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Topological principles of thinning

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Abstract:
In this work, we study the topological aspects of the deletion of a subset from a binary image in a square grid. We derive then criteria for the topological validity of usual thinning algorithms based on a succession of deletions of pixels (sequential or parallel). We also survey briefly the geometrical requirements of the thinning process. Finally, we apply our results to the analysis of two existing thinning algorithms, one sequential (Hilditch's) and one parallel (Rosenfeld's).

I. Introduction

In optical character recognition (OCR), a common practice is to reduce a digitized representation of a character to a one pixel thick figure called the skeleton, which retains all its significant topological and geometrical features. The assumption underlying this process is that the skeleton, by containing a smaller amount of data than the original character while retaining its basic properties, is easier to analyze and thus to recognize than a thicker figure. Indeed, practice proves that it is easier to extract geometrical and topological features from images built with digital curves than with images of arbitrary width.

One can find in the literature a large number of algorithms producing skeletons from arbitrary figures. Most of them use the method of thinning: successive layers of pixels are deleted from the figure until it becomes one pixel thick. In general the pixels are deleted according to certain criteria based on the configuration of white and black pixels in their 8-neighbourhood. These algorithms divide themselves into two classes: those in which at each pass the deletion criterion is applied to all pixels at the same time and pixels which satisfy it are deleted together *(parallel algorithms)*, and those in which the criterion is applied successively to all pixels and the deletion of a pixel occurs before that criterion is applied to another pixel *(sequential algorithms)*.

We do not intend to survey here the various thinning algorithms available in the literature. The most representative ones are compared in [8]. Our aim is rather to describe the principles which underly all these algorithms. As said above, the features which must be retained in the skeletonization process are of a topological or geometrical nature. We will show that the topological requirements can be stated in very rigorous terms, while geometrical requirements are more vague and admit different mathematical formulations. We have thus concentrated our analysis on the topological aspects of thinning, while touching briefly on the geometrical ones. Our results include a characterization of the conditions to be fulfilled by the deletion criterion of a thinning algorithm in order to be topologically valid.

Our report is organized as follows:

In Section 2 we analyze the topological properties of the set K of pixels to be deleted from a figure F in order to get a skeleton. This leads to the concept of k-deletability and strong k-deletability ($k = 4$ or 8), with the latter being more suitable than the former.

In Section 3 we show that a strongly k-deletable subset K of a figure F can be deleted by a succession of deletions of individual pixels $p_1, \dots, p_t$, where each p_i is k-deletable from $F \backslash \{p_j \mid j < i\}$. A consequence of this result is that any topologically valid skeleton can be obtained by some thinning process.

In Section 4 we recall well-known results characterizing the deletability of a pixel, while in Section 5 we give the condition for the strong deletability of a group of deletable pixels. These results allow us to verify the topological validity of any thinning algorithm, sequential or parallel.

In Section 6 we give a brief description of the geometrical conditions that can be combined with the topological ones in order to obtain a performing algorithm. Finally we analyze in Section 7 two algorithms chosen from the literature, one sequential and one parallel, and show how they satisfy in practice the requirements established theoretically.

Our study should allow a better understanding of the thinning process and give mathematical tools for the verification of the validity of any thinning algorithm.

II. Deletability and Strong Deletability

Let G be a rectangular or quadruled grid with *frame* FG. Let F be a *figure* on G with *background* $B = G\backslash F$. We make the *Restricted Frame Assumption (RFA)* which states that the frame FG is entirely included either in B or in F, [4]. Given two arbitrary subsets S and T of G, let $\mathcal{C}(k,T)$ denote the set of k-connected components of T with

$$C_k(T) \doteq | \mathcal{C}(k,T) |, \tag{1}$$

and $\mathcal{C}(k,T,S)$ denote the set of k-connected components of T which are k-adjacent or k-connected to S with

$$C_k(T,S) \doteq | \mathcal{C}(k,T,S) | . \tag{2}$$

Let K designate an arbitrary subset of figure F. Hereafter, we shall contemplate deletion of K from F, thereby obtaining a new figure $F' = F\backslash K$, with background $B' = B \bigcup K$. Recall that if k-connectedness is used for F and F', k'-connectedness must be used for B and B', with $k = 4$ or 8, and $k' = 12 - k$.

One of the purposes of this paper is to examine the behavior of thinning algorithms from a topological viewpoint. It is evident that deletion of some subset K from F may change the topology of F. Therefore, our first step will be to determine the conditions to be imposed on K in order that the topological structure of F be preserved.

The topological structure of F is characterized by the set of k-connected components of F, the set of k'-connected components of B, their adjacency and surrounding relations, and the relation of FG to F and B, [4,5].

Invariance of the numbers of connected components of the figure and the background is, clearly, the first necessary condition to be satisfied by the deletion of K from F. So, we must have

$$C_k(F) = C_k(F'), \tag{3}$$

and

$$C_{k'}(B) = C_{k'}(B'). \tag{4}$$

In general, conditions (3) and (4) are not sufficient to guarantee the invariance of the topological structure of F under deletion of K. Figure 1 displays three examples where K satisfies (3) and (4), and F and F' are not topologically equivalent: Connected components of the figure are split or erased, while connected components of the background are merged or created. Moreover, the geometrical structure of the figure is distorted by the absence of correspondence between connected components of F and F'. In fact, one expects from the thinning process that it maintains a natural correspondence between the connected components of F and B on the one hand, and those of F' and B' on the other hand.

We note that $F' \subseteq F$ and we define a map

$$\varphi : \mathcal{C}(k, F') \to \mathcal{C}(k, F), \tag{5}$$

with $X'\varphi = X$ iff $X' \subseteq X$. Similarly, as $B \subseteq B'$, we have a map

$$\beta : \mathcal{C}(k', B) \to \mathcal{C}(k', B'), \tag{6}$$

with $Y\beta = Y'$ iff $Y \subseteq Y'$.

Now, four common sense requirements translate readily into an equal number of formal conditions.

- A connected component of the figure cannot be split. Formally, an element of $\mathcal{C}(k, F)$ contains at most one element of $\mathcal{C}(k, F')$, i.e.,

$$\varphi \text{ is injective.} \tag{7.a}$$

- A connected component of the figure cannot be erased. Formally, an element of $\mathcal{C}(k, F)$ contains at least one element of $\mathcal{C}(k, F')$, i.e.,

$$\varphi \text{ is surjective.} \tag{7.b}$$

- Two or more connected components of the background cannot be merged into a single one. Formally, an element of $\mathcal{C}(k', B')$ contains at most one element of $\mathcal{C}(k', B)$, i.e.,

$$\beta \text{ is injective.} \tag{8.a}$$

- A connected component of the background cannot be created. Formally, an element of $\mathcal{C}(k', B')$ contains at least one element of $\mathcal{C}(k', B)$, i.e.,

$$\beta \text{ is surjective.} \tag{8.b}$$

It is clear that any two of (3), $(7, a)$ and $(7, b)$ imply the third, and similarly for (4), $(8, a)$ and $(8, b)$.

The following lemma translates these four conditions in terms of K.

<u>**Lemma 1**</u>: *Consider the following four conditions*

$$\forall A \in \mathcal{C}(k, K), \quad C_k(F', A) \leq 1. \tag{9.a}$$

$$\forall A \in \mathcal{C}(k, K), \quad C_k(F', A) > 0. \tag{9.b}$$

$$\forall Z \in \mathcal{C}(k', K), \quad C_{k'}(B, Z) \leq 1. \tag{10.b}$$

$$\forall Z \in \mathcal{C}(k', K), \quad C_{k'}(B, Z) > 0. \tag{10.b}$$

Then, (i) $(7.a) \Leftrightarrow (9.a)$, *(ii)* $(7.b) \Leftrightarrow (9.b)$, *(iii)* $(8.a) \Leftrightarrow (10.a)$, *and (iv)* $(8.b) \Leftrightarrow (10.b)$.

Proof. We need only prove *(i)* and *(ii)*. For *(iii)* and *(iv)* we have only to interchange F and B', F' and B, k and k', φ and β in the following proof.

(i) If (9.a) does not hold, then $C_k(F', A) \geq 2$ for some $A \in \mathcal{C}(k, K)$. Let $X, Y \in \mathcal{C}(k, F', A)$, $X \neq Y$. Then, $X \bigcup A \bigcup Y$ is k-connected and it belongs to some $W \in \mathcal{C}(k, F)$. As $X, Y \subseteq W$, (7.a) does not hold.

If (7.a) does not hold, then there is some $W \in \mathcal{C}(k, F)$ containing at least two distinct $X, Y \in \mathcal{C}(k, F')$. As $F = F' \bigcup K$, W is a union of elements of $\mathcal{C}(k, F')$ and elements of $\mathcal{C}(k, K)$. Consider a k-path P in W joining X to Y. It goes successively through elements of $\mathcal{C}(k, F')$ and $\mathcal{C}(k, K)$ in alternance. As $X \neq Y$, there is some $A \in \mathcal{C}(k, K)$ and some $X_0, X_1 \in \mathcal{C}(k, F')$ such that $X_0 \neq X_1$ and P goes successively through X_0, A and X_1. Then, $X_0, X_1 \in \mathcal{C}(k, F', A)$ and so, (9.a) does not hold.

(ii) If (9.b) does not hold, then $C_k(F', A) = 0$ for some $A \in \mathcal{C}(k, K)$. As $F = K \bigcup F'$, and $C_k(K, A) = 0$ (by definition), $A \in \mathcal{C}(k, F)$ and so (7.b) does not hold since A contains no element of $\mathcal{C}(k, F')$.

If (7.b) does not hold, then there is some $W \in \mathcal{C}(k, F)$ such that W contains no element of $\mathcal{C}(k, F')$. As $F = F' \bigcup K$, $W \subseteq K$ and so $W \in \mathcal{C}(k, K)$ and $C_k(F', W) = 0$ (since $W \in \mathcal{C}(k, F)$). Thus, (9.b) does not hold. ∎

We may conclude from the above that to guarantee the invariance of the topological structure of F and B, K must be subjected to the following conditions

$$\varphi \text{ is bijective,} \tag{7.c}$$

$$\beta \text{ is bijective;} \tag{8.c}$$

which, by Lemma 1 are equivalent to

$$\forall A \in \mathcal{C}(k, K), \quad C_k(F', A) = 1, \tag{9.c}$$

and

$$\forall Z \in \mathcal{C}(k', K), \quad C_{k'}(B, Z) = 1, \tag{10.c}$$

respectively.

We shall presently show that if K is 4-connected, the conditions (3) and (4) which expressed the invariance of the numbers of connected components of the figure and the background suffice to imply (9.c) and (10.c), hence, (7.c) and (8.c) also. To this end we need the following lemma.

Lemma 2: *Let S and T be two disjoint subsets of G. Suppose that S is k-connected. Then*

$$C_k(T) = C_k(T \bigcup S) + C_k(T, S) - 1. \tag{11}$$

In particular, $C_k(T) = C_k(T \bigcup S)$ if and only if

$$C_k(T, S) = 1. \tag{12}$$

Proof. Let $m \doteq C_k(T,S)$ and $n \doteq C_k(T) - C_k(T,S)$, $(m,n \geq 0)$. Let $T_1, \ldots, T_m$ be the elements of $\mathcal{C}(k,T,S)$, and $T'_1, \ldots T'_n$ be those of $\mathcal{C}(k,T) \backslash \mathcal{C}(k,T,S,)$. Then, the elements of $\mathcal{C}(k, T \bigcup S)$ are

$$(T_1 \bigcup .. \bigcup T_m) \bigcup S, T'_1, \ldots, T'_n.$$

Thus we have:

$$C_k(T) = m + n,$$

$$C_k(T \bigcup S) = n + 1,$$

$$C_k(T,S) = m.$$

Then, (11) follows with (12) as an immediate consequence. ∎

Next, let us apply Lemma 2 to K.

