots, n-1)$ on a n-dimensional grid.

Let us note finally that it is possible to study infinite digital straight lines. For this purpose one can generalize Santaló's Theorem to the case of an infinite number of parallel straight line segments by requiring that these segments are bounded and closed.

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