In the first place, if K is k-connected, we set $S = K$, and $T = F'$. From (12) we find that

$$C_k(F) = C_k(F') \tag{3}$$

if and only if

$$C_k(F', K) = 1. \tag{13}$$

In the second place, if K is k'-connected, we set $S = K$, and $T = B$. From (12) we find that

$$C_{k'}(B) = C_{k'}(B') \tag{4}$$

if and only if

$$C_{k'}(B, K) = 1. \tag{14}$$

Now, K is 4-connected if and only if it is both k- and k'-connected. We get thus the following result:

Lemma 3: *If K is 4-connected, then* (3) *and* (4) *are equivalent to* (13) *and* (14) *respectively.*

We have characterized a condition for a one to one correspondence between the connected components of F and F', and those of B and B'. One of our goals in the next section will be to demonstrate that this correspondence preserves the neighborhood relations between these connected components.

We need also consider the relation of figure F to the frame FG of G. In order to be consistent with the assumption that F satisfies the *RFA*, we shall also assume that F' satisfies the *RFA*. Moreover, we shall require that the deletion of K from F does not change the colour of FG. In other words, we impose one of the following two conditions:

If $FG \subseteq B$, then $FG \subseteq B'$.

$$\text{If } FG \subseteq F, \text{ then } FG \subseteq F'.$$

Clearly, this holds if and only if

$$K \bigcap FG = \emptyset. \tag{15}$$

Let us now summarize our results thus far. We have established two different sets of conditions which allow the deletion of K from F. One set is stronger than the other, but if K is 4-connected, then they are equivalent. This leads us to make the following definitions.

Definition 1: *Given $K \subseteq F$, $F' = F \backslash K$, and $B' = B \bigcup K$. We shall say that K is k-deletable from F if and only if $C_k(F) = C_k(F')$, $C_{k'}(B) = C_{k'}(B')$, and $K \bigcap FG = \emptyset$.*

Definition 2: *Given $K \subseteq F$, $F' = F \backslash K$, and $B' = B \bigcup K$. We shall say that K is strongly k-deletable from F if and only if φ and β are both bijective (or, equivalently, $\forall A \in \mathcal{C}(k, K)$, $C_k(F', A) = 1$, and $\forall Z \in \mathcal{C}(k', K)$, $C_{k'}(B, Z) = 1$) and $K \bigcap FG = \emptyset$.*

Figure 2 displays examples of sets K which are either strongly 8-deletable but not 4-deletable, (2.a), or strongly 4-deletable but not 8-deletable, (2.b). In simple configurations, such as that in (2.c), the connected components of K are both strongly 4-deletable and strongly 8-deletable. This observation prompts us to make the following definition:

Definition 3: *Given $K \subseteq F$, $F' = F \backslash K$, and $B' = B \bigcup K$. We shall say that K is (strongly) (4,8)-deletable from F if it is both (strongly) 4-deletable from F, and (strongly) 8-deletable from F.*

We have shown that if K is strongly k-deletable from F, then it is k-deletable from F; but the converse is not true, except if K is 4-connected. In the next section we will show that if K is strongly k-deletable, the deletion of K from F can be realized by a succession of deletions of k-deletable pels from F. This is, indeed, what happens in most actual thinning algorithms.

III. A Characterization of Strong Deletability

In this Section, our goal is to show that if K is strongly k-deletable from F then K contains strongly k-deletable pels, and the deletion of K from F can be realized by a succession of deletions of these pels. This property is, in fact, a characterization of strongly k-deletable subsets. To achieve this goal we decompose K into smaller and smaller strongly deletable subsets. In the first place we characterize deletable subsets of K. Next, the analysis focuses on 8-connected, and 4-connected components of K. Eventually, we exhibit the characterization of deletability at the *pel* level.

Let us first remark that the following result is obvious:

Lemma 4: *Let $K \subseteq F \backslash FG$, $K' \subseteq K$, and $K'' = K \backslash K'$. Then any two of the following statements imply the third:*

(i) K' is k-deletable from F.
(ii) K'' is k-deletable from $F \backslash K'$.
(iii) K is k-deletable from F.

The corresponding result in terms of strong deletability is the following:

Lemma 5: *Let K, K', and K'' be the three subsets of K which satisfy the hypothesis and the three statements of Lemma 4. Then, the following two statements are equivalent:*

(i) K is strongly k-deletable from F.
(ii) K' is strongly k-deletable from F and K'' is strongly k-deletable from F.

Proof. We have the following six maps:

$$\begin{aligned} \phi_K &: \mathcal{C}(k, F\backslash K) \to \mathcal{C}(k, F), \\ \phi_{K''} &: \mathcal{C}(k, F\backslash K) \to \mathcal{C}(k, F\backslash K'), \\ \phi_{K'} &: \mathcal{C}(k, F\backslash K') \to \mathcal{C}(k, F), \\ \beta_K &: \mathcal{C}(k', B) \to \mathcal{C}(k', B \bigcup K), \\ \beta_{K''} &: \mathcal{C}(k', B \bigcup K') \to \mathcal{C}(k', B \bigcup K), \\ \beta_{K'} &: \mathcal{C}(k', B) \to \mathcal{C}(k', B \bigcup K'), \end{aligned} \tag{16}$$

corresponding to (5) and (6). It is clear that

$$\phi_K = \phi_{K''} \cdot \phi_{K'},$$

and

$$\beta_K = \beta_{K'} \cdot \beta_{K''}. \tag{17}$$

As the three statements of Lemma 4 hold, we have:

$$C_k(F) = C_k(F\backslash K') = C_k(F\backslash K),$$

and

$$C_{k'}(B) = C_{k'}(B \bigcup K') = C_{k'}(B \bigcup K). \tag{18}$$

By (16), (17), and (18) it is clear that ϕ_K is bijective if and only if both $\phi_{K''}$ and $\phi_{k'}$ are bijective, and likewise in terms of β. Now, statement *(i)* means that ϕ_K and β_K are bijective, while statement *(ii)* means that $\phi_{K'}$, $\beta_{K'}$, $\phi_{K''}$, and $\beta_{K''}$ are bijective. Therefore these two statements are equivalent. ∎

Our next step will be to show that if K is strongly k-deletable, every 8-connected component of K is strongly k-deletable.

Proposition 6: *If K is strongly k-deletable from F, then for every $X \in \mathcal{C}(8, K)$, X is strongly k-deletable from F.*

Proof. For any $A \in \mathcal{C}(k, X)$, and $Z \in \mathcal{C}(k', X)$ we have $A \in \mathcal{C}(k, K)$, and $Z \in \mathcal{C}(k', K)$ by virtue of the 8-connectedness of X. We also have

$$C_k(F \backslash K, A) = 1, \tag{19}$$

and

$$C_{k'}(B, Z) = 1, \tag{20}$$

for K is strongly k-deletable. As $A \in \mathcal{C}(k, X)$, A is not k-adjacent to $K \backslash X$, hence

$$C_k(K \backslash X, A) = 0. \tag{21}$$

Now, $F \backslash X = (F \backslash K) \bigcup (K \backslash X)$. Thus A can be k-connected to $F \backslash X$ only through $F \backslash K$. This fact together with (19) and (21) imply that

$$C_k(F \backslash X, A) = 1. \tag{22}$$

Then, by (20) and (22), X is strongly k-deletable from F. ∎

Now, it follows from Lemma 5 that the deletion of K from F can be realized by successive deletions of its 8-connected components. Moreover, it is readily verified that Lemma 5 and Proposition 6 remain true if we substitute "strong $(4, 8)$-deletability" for "strong k-deletability".

In parallel with Proposition 6, we will presently exhibit the characterization of 4-connected components which are (strongly) k-deletable. In our discussion, we will have to make a distinction between 4-, 8-, and (4,8)-deletability.

Proposition 7: *Let $X \subseteq F \backslash FG$, and suppose that X is 8-connected. If X is strongly k-deletable from F, then there exists some $Y \in \mathcal{C}(4, X)$ such that Y is (strongly) k-deletable from F.*
If X is strongly (4,8)-deletable from F, then every $Y \in \mathcal{C}(4, X)$ is (strongly) 4-deletable from F, and one of them is (strongly) (4,8)-deletable from F.
(Note: For such a set Y, deletability is equivalent to strong deletability by Lemma 2.)

Proof. We distinguish three cases, namely, $k = 4$, $k = 8$, and (4,8)-deletability.

(a) $k = 4$. As $C_8(B, X) = 1$, there exists some $Y \in \mathcal{C}(4, X)$ such that Y is 8-adjacent to B. We will show that any such Y is strongly 4-deletable from F. Clearly, as $C_8(B, X) = 1$, $C_8(B, Y) \neq 0$, and $Y \subseteq X$, we have

$$C_8(B, Y) = 1. \tag{23}$$

By definition of the strong 4-deletability of X, we have $C_4(F \backslash X, Y) = 1$. As $Y \in \mathcal{C}(4, X)$, Y is not 4-adjacent to $X \backslash Y$. Now, $F \backslash Y = (F \backslash X) \bigcup (X \backslash Y)$. Thus, Y can be 4-connected to $F \backslash Y$ only through $F \backslash X$ and this implies

$$C_4(F \backslash Y, Y) = 1. \tag{24}$$

By (23) and (24), Y is (strongly) 4-deletable from F.

(b) $\underline{k = 8}$. By definition of the strong 8-deletability, we have $C_8(F\backslash X, X) = 1$, and

$$C_4(B, Z) = 1 \tag{25}$$

for every $Z \in \mathcal{C}(4, X)$. As $C_8(F\backslash X, X) = 1$, and X is 8-connected, every $Z \in \mathcal{C}(4, X)$ is 8-connected to $F\backslash X$ in F. In other words there exists an 8-path through F connecting Z to $F\backslash X$. Let d denote the restriction of the distance d_8 to F, (see [?]). Choose $Y \in \mathcal{C}(4, X)$ such that $d(Y, F\backslash X) \geq d(Z, F\backslash X)$ for any other $Z \in \mathcal{C}(4, X)$. (We may assume that $C_4(X) \geq 1$.) Then, the 8-connected component of $F\backslash X$ which is 8-adjacent to X must be 8-connected to some $Z \neq Y$. Therefore,

$$C_8(F\backslash X, X\backslash Y) = C_8(F\backslash X, X) = 1. \tag{26}$$

Now, combining (26) with Lemma 2, we obtain:

$$C_8(F\backslash X) = C_8(F) + C_8(F\backslash X, X) - 1 = C_8(F),$$

$$C_8(F\backslash X) = C_8(F\backslash Y) + C_8(F\backslash X, X\backslash Y) - 1 = C_8(F\backslash Y).$$

By Lemma 3 again, we get

$$C_8(F\backslash Y, Y) = C_8(F\backslash Y) - C_8(F) + 1 = 1. \tag{27}$$

By (25) and (27), Y is (strongly) 8-deletable from F.

(c) $\underline{(4, 8)}$. X is strongly (4,8)-deletable from F. Thus, it is strongly 8-deletable and so (25) holds for every $Z \in \mathcal{C}(4, X)$. But X is also (strongly) 4-deletable, and we may apply the result of (a): For every $Z \in \mathcal{C}(4, X)$, as Z is 8-adjacent to B (by (25)), Z is strongly 4-deletable from F. Next, as X is strongly 8-deletable from F, we may apply the result of (b): There is some $Y \in \mathcal{C}(4, X)$ such that Y is (strongly) 8-deletable from F. Thus, Y is (strongly) (4,8)-deletable from F. This completes the proof of Proposition 7. ∎

One consequence of Lemma 5 and Proposition 7 is that the deletion of K from F can be realized by a succession of deletions of its 4-connected components.

Our last step in this section brings us down at the pel level. We shall make use hereafter of the following notation: Given two sets $S \subseteq G$ and $T \subseteq G$. Suppose that S, and T are k-adjacent. Let $\delta_k(S, T)$ designate the "k-border of S with respect to T", viz., $\delta_k(S, T) \doteq \{p \in S \mid d_k(p, T) = 1\}$. If $T = G\backslash S$, then we use the notation $\delta_k(S)$ instead of $\delta_k(S, T)$.

Proposition 8: *Let $Y \subseteq F\backslash FG$. Suppose that Y is 4-connected. Let x stand for 4, 8, or (4,8). If Y is x-deletable from F, then there exists a pixel $p \in Y$ such that p is x-deletable from F.*

Proof. We assume $|Y| > 1$, otherwise the result is obvious. We define two integers k_1, and k_2 as follows:

(i) If $x = k$, then $k_1 \doteq k'$, and $k_2 \doteq k$.

(ii) If $x = (4,8)$, then $k_1 \doteq k_2 \doteq 4$.

As Y is x-deletable from F, we have $C_{k_1}(B,Y) = C_{k_2}(F\backslash Y, Y) = 1$. It follows thus that $\delta_{k_1}(Y,B)$ and $\delta_{k_2}(Y, F\backslash Y)$ are non-void. Moreover, there always exist pels $p \in \delta_{k_1}(Y,B)$, and $q \in \delta_{k_2}(Y, F\backslash Y)$ such that $p \neq q$ for otherwise we would have

$$|\delta_4(Y)| \leq |\delta_{k_1}(Y,B) \bigcup \delta_{k_2}(Y, F\backslash Y)| = 1,$$

which is impossible. After this preamble, we examine two cases:

(a) $\underline{x = k}$. As Y is k-deletable, $C_{k'}(B,Y) = 1$. In addition, $C_k(F\backslash Y, Y) = 1$. As $q \in \delta_k(Y, F\backslash Y)$, the unique k-connected component of $F\backslash Y$ which is k-adjacent to Y is k-adjacent to q, and so to $Y - p$, where $Y - p$ denotes the set Y with p deleted. Thus,

$$C_k(F\backslash Y, Y - p) = C_k(F\backslash Y, Y) = 1. \tag{29}$$

We next apply Lemma 3 three times, in combination with (29), and we get:

$$C_k(F\backslash Y) = C_k(F) + C_k(F\backslash Y, Y) - 1 = C_k(F);$$

$$C_k(F\backslash Y) = C_k(F - p) + C_k(F\backslash Y, Y - p) - 1 = C_k(F - p);$$

$$C_k(F - p, p) = C_k(F - p) - C_k(F) + 1 = 1. \tag{30}$$

Thus, p is k-deletable from F.

(b) $\underline{x = (4,8)}$. Then the argument of (a) can be applied for$k = 4$ and $k = 8$, since we have $p \in \delta_4(Y,B)$ and $q \in \delta_4(Y, F\backslash Y)$. ∎

Proposition 8 brings to an end this part of our analysis. By iterative applications of Lemma 5, and Propositions 6, 7, and 8, we obtain our fundamental result which characterizes strong deletability:

Theorem 9: *Let $K \subseteq F\backslash FG$. Let $t = |K|$ (we assume $t > 1$). Let $x = 4$, 8 or $(4,8)$. Then the following two statements are equivalent:*

(i) K is strongly x-deletable from F.

(ii) We can label the elements of K, $p_1, \ldots, p_t$, in such a way that each p_i, $1 \leq i \leq t$, is x-deletable from $F\backslash\{p_j \mid j < i\}$.

Moreover, in (ii), we can choose the p_i's in such a way that for $k = 4$ or 8, every k-connected component of K is of the form $\{p_s \mid i \leq s \leq j\}$ for some i, j such that $1 \leq i \leq j \leq t$.

Now we remark that most thinning algorithms proceed by successive deletions of individual pels, and our Theorem states that any strongly deletable part of the figure can be deleted in such a way. Aside from the topological considerations put forth in Section II, this provides additional justification for our choice of strong deletability conditions.

An interresting consequence of the Theorem is the preservation of the *adjacency tree* [4] of a figure by the deletion of a strongly deletable set. Indeed, it is easily seen that if a k-deletable pel p is deleted from figure F, then F and $F - p$ have the same adjacency tree. As the argument can be repeated iteratively, it follows that F and $F \setminus K$ have the same adjacency tree.

Henceforth, we shall consider that the topological condition for the deletion of K from F in the thinning process is the strong deletability of K with $x = k$ if one considers k-connectedness on the figure, and $x = (4, 8)$ if one considers both 4- and 8-connectedness on the figure.

IV. Deletability condition for a pixel

In this section, we will survey well-known results [6,9,10] expressing the deletability condition for a single pixel, namely a criterion expressed in terms of connected components of the intersection of F and B with the 8-neighbourhood of that pixel [6], and a numerical expression for this criterion [9,10].

We will explain these results in detail, and we will give a very simple proof of Rosenfeld's deletability criterion [6]. In fact, we will prove a more general result.

The analysis of the 8-neighbourhood of a pixel will require some additional notations. Let p be a pixel in $G \setminus FG$, and let T be any subset of G. Let $k = 4$ or 8. We recall from [4] that $N_k(p)$ is the k-neighbourhood of p. We will then write:

$$NC_k(T, p) \doteq C_k(T \bigcap N_8(p), p), \tag{31}$$

i.e., the set of k-connected components of the restriction of T to $N_8(p)$ which are k-connected to p. Note that for $k = 8$, we have:

$$NC_8(T, p) = C_8(T \bigcap N_8(p)). \tag{32}$$

In Figure 3 we give a few examples of the values taken by $NC_4(T, p)$ and $NC_8(T, p)$ for several configurations.

Let us return to our figure F. Let $p \in F \setminus FG$ and suppose that we wish to delete p from F. We must check whether p is k-deletable from F. We will show that the answer depends upon the two numbers

$$NC_k(F, p)$$

and

$$NC_{k'}(B, p).$$

A first look at them shows us three possibilities:

(*a*) $B \bigcap N_{k'}(p) = \emptyset$, $NC_{k'}(B,p) = 0$ and $NC_k(F,p) = 1$.

(*b*) $F \bigcap N_k(p) = \emptyset$, $NC_{k'}(B,p) = 1$ and $NC_k(F,p) = 0$.

(*c*) $B \bigcap N_{k'}(p) \neq \emptyset \neq F \bigcap N_k(p)$ and every $X \in \mathcal{C}(k, F \bigcap N_8(p), p)$ neighbours some $Y \in \mathcal{C}(k', B \bigcap N_8(p), p)$ and vice-versa.

In Figure 4 we illustrate cases (*a*) and (*b*) for both $k = 4$ and $k = 8$.

Let us now define two more numbers; in fact, we will show that they are equal. These numbers will be used in the deletability criterion for the pixel p:

$CN_k(F,p)$ is the number of $X \in \mathcal{C}(k, F \bigcap N_8(p), p)$ neighbouring some $Y \in \mathcal{C}(k', B \bigcap N_8(p), p)$. (33)

$CN_{k'}(B,p)$ is the number of $Y \in \mathcal{C}(k', B \bigcap N_8(p), p)$ neighbouring some $X \in \mathcal{C}(k, F \bigcap N_8(p), p)$. (34)

Then the three cases (*a*), (*b*) and (*c*) considered above give the following:

— In (*a*) and (*b*) we have

$$CN_k(F,p) = CN_{k'}(B,p) = 0. \tag{35}$$

— In (*c*) we have

$$CN_k(F,p) = NC_k(F,p) > 0$$

and

$$CN_{k'}(B,p) = NC_{k'}(B,p). \tag{36}$$

As we announced it, we have the following result:

Lemma 10. $CN_k(F,p) = CN_{k'}(B,p)$.

Proof. By (35), we may assume that we are in case (*c*). Now when we scan $N_8(x)$ circularly, then we encounter alternately the elements of $\mathcal{C}(k, F \bigcap N_8(p), p)$ and $\mathcal{C}(k', B \bigcap N_8(p), p)$. (We encounter also elements of $\mathcal{C}(k, F \bigcap N_8(p)) \setminus \mathcal{C}(k, F \bigcap N_8(p), p)$ when $k = 4$, or of $\mathcal{C}(k', B \bigcap N_8(p)) \setminus \mathcal{C}(k', B \bigcap N_8(p), p)$ when $k = 8$, but they do not separate distinct elements of $\mathcal{C}(k, F \bigcap N_8(p), p)$ and $\mathcal{C}(k', B \bigcap N_8(p), p)$ respectively.)

Thus the relation of neighbourhood between $\mathcal{C}(k, F \bigcap N_8(p), p)$ and $\mathcal{C}(k', B \bigcap N_8(p), p)$ forms a cycle, and so the result holds. ∎

We are now in position to state Rosenfeld's deletability criterion for a pixel p. We will first prove a more general result:

Proposition 11. *Assume that F satisfies the RFA and that $p \in F \backslash FG$. Let $F' = F \backslash \{p\}$ and $B' = B \bigcup \{p\}$. Then:*

$$C_k(F') - C_k(F) + C_{k'}(B) - C_{k'}(B') = CN_k(F,p) - 1. \tag{37}$$

Moreover, one of the following holds:

(*i*) $CN_k(F,p) = 0$ *and* (*a*) *holds,*
$C_k(F') = C_k(F)$, *and*
$C_{k'}(B') = C_{k'}(B) + 1.$

(*ii*) $CN_k(F,p) = 0$ *and* (*b*) *holds,*
$C_k(F') = C_k(F) - 1$, *and*
$C_{k'}(B') = C_{k'}(B).$

(*iii*) $CN_k(F,p) > 0,$
$C_k(F') \geq C_k(F)$, *and*
$C_{k'}(B') \leq C_{k'}(B).$

Proof. In case (*a*), the deletion of p from F creates a new hole in F. Thus (*i*) holds then. In case (*b*), p is an isolated pixel of F and so (*ii*) holds then. In case (*c*), as p is k-adjacent to pixels of F and k'-adjacent to pixels of B, the deletion of x from F can only split k-connected components of F or merge k'-connected components of B. Thus (*iii*) holds in this case.

There remains us only to prove (37). Given a figure F^* with background B^*, such that F^* satisfies the *FA*, the number

$$C_k(F^*) - C_{k'}(B^*) + 1$$

is the Euler number or genus of F^*, and we write it $g_{(k,k')}(F^*)$ (see [4]).

We may suppose that F satisfies the *Frame Assumtion (FA)*, which states that the frame FG is entirely included in B [4]; otherwise we extend the grid G by a layer of white pixels on each side of FG, and so the two sides of equation (37) remain unchanged, while the new figure satisfies the *FA*.

Then the left-hand side of (37) is equal to:

$$g_{(k,k')}(F') - g_{(k,k')}(F).$$

But in [4], the formulas (95) and (96) give the genus of a figure satisfying the *FA* in terms of the number of occurrences of certain types of configurations of black and white pixels among all 2×2-squares in G. Now the only 2×2-squares in G which form distinct configurations of black and white pixels in F and F' are those containing p. Thus $g_{(k,k')}(F') - g_{(k,k')}(F)$ depends only upon the squares containing p, in other words upon $N_8(p)$.

We have thus shown that the left-hand side of (37) depends only upon $N_8(p)$. Thus we may assume that $F \subseteq N_8(p)$ and so we have:

$$CN_k(F,p) = NC_k(F,p) = C_k(F \bigcap N_8(p), p) = C_k(F,p) = C_k(F',p)$$

and $C_{k'}(B) = C_{k'}(B') = 1$. Thus (37) becomes equivalent to

$$C_k(F') - C_k(F) = C_k(F', p) - 1,$$

which is none other than (11) with $T = F'$ and $S = \{p\}$.

Thus (37) holds in this case, and so it holds in general. ∎

Now we state Rosenfeld's deletability criterion for the pixel p. In [6] he proved that: p is k-deletable from F if and only if

$$CN_k(F, p) = 1. \tag{38}$$

It is easily seen that it is a consequence of (37).

Thinning algorithms consider the neighbourhoods of the pixels and delete these pixels when their 8-neighbourhood belongs to a certain set of configurations (satisfying the condition (38) of course). These configurations can be concisely described if the 8 pixels of the 8-neighbourhood $N_8(p)$ of a pixel p are numbered. The general use is to number them from 0 to 7 (or sometimes from 1 to 8) in circular order. We will assume this, and we will also suppose that the 4-neighbours of that pixel have even numbers. The situation is illustrated in Figure 5.

With this numbering of the 8-neighbours of a pixel p, it is possible to write the numbers $CN_4(F, p)$ and $CN_8(F, p)$ as an algebraic expression involving the values taken by the pixels numbered in this way.

Indeed, take any pixel $p \in F \backslash FG$; we define then the 8 numbers $f_p(i)$ (where $i = 0, \dots 7$) by

$$\begin{aligned} f_p(i) &= 1 \quad \text{if the 8-neighbour of } p \text{ numbered } i \text{ belongs to } F. \\ &= 0 \quad \text{otherwise.} \end{aligned} \tag{39}$$

Now the two numbers $CN_4(F, p)$ and $CN_8(F, p)$ can be expressed with the numbers of the form (39) thanks to the formulas of [9,10], which are called *(Yokoi's) 4- and 8-connectivity numbers.* We state them here:

Proposition 12 [9,10].

$$CN_4(F, p) = \sum_{j=0}^{3} \Big(f_p(2j) - f_p(2j) \cdot f_p(2j \oplus 1) \cdot f_p(2j \oplus 2) \Big) \tag{40}$$

and

$$CN_8(F, p) = \sum_{j=0}^{3} \Big(\overline{f_p(2j)} - \overline{f_p(2j)} \cdot \overline{f_p(2j \oplus 1)} \cdot \overline{f_p(2j \oplus 2)} \Big), \tag{41}$$

where $\oplus$ is the modulo 8 sum and $\overline{x} = 1 - x$ for every $x \in \{0, 1\}$.

Proof. Every $X \in \mathcal{C}(4, F \cap N_8(p), p)$ which neighbours some $Y \in \mathcal{C}(8, B \cap N_8(p), p)$ is characterized by the last pixel $y \in X \cap N_4(p)$ in the circular order determined by the numbering of the pixels of $N_8(p)$. If y is numbered i, then that number i is characterized by the following 3 properties:

(a) i is even (i.e., $y \in N_4(p)$).

(b) $f_p(i) = 1$ (i.e., $y \in F$).

(c) $f_p(i \oplus 1) = 0$ or $f_p(i \oplus 2) = 0$ (i.e., y is the last one in the circular order).

Now (a) means that $i = 2j$ and (b) and (c) together are equivalent to the statement:

$$f_p(i) - f_p(i) \cdot f_p(i \oplus 1) \cdot f_p(i \oplus 2) = f_p(i) \cdot (1 - f_p(i \oplus 1) \cdot f_p(i \oplus 2)) = 1. \tag{42}$$

As the right-hand side of (40) is the number of numbers $i = 2j$ satisfying (42), equation (40) must hold.

Finally we remark that $CN_8(F,p) = CN_4(B,p)$ and so (41) follows from (40) by interverting F and B and so by replacing every $f_p(i)$ by $\overline{f_p(i)}$. ∎

We can now sum up: a pixel $p \in F \backslash FG$ is k-deletable from F if and only if it satisfies Rosenfeld's criterion (38), which can be expressed with the help of Yokoi's connectivity numbers defined by (39), (40) and (41). This condition is the basis for the topological part of the deletion criterion in a sequential thinning algorithm.

However this condition does not allow us to check whether a subset K of F consisting of k-deletable pixels is itself k-deletable, a situation which arises in the deletion stages of a parallel thinning algorithm. We will study this problem in the next section and show that we can decide upon the k-deletability of K by looking at some small configurations of pixels, namely pairs, triples and quadruples of pairwise 8-adjacent pixels of K together with their 8-neighbourhood.

V. Parallel deletion and minimal non-deletable subsets

In Section III we showed that a strongly deletable subset of a figure can be deleted from it by a succession of deletions of individual pels. In Section IV we expressed the criterion for the deletability of a single pel in terms of the configuration of black and white pels in its 8-neighbourhood. With this criterion it is possible to check the topological validity of a *sequential* thinning algorithm.

But there is another class of thinning algorithms: *parallel* algorithms, in which at every deletion stage a criterion is applied to the pels of the figure, and *all* pels which satisfy it are deleted *together*. Such a criterion must contain as necessary condition the deletability of the pel, but this is not sufficient, because a figure can contain a non-deletable subset whose pels are all deletable individually. In fact, the necessary and sufficient condition for the topological validity of a parallel thinning algorithm is the following:

Given a figure F and a (parallel) deletion criterion $\mathcal{X}$, let K be a subset of F whose pixels satisfy $\mathcal{X}$; then K is deletable from F. (43)

Note that in this case, K is strongly deletable.

In this section we will determine when a deletion condition (of a parallel thinning algorithm) satisfies the criterion (43). It will then be possible to check the topological validity of every possible thinning algorithm. Finally, in Section VI, we will show how topological conditions can be combined with geometrical ones to give valid thinning algorithms.

Note: by "deletability", one means in fact x-deletability ($x = 4$, 8 or $(4,8)$).

We will attack our problem by the "minimal counterexample" approach. Then we will get "forbidden configurations" for parallel thinning algorithms.

Let U be a subset of $F \backslash FG$ such that U does not satisfy (43), but every proper subset of U does. Then U is characterized by the following 3 properties:

(i) $U \neq \emptyset$.
(ii) U is not x-deletable from F.
(iii) Every proper subset V of U is x-deletable from F (and is in fact strongly x-deletable by Section III).

Such a set U is a *minimal non-x-deletable subset of F*. In brief, we call it an *(x,F)-set*. For example, a pel which is not x-deletable from F is an (x,F)-set; it is a *trivial* (x,F)-set. We will say say that an (x,F)-set is *non-trivial* if it contains more than one pel.

It is clear that in a parallel thinning algorithm, condition (43) is satisfied if and only if the set K of pixels satisfying the deletion criterion cannot contain a (x,F)-set.

Such a subset of F which does not contain a (x,F)-set is called a *perfectly x-deletable* subset of F. It has an interresting property in view of Theorem 9, which characterized strongly x-deletable sets:

Let K be a subset of size t of F, where $t > 1$. Then K is perfectly x-deletable from F if and only if for every labelling $p_1, \dots, p_t$ of the elements of K, each p_i $(1 \leq i \leq t)$ is x-deletable from $F \backslash \{p_j \mid j < i\}$. (44)

This contrasts with strong deletability, where we know only that there is at least one labelling of the elements of K having that property.

The following result characterizes non-trivial (x,F)-sets:

Theorem 13: *Let U be a non-trivial (x,F)-set. Then one of the following holds:*

(i) U is a pair of 8-adjacent pels.
(ii) $x = 8$ and U is an isolated triangle (in other words, U is a triple of pairwise 8-adjacent pels and $U \in \mathcal{C}(8,F)$).
(iii) $x = 8$ and U is an isolated square (in other words U is a quadruple of pairwise 8-adjacent pels and $U \in \mathcal{C}(8,F)$).

The cases (ii) and (iii) are illustrated in Figure 6.

Proof. Let p and q be two arbitrary distinct elements of U. Write $U_p = U\setminus\{p\}$, $U_q = U\setminus\{q\}$, $U_{pq} = U\setminus\{p,q\}$, $F_p = F\setminus U_p$, $F_q = F\setminus U_q$, and $F_{pq} = F\setminus U_{pq}$.

As U_{pq} and U_q are both x-deletable from F and $\{p\} = U_q\setminus U_{pq}$, p is x-deletable from F_{pq} (by Lemma 4). As U_p is x-deletable from F, U is not and $\{p\} = U\setminus U_p$, p is not deletable from F_p (again by Lemma 4). It follows then by Rosenfeld's deletability criterion that $N_8(p)\bigcap F_p \neq N_8(p)\bigcap F_{pq}$. But $F_{pq} = F_p \bigcup \{q\}$, and so we deduce that $q \in N_8(p)$.

As p and q were arbitrarily chosen, the pels of U are all pairwise 8-adjacent, and so either (i) holds, U is a triangle or U is a square. We have only to show that in the last two cases (ii) and (iii) respectively:

(a) *U is a triangle*: Write $U = \{p_1, p_2, p_3\}$. We can write $N_8(U) = \{q_1, ..., q_{12}\}$. We illustrate U and $N_8(U)$ in Figure 7 (up to a rotation).

Suppose first that $q_4 \in F$. We know that p_3 is x-deletable from $F\setminus\{p_2\}$, but not from $F\setminus\{p_1,p_2\}$. In Figure 8 we show $N_8(p_3)$ in $F\setminus\{p_2\}$ and in $F\setminus\{p_1,p_2\}$. But it is then easily seen that Yokoi's connectivity numbers (see formulas (40) and (41)) are the same in both cases, which is a contradiction. Thus $q_4 \in B$.

Now suppose that $x \neq 8$. In Figure 9 we show $N_8(p_2)$ in F and $F\setminus\{p_1,p_3\}$. As p_2 is 4-deletable from F, it is easily seen from Figure 9(a) that we must have: q_8, q_9, q_{10}, q_{11}, $q_{12} \in F$. But it follows then from Figure 9(b) that p_2 is $(4,8)$-deletable from $F\setminus\{p_1,p_3\}$, in other words that U is $(4,8)$-deletable from F, a contradiction. Thus $x = 8$.

In Figure 10 we show $N_8(p_1)$ and $N_8(p_3)$ in $F\setminus\{p_2\}$. As both p_1 and p_3 are 8-deletable from $F\setminus\{p_2\}$, Rosenfeld's criterion implies that:

(1°) q_{11}, q_{12}, q_1, q_2, $q_3 \in B$ (see Figure 10(a)).
(2°) q_5, q_6, q_7, q_8, $q_9 \in B$ (see Figure 10(b)).

Thus $q_i \in B$ for $i \neq 10$. In Figure 11 we show $N_8(p_2)$ in F. The 8-deletability of p_2 from F implies that $q_{10} \in B$. Thus (ii) holds in this case.

(b) *U is a square*: We display U and its 8-neighbourhood in Figure 12.

Suppose first that $x \neq 8$. Then p_1 is 4-deletable from F and also from $F\setminus\{p_3\}$. But in Figure 13 we show $N_8(p_1)$ in $F\setminus\{p_3\}$ and in F, and it is easily seen that in Figure 13(a) and (b), Yokoi's 4-connectivity number is not the same, which is a contradiction. Thus $x = 8$.

In Figure 14 we show $N_8(p_1)$ in $F\setminus\{p_2,p_4\}$. As p_1 is 4-deletable from $F\setminus\{p_2,p_4\}$, Rosenfeld's criterion implies that q_{11}, q_{12}, q_1, q_2, $q_3 \in B$. A similar argument applied to p_2, p_3 and p_4 shows that $q_i \in B$ for every i. Thus (iii) holds in this case. ∎

This result allows us to describe in a rigorous way the requirements to be fulfilled by the deletion criterion of a parallel algorithm. Indeed, it must satisfy (43). Thus the algorithm is topologically valid if and only if the set of pels satisfying that deletion criterion contains no (x,F)-set as a subset. With Theorem 13, this condition is expressed as follows:

(i) A pel p satisfying the deletion criterion is a x-deletable pel of F.
(ii) If p and q are two 8-adjacent pels satisfying the deletion criterion, then p is x-deletable from $F\setminus\{q\}$.
(iii) If $k = 8$ *and U is an isolated triangle of F, then at least one pel of U does not satisfy the deletion criterion.*
(iv) Same as in (iii), but with "square" instead of "triangle".

As we explained in the preceding section, a deletion criterion is in general expressed in terms of the configuration of white and black pels in the 8-neighbourhood of the pel which is examined, and these 8 pels are numbered as in Figure 5. Let us write $\mathcal{S}$ for the set of all configurations for which the deletion criterion is positive. Then the conditions (*i*) to (*iv*) above become the following:

(i) Every $C \in \mathcal{S}$ *has its connectivity number* (40) *or/and* (41) *equal to* 1.
(ii) Consider the 4 grid portions of Figure 15. For any configuration D of black and white pels on anyone of them, with p and q being both black, we write C for the portion of D in $N_8(p)$, *C′ for the one in* $N_8(q)$, *and C″ for the configuration formed from C by changing q from black into white; if C,* $C' \in \mathcal{S}$, *then C″ has connectivity number* (40) *or/and* (41) *equal to* 1. (We give an example in Figure 16).
(iii–iv) If $k = 8$, *then the five sets* $\mathcal{T}_0$, $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$ *and* $\mathcal{T}_4$ *of configurations shown in Figure 17 contain each one configuration which is not in* $\mathcal{S}$.

Let us sum up: With the results of this section and the preceding one, we know the requirements for the topological validity of a thinning algorithm, sequential or parallel, whose deletion criterion is based on the configuration of white and black pels in the 8-neighbourhood of a pel of the figure: In a parallel algorithm, we require that all pixels satisfying the deletion criterion are deletable and do not form together minimal non-deletable sets. In sequential algorithms, we require that each pixel satisfying the deletion criterion is deletable from the portion of the figure which remains at the moment where that criterion is applied to that pixel. However, the result of a sequential thinning relies in general heavily on the scanning order of the figure, something which can create geometric distortions in the resulting skeleton, because a pixel which might be deleted in some scanning order might not in another one. As we will see in Section 7 which analyses two concrete examples, Hilditch's sequential thinning algorithm has a stronger topological requirement, which means in fact that the set of pixels which are deleted in a thinning stage form a perfectly deletable set as in parallel algorithms. Thus one can consider that the absence of minimal non-deletable subsets is also useful for the deletion criterion of a sequential thinning algorithm.

We have left out of consideration the geometrical requirements of a thinning algorithm. In the next section, we will explain briefly how geometrical conditions can be combined with topological ones to give algorithms which produce a good skeleton. Then in Section 7, we will see how our theory applies to two examples of thinning algorithms, one parallel and one sequential.

VI. Topological and geometrical conditions in thinning algorithms

Up to now we have made a detailed study of the topological requirements of a thinning algorithm, but we have left aside the geometrical ones. The reader might then wonder if there can be such a detailed analysis for these geometrical conditions.

Unfortunately, geometrical features are more intuitive and vaguely defined than topological ones. For example, it is difficult to explain in precise mathematical terms the geometrical similarity between the two figures shown in Figure 18, the second one being a thinned version of the first one.

In fact, the geometrical conditions used in most thinning algorithms are of a heuristic nature, and their validity is tested experimentally rather than mathematically.

The standard organisation of a thinning algorithm is in a succession of deletion stages, and in each one of them, a deletion criterion based on the 3×3-window is applied to all border pixels, sequentially or in parallel (according to the algorithm). The algorithm ends when no further pixel is deleted in a deletion stage.

The deletion criterion must combine the topological conditions that we have studied so far with some geometrical conditions to be defined.

We propose here two types of geometrical conditions for thinning algorthms. The first one, "no deletion of end-pixel", is used in most algorithms and is easy to implement, but it does not correspond to a very precise mathematical concept. The second one, "preservation of the distance-skeleton", was proposed in [2] and found independently by the author; its implementation is more time-consuming, but it relies on a precise mathematical formulation. Any study of geometrical requirements of thinning algorithms must take into account this trade-off between mathematical rigor and computational speed.

Let us now consider the first type of geometrical requirement. Suppose that at the beginning of a deletion stage we have a partially thinned figure T, and let $p \in T$. The condition can be formulated as follows:

If p is an end-pixel of T, then p cannot be removed from T during that deletion stage.

Our problem is precisely to define that vague notion of "end-pixel". For example, the two bottom pixels of Figure 18.b are end-pixels. In Figure 19 we give other examples of end-pixels in 8-connected skeletons. Roughly speaking, one can say that an end-pixel is a pixel at the extremity of a one-pixel thick portion of the figure.

Let us be more precise. A one-pixel thick set having extremities is what we call a simple open k-path; it is a k-path $(p_1, \dots, p_n)$ such that for $1 \leq i < j \leq n$, p_i is k-adjacent to p_j if and only if $j = i + 1$; its extremities are p_1 and p_n. Let us now say that p is a k-end-pixel of T if and only if there is a simple open k-path P such that $N_8(p) \bigcap T = N_8(p) \bigcap P$; in other words, the relation of p with T is the same as with a simple open k-path. Then it is easily seen that:

(*i*) *p is a 4-end-pixel if and only if $T \bigcap N_8(p)$ contains only one 4-connected component X which is 4-adjacent to p, and $|X| = 1$ or 2.* (45)

(ii) *p is an 8-end-pixel if and only if* $\mid T \bigcap N_8(p) \mid = 1$. (46)

Note that a k-end-pixel of T is k-deletable from it.

One can now take the following "no deletion of end-pixel" condition in a deletion stage. If the pixel p is to be deleted from T during that deletion stage, then it must satisfy the following:

— *For $k = 4$: The 4-connected components of $T \bigcap N_8(p)$ which are 4-adjacent to p contain together at least 3 pixels.* (47)

— *For $k = 8$: $T \bigcap N_8(p)$ contains at least 2 pixels.* (48)

Most algorithms take $k = 8$ because the resulting skeleton contains fewer pixels than with $k = 4$. It is interesting to note that in many cases (for example in the algorithms of Hilditch [3] and Rosenfeld [7] studied in the next section, which assume $k = 8$), it is precisely this condition which is chosen, namely that $\mid T \bigcap N_8(p) \mid \geq 2$.

Of course, the definition of end-pixels that we have given here is only a particular one. The concept is vague enough to admit other definitions.

Now we can consider the second type of geometrical requirement (from [2]). The idea is the following. Before applying the thinning algorithm to the figure, one computes its discrete distance-skeleton for the 4-distance or the 8-distance. Then one applies to that figure a thinning algorithm whose geometrical condition is simply:

If p is a pixel of the discrete distance skeleton of T, then p cannot be removed from T during that deletion stage.

This method is more rigorous than the one involving "end-pixels", but also more time-consuming, because the thinning process becomes a sequence of two distinct processes. This is a clear example of the trade-off between rigor and computational speed which appears in the study of geometrical requirements of thinning algorithms.

In most thinning algorithms, the geometrical condition used in the deletion criterion is of one of the two types described above. Of course the two can be combined, even partially. For example one can require the preservation of some significant parts of the distance-skeleton (for example those whose distance to the background is relatively large) together with end-pixels. In Hilditch's algorithm the possibility is given to reserve in advance a set of pixels of the figure which may not be deleted during any stage of the thinning process.

Now that we have described the standard geometrical prerequisites of most thinning algorithms, we have still some geometrical problems to consider.

We said at the end of the previous section that in sequential algorithms, the scanning order can slant the thinning process in a corresponding direction, because some pixels which are deleted in a particular scanning order are not in another one. We hinted at a possible solution by requiring that the pixels which are deleted during a deletion stage form a perfectly deletable set, as in Hilditch's algorithm. Another possibility is to apply successively all permutations of the group of symmetries of the square to the scanning order and then to use every distinct permuted version of the scanning order in turn during the successive deletion stages.

A similar problem arises in parallel algorithms due to the lack of symmetry of the set S of deletable configurations on a 3×3-window. Indeed, the need to prevent the deletion of non-deletable pairs of 8-adjacent deletable pixels implies that we must introduce some dissymmetry in S. For example we show a two pixel-thick horizontal portion of a figure in Figure 20.a. Clearly every pixel of it is deletable, but every pair of vertically adjacent pixels is not. Now a symmetrical deletion criterion would either delete all these pixels or none. We must thus choose and so either delete the bottom row or the top one (Figure 20.b). But then, as shows Figure 21, such a choice leads to a shift of the skeleton towards one side of the figure.

A classical solution is to apply successively all permutations of the group of symmetries of the square to the set S and then to use every permuted version of S in turn during the successive deletion stages.

Let us be more precise. Suppose that among the 8 symmetries of the square, there are exactly n ones which preserve S. Then n divides 8 and the action of the 8 symmetries of the square upon S produce m permuted versions $S_1 = S, \dots, S_m$ of S, where $m = 8/n$. Then every deletion stage divides itself into m deletion substages, where for every $r = 1, \dots, m$, the r^{th} deletion substage is simply a deletion stage in which the deletion criterion uses the configuration S_r.

Rosenfeld [7] popularized a particular version of this method (with $m = 4$) by the successive deletion of "north", "west", "south" and "east" pixels. A north pixel is a black pixel 4-adjacent to a white pixel just above it. West, south and east pixels are determined similarly (see Figure 22). Then one chooses a set S of deletable configurations such that:

(i) *Every configuration in S has a north pixel in its center.*

(ii) *S is preserved by the left-right symmetry (but not by any non-identity symmetry).*

Then $m = 4$ and the distinct sets S_r ($r = 2, 3, 4$) are obtained by successive quarter-turn rotations of S, and consist then in west, south and east pixels respectively. Then the thinning process becomes an alternance of deletions of north, then east, south and west pixels satisfying certain properties.

As will be seen in the next section with the analysis of Rosenfeld's algorithm, this method can lead to very simple deletion criteria.

There are other algorithms in which $m = 2$, but even then it is sometimes sufficient to use S only without dividing each deletion stage into 2 substages.

The geometrical aspects of the process of thinning that we have exposed in this section are, with the topological criteria outlined in the preceding sections, quite sufficient for the understanding of most thinning algorithms found in the litterature. In the next section we will study two particular thinning algorithms [3,7], and show how they verify in practice our theoretical requirements.

A survey of the properties of various thinning algorithms can be found in [8].

VII. Two examples

As an illustration of our characterization of the requirements of the thinning process, we will describe here two thinning algorithms chosen from the litterature. The first one is due to Hilditch [3] and is sequential, while the second one is parallel and was found by Rosenfeld [7]. We will analyse them and show that they satisfy the respective topological deletability conditions that we gave in Sections 4 and 5 for sequential and parallel algorithms, and also the geometrical requirement forbidding the deletion of end-pixels (see Section 6).

Hilditch's algorithm [3]

This algorithms assumes the 8-adjacency for the figure (and the 4-adjacency for the background). It produces thus an 8-connected skeleton (see [8]). Let us describe here the deletion criterion applied during each deletion stage.

We start with the original figure F. The algorithm allows us to reserve a subset U of F whose pixels cannot be deleted during any deletion stage. For example we can choose for U a significant portion of the discrete distance skeleton (see Section 6). Suppose that at the beginning of a deletion stage we have a partially thinned figure T. We scan successively the pixels of T, but as we will see, only those in the 4-border of T can be candidates for deletion. Suppose that we are on pixel p and that since the beginning of that deletion stage we have obtained a subset D of T consisting of pixels to be deleted during this deletion stage. Then the pixel p is to be deleted from T, in other words p must be added to D if and only if it satisfies all the following 6 conditions:

(1°) *p lies in the 4-border of T, i.e., p is 4-adjacent to a pixel of $G\setminus T$.*

(2°) *p does not belong to U.*

(3°) *p has at least two 8-neighbours in T.*

(4°) *p has at least one 8-neighbour in $T\setminus D$.*

(5°) *p is 8-deletable from T. (This can be seen from $N_8(p) \bigcap T$).*

(6°) *For every pixel $q \in D \bigcap N_8(p)$, p is 8-deletable from $T\setminus\{q\}$. (This can be seen from $N_8(p) \bigcap (T\setminus\{q\})$).*

As Hilditch noted, condition (6°) can be simplified as follows:

— First, one can assume that the scan of the image is done along some predefined order, for example from the top row to the bottom row and from left to right on each row; then only 4 among the eight 8-neighbours of p can belong to D, namely the pixels numbered 6, 7, 0 and 1 in Figure 5.

— Second, conditions (3°) and (5°) ensure that if one of the neighbours of p at a corner of $N_8(p)$ (in other words with an odd number in Figure 5) belongs to D, then it is not isolated in $N_8(p) \bigcap T$, and so its deletion from T does not alter the 8-connectivity number of p. Thus condition (6°) must only be tested with $q \in N_4(p)$.

Thus by combining these two simplifications, it follows that condition (6°) must be tested only for two pixels in $N_8(p)$.

We will now study the properties of this algorithm. In particular, we will show why this algorithm is topologically valid; note that from Section 4 it is clear that for this purpose we have only to show that if p satisfies the conditions (1°) to (6°), then p is 8-deletable from $T \backslash D$.

Suppose that during the deletion stage, the set D is successively built with the pixels $p_1, \ldots, p_n$; in other words each p_i satisfies the conditions (1°) to (6°) with the set $D_i = \{p_j \mid j < i\}$ in the place of D. Let $D = \{p_1, \ldots, p_n\}$. Condition (5°) implies that each $p_i \in D$ is 8-deletable from T. We claim now that each pair $\{p_i, p_j\}$ $(i \neq j)$ is also 8-deletable from T. Indeed, if p_i and p_j are not 8-adjacent, then this follows from Rosenfeld's criterion for deletability (see Section 4), while if they are 8-adjacent, this follows from condition (6°) applied to $p_{max(i,j)}$.

Now we can show that D contains no minimal non-8-deletable subset. Indeed, by Theorem 13 and by what we have said in the preceding paragraph, we have only to show that D does not contain a triangle or square X forming an 8-connected component of T. Now if p_i is the element of X having the largest index i, then $X \subseteq D_i \bigcup \{p_i\}$ and condition (4°) implies that $T \backslash D_i$ intersects $N_8(p_i)$. Thus X is 8-adjacent to $T \backslash X$ and so X is not an 8-connected component of T.

Thus D is perfectly 8-deletable from T, a situation which is analogous to the one in parallel algorithms, where one obtains a set D of pixels to be deleted, and D or any of its subsets is deletable.

This property of Hilditch's algorithm limits the dependence of the result of a deletion stage on the order in which the pixels are scanned; this dependence constitutes one of the main defects of most sequential algorithms, because a pixel which might be deleted in some scanning order might not in another order. It is therefore an important quality in the appraisal of this algorithm.

Now that we have analyzed the deletability properties of Hilditch's algorithm, justifying its topological validity, we will consider its efficiency, particularly in regards of the preservation of geometrical features.

It is clear that conditions (1°), (4°), (5°) and (6°) deal with the preservation of connectivity, and we used them in our analysis of the topological validity of the algorithm. Now condition (3°) is the classical condition used to prevent the deletion of end-pixels in the case where the 8-adjacency is chosen for the figure (see (..) in Section 6).

Moreover, condition (2°), which is optional, allows for the preservation of a set chosen in advance, for example a portion of the discrete distance skeleton. Thus the algorithm garantees the preservation of geometrical features. As was noted in [8], the result of the algorithm is a perfect 8-curve, in other words at the end of the thinning process, no further pixels must be deleted. (In several algorithms, the resulting skeleton contains 4-connected portions, which can be further thinned into 8-connected portions; for example one finds often an unnecessary pixel 4-adjacent to two 8-adjacent pixels, see Figure 23).

Hilditch's algorithm could be adapted to the 4-adjacency instead of the 8-adjacency. Then the whole of our argument on topological requirements would still be valid if we replaced 8 by 4, but condition (3°) would have to be changed, because the geometrical condition preventing the deletion of end-pixels takes another form in the 4-connected case. One can for example (see (47) in Section 6) suggest the following condition instead of (3°):

(3°') *The 4-connected components of* $N_8(p) \bigcap T$ *which are 4-adjacent to* p *contain together at least 3 pixels.*

Rosenfeld's algorithm [7]

Rosenfeld's paper [7] deals with the characterization of a particular type of parallel thinning algorithms, those based on a decomposition of the deletion stage into 4 deletion stages of north, west, south and east pixels respectively (see Section 6). While our characterization of parrallel thinning algorithms deals only with their topological requirements, Rosenfeld's characterization of that particular case considers not only topological conditions, but also geometrical ones. It gives a very simple general form for such a type of parrallel algorithms.

As Hilditch's algorithm, Rosenfeld's general form of parrallel algorithms assumes the 8-adjacency on the figure. We will deal with the 4-connected case later.

Consider a parallel deletion stage applied to north pixels in a figure T, which deletes all north pixels of T whose 8-neighbourhood in T form certain configurations. We require two things:

(1°) That this deletion does not alter the topological structure of T. *(Topological requirement).*

(2°) That if T is a simple 8-path , then no pixel of T is deleted. *(Geometrical requirement).*

(Note: A simple k-path is a simple closed k-path in the sense of [4,6] or a simple open k-path in the sense of Section 6; in other words, it is an 8-path $(p_1, \dots, p_n)$ such that $p_i \in N_8(p_j)$ if and only if $| i - j |= 1$ or $\{i, j\} = \{1, n\}$ if that path is closed)

Rosenfeld showed that (1°) and (2°) are equivalent to the following: For every pixel p satisfying the deletion condition of that deletion stage,

(a) *p is 8-deletable,* and

(b) *p has at least two 8-neighbours in T.*

With Theorem 13, Rosenfeld's result is easy to prove. We first show that conditions (a) and (b) are necessary. It is obvious that condition (a) is necessary for (1°). Now condition (b) is necessary for (2°). Indeed, suppose that a pixel having only one 8-neighbour in T along some direction can be deleted. Take thus a simple path having one end-pixel whose sole neighbour in that path stands in that direction; then that pixel can be removed from that path, and (2°) is contradicted.(This argument is illustrated in Figure 24).

Now we will prove that (a) and (b) are sufficient. It is clear that (b) garantees (2°). Now we must prove (1°). For this purpose, we must only prove that with (a) and (b), the set of north pixels which are deleted does not contain a minimal non-8-deletable subset.

We know by Theorem 13 that such a set must be a pair of 8-adjacent 8-deletable north pixels or a triangle or a square. But it is easily seen that in a triangle or a square, at least one pixel is not a north pixel, and so it cannot belong to the set of deleted pixels. Thus it must be a pair $\{p, q\}$ of 8-adjacent 8-deletable pixels. Up to a symmetry, the configuration formed by $N_8(p) \bigcup N_8(q)$ is one of the two configurations shown in Figure 25. We have only to show that in each of the two configurations, p has the same 8-connectivity number (41) in T as in $T \backslash \{q\}$.

Consider first the case where p and q are 4-adjacent (Figure 25.a). Let x and y be the respective values (0 or 1) taken by the two respective south neighbours of p and q. By (b), p has an 8-neighbour $r \neq q$ in T. By (a), r and q are in the same 8-connected component of $T \bigcap N_8(p)$. It follows then that we must have $x = 1$ or $y = 1$, in other words $\overline{x} \cdot \overline{y} = 0$. The portion of the 8-connectivity number of p in which the value $v(q)$ (0 or 1) of q intervenes is:

$$\overline{x} - \overline{x} \cdot \overline{y} \cdot \overline{v(q)} + \overline{v(q)} - \overline{v(q)} \cdot \overline{0} \cdot \overline{0} = \overline{x} - 0 \cdot \overline{v(q)} + \overline{v(q)} - \overline{v(q)} \cdot 1 = \overline{x},$$

and so the connectivity number of p is independent of the value $v(q)$ of q.

Consider now the case where p and q are diagonally adjacent (Figure 25.b). Let z be the value (0 or 1) taken by the south neighbour of p. Here (b) and (a) imply that $z = 1$. The portion of the 8-connectivity number of p in which $v(q)$ intervenes is:

$$\overline{z} - \overline{z} \cdot \overline{v(q)} \cdot \overline{0} = 0 - 0 \cdot \overline{v(q)} \cdot 1 = 0,$$

and so the connectivity number of p is again independent of the value $v(q)$ of q.

Thus, as p is 8-deletable from T, it is also 8-deletable from $T \backslash \{q\}$, and so $\{p, q\}$ cannot be a minimal non-8-deletable subset of T. Thus (1°) must hold. ∎

The reader will note that the deletion criterion (a, b) is a very simple one, and this makes Rosenfeld's algorithm attractive.

If one chooses the 4-adjacency for the figure, then condition (b) becomes:

(b') *The 4-connected components of $N_8(p) \bigcap T$ which are 4-adjacent to p contain together at least 3 pixels.*

This change is the same as from condition (3°) to (3°′) in Hilditch's algorithm. Then our proof of the equivalence between the two sets of 2 conditions remains the same, with the following changes:

— In Figure 25.a, we must have $x = y = 1$, and the portion of the 4-connectivity number of p in which $v(q)$ intervenes becomes:

$$x - x \cdot y \cdot v(q) + v(q) - v(q) \cdot 0 \cdot 0 = 1 - v(q) + v(q) - 0 = 1,$$

which is independent of $v(q)$.

— In Figure 25.b, z can take any value, and the portion of the 4-connectivity number of p in which $v(q)$ intervenes becomes:

$$z - z \cdot v(q) \cdot 0 = z - 0 = z,$$

which is again independent of $v(q)$.

VIII. Conclusion

In this report, we have studied in detail the topological aspects of the thinning process, and touched briefly upon its geometrical aspects. As shows our analysis of two examples of thinning algorithms in Section 7, namely Hildtch's [3] and Rosenfeld's [7], our results allow us to verify the validity of the various algorithms existing in the literature and to understand their features.

We recall that, starting with topological requirements of thinning, we arrived in Section 2 at the concepts of *k-deletable* and *strongly k-deletable* subsets of a figure F. We explained why the portion of F which is deleted during the thinning process should be strongly k-deletable from it. In Section 3 we showed that a subset K of F is strongly k-deletable from it if and only if it can be deleted by a succession of deletion of k-deletable pixels. In particular, it follows that the deletion of a strongly k-deletable subset does not change the topological structure of the figure. A major consequence of this result is that the classical method used for thinning, the succession of deletions of individual pixels, is theoretically justified.

The deletability condition for a single pixel and its numerical expression is well-known, we recall it in Section 4. This criterion allows us to check the topological validity of sequential thinning algorithms. On the other hand, the topological validity of parallel algorithms require a stronger criterion, namely that the pixels which are deleted during one pass form a deletable set whose subsets are all deletable. This type of sets is characterized in Section 5. Thus we are able to state whether a given thinning algorithm, sequential or parallel, preserves the topological structure of the figure or not.

We can thus conclude that with our results, the topological features of the usual form of thinning algorithms, that is a succession of deletion stages consisting each in a parallel or sequential combination of deletions of individual pixels, are completely known.

We think that an important challenge for further researches on the subject of thinning would be to characterize in a rigorous manner the geometrical requirements of that process and their relations with the topological ones, a problem which was briefly considered in Section 6. In particular, one can ask if these requirements can be expressed in another form than "no end-pixel deletion" or "no deletion of a pixel of the discrete distance-skeleton".

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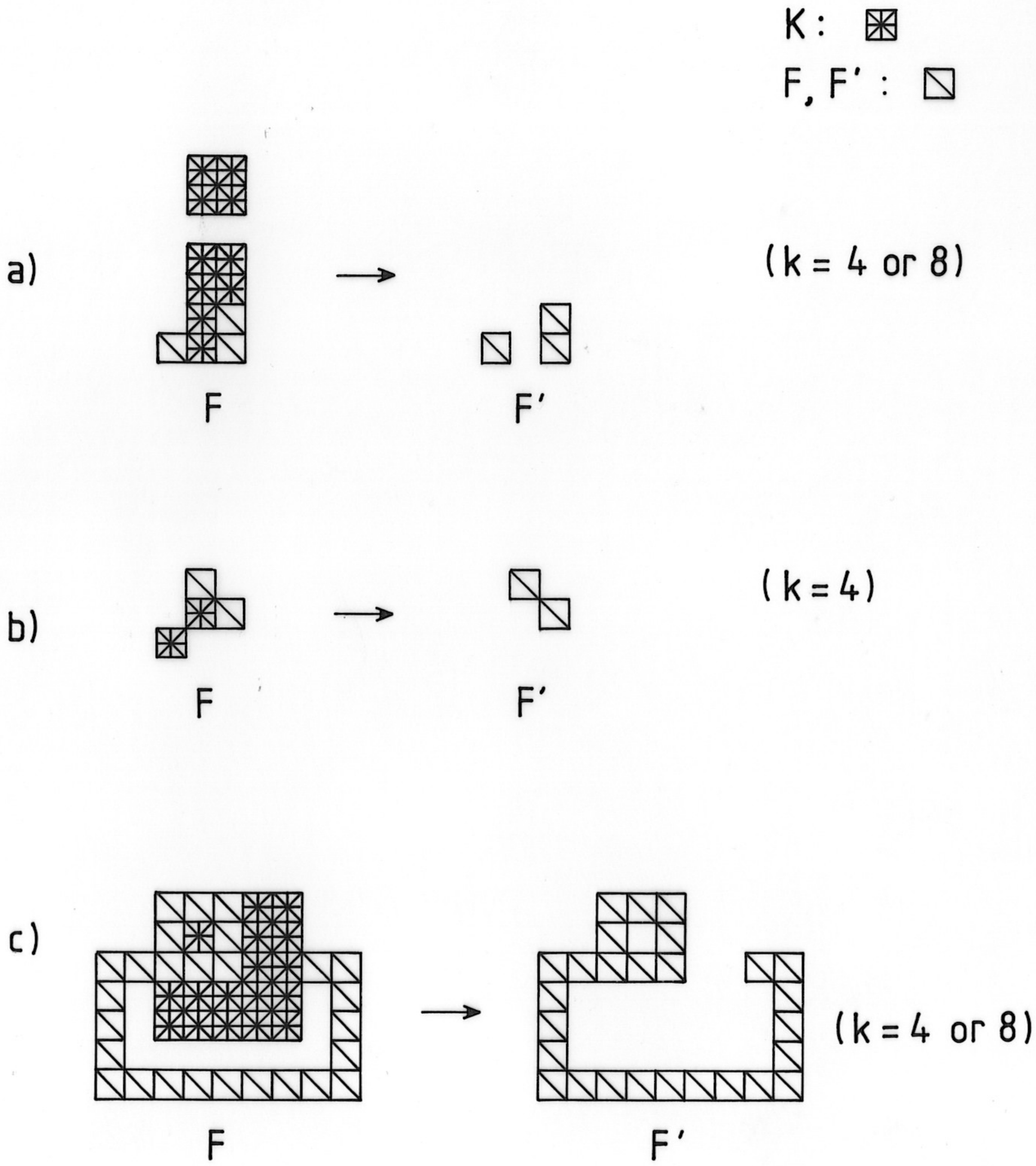
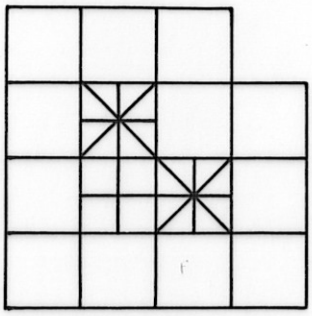
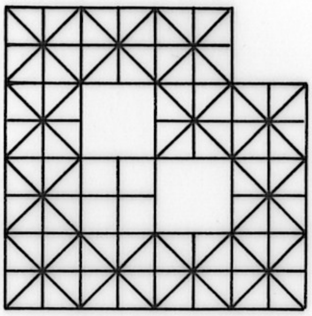


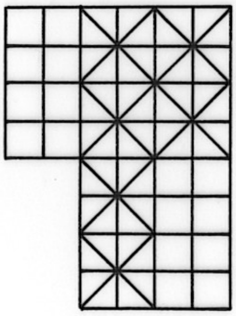
FIG. 1



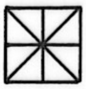
8-deletable but
not 4-deletable.

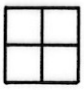


4-deletable but
not 8-deletable.



both 4- and 8- deletable

 : F'

 : K

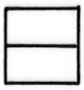
 : B

FIG. 2

T	T	
	p	T

$NC_4(T,p) = 2$

$NC_8(T,p) = 1$

		T
T	p	
T	T	

$NC_4(T,p) = 1$

$NC_8(T,p) = 2$

T		
	p	T
	T	

$NC_4(T,p) = 2$

$NC_8(T,p) = 2$

T	T	T
	p	T

$NC_4(T,p) = 1$

$NC_8(T,p) = 1$

FIG. 3

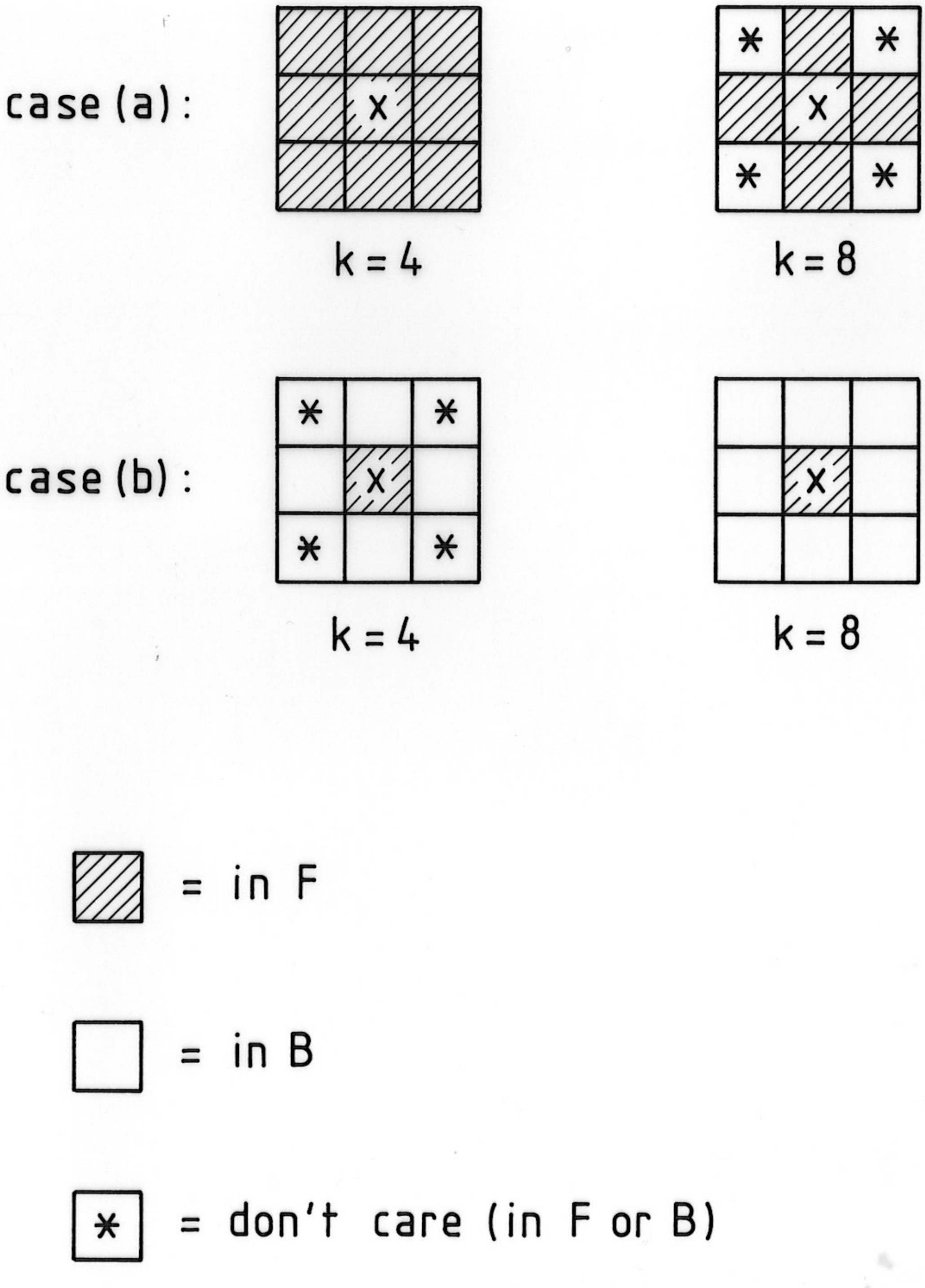


FIG. 4

P_7	P_0	P_1
P_6	P	P_2
P_5	P_4	P_3

FIG. 5

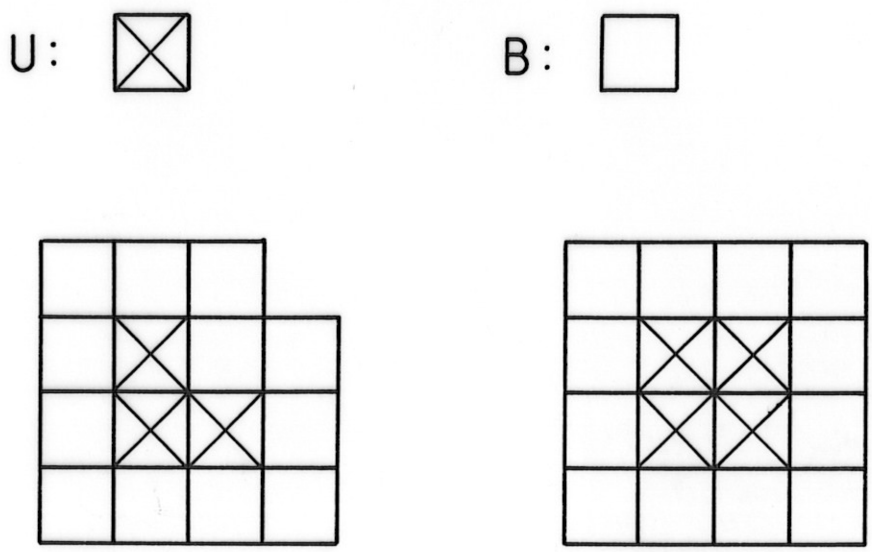


FIG. 6 Two (8, F) sets.

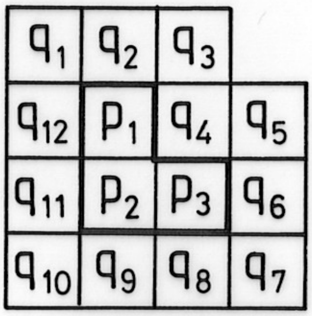


FIG. 7

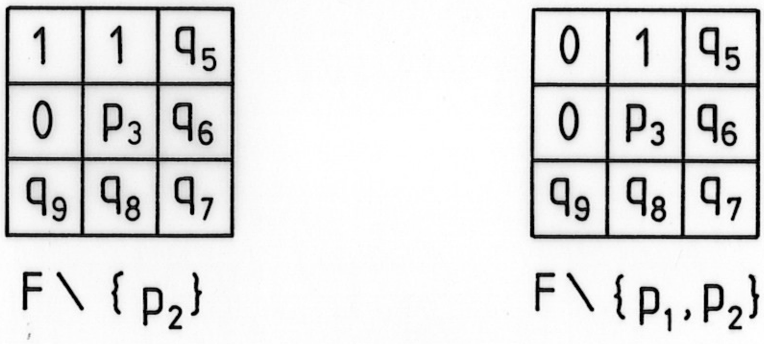


FIG. 8

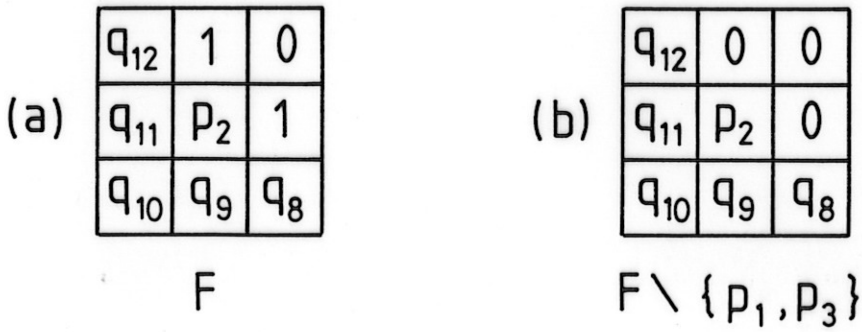


FIG. 9

(a)

q_1	q_2	q_3
q_{12}	p_1	0
q_{11}	0	1

$F \setminus \{p_2\}$

(b)

1	0	q_5
0	p_3	q_6
q_9	q_8	q_7

$F \setminus \{p_2\}$

FIG. 10

0	1	0
0	p_2	1
q_{10}	0	0

F

FIG. 11

q_1	q_2	q_3	q_4
q_{12}	p_1	p_2	q_5
q_{11}	p_4	p_3	q_6
q_{10}	q_9	q_8	q_7

FIG. 12

(a)

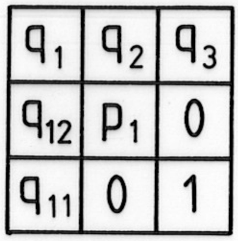
q_1	q_2	q_3
q_{12}	p_1	1
q_{11}	1	0

(b)

q_1	q_2	q_3
q_{12}	p_1	1
q_{11}	1	1

FIG. 13

FIG. 14



$F \setminus \{p_2, p_4\}$

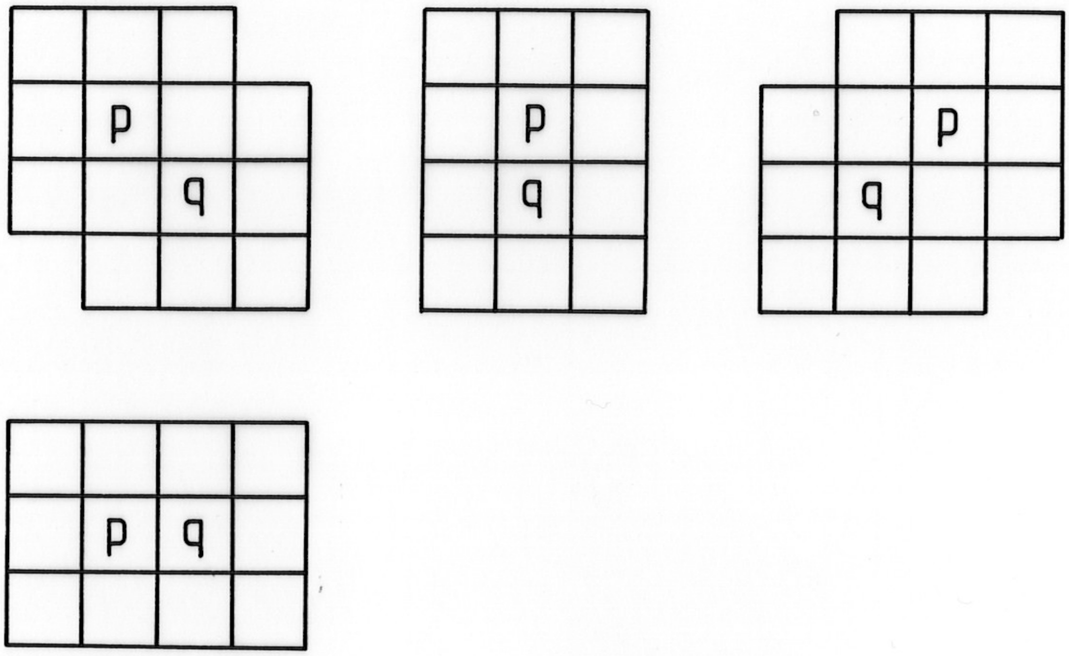


FIG. 15

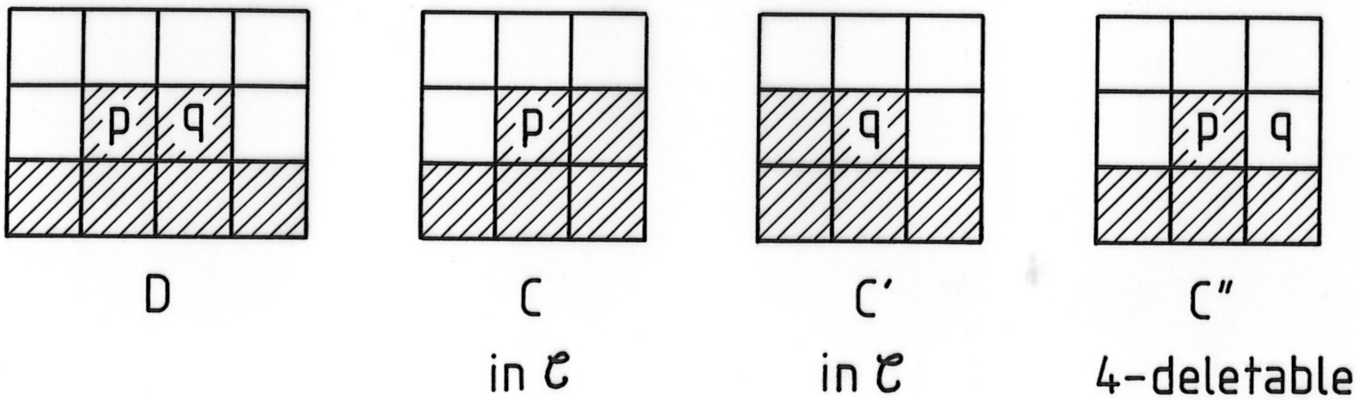


FIG. 16 Example with x= 4

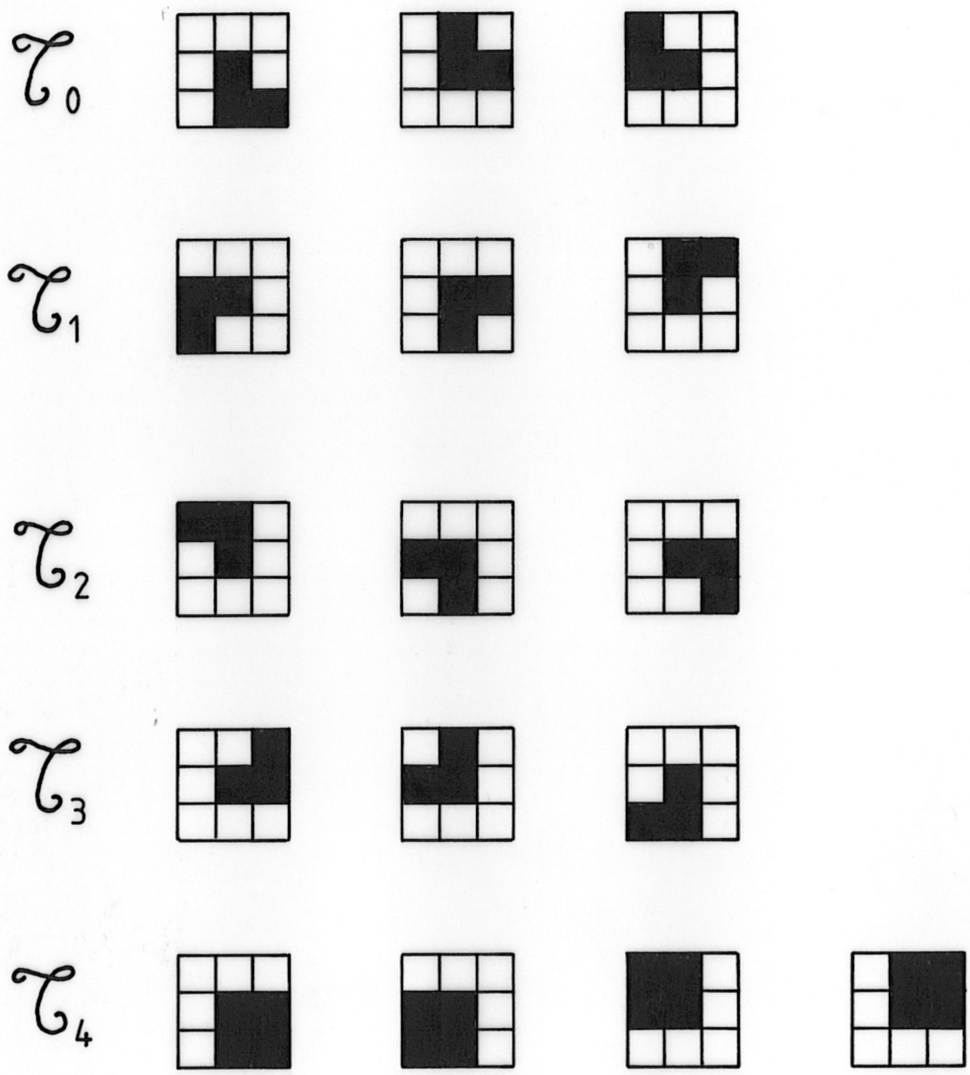


FIG. 17

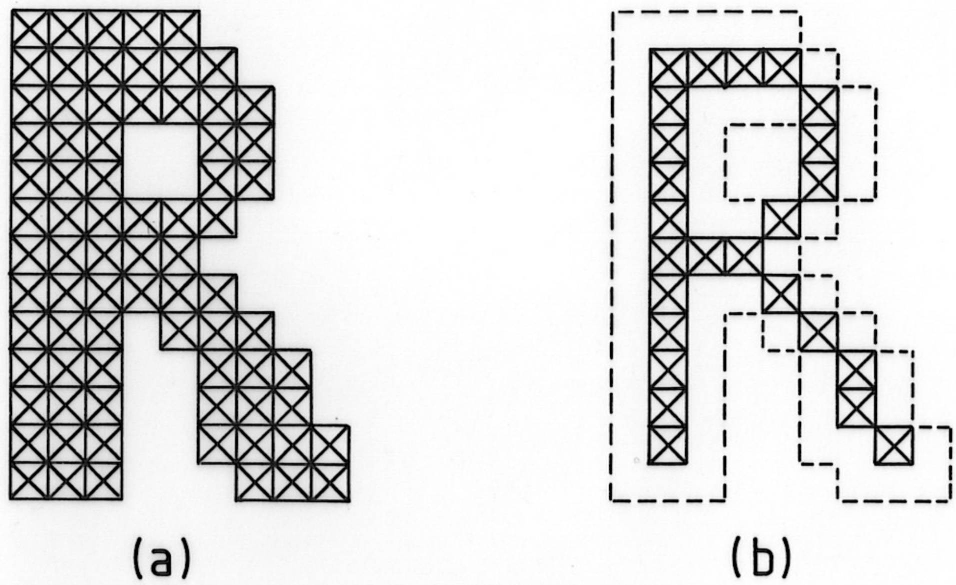


FIG. 18

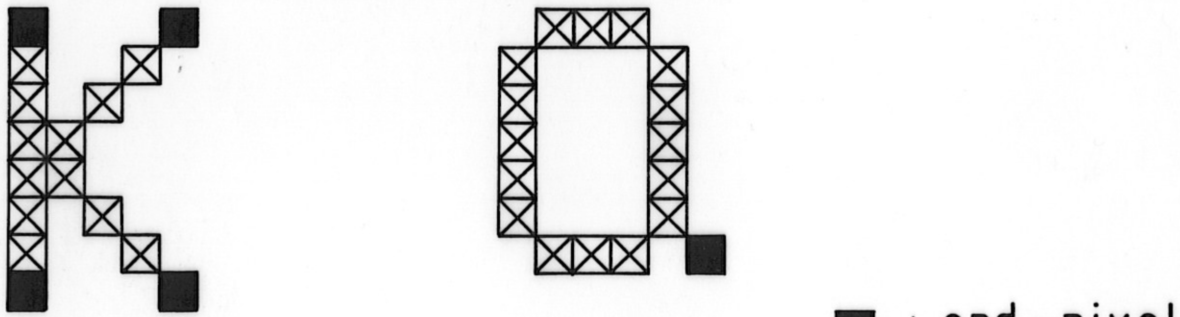


FIG. 19

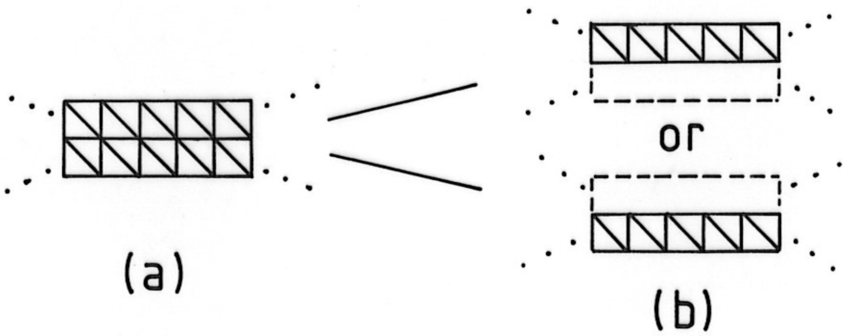


FIG. 20

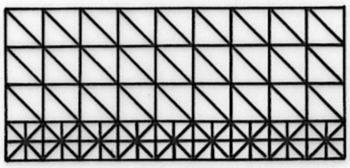


FIG. 21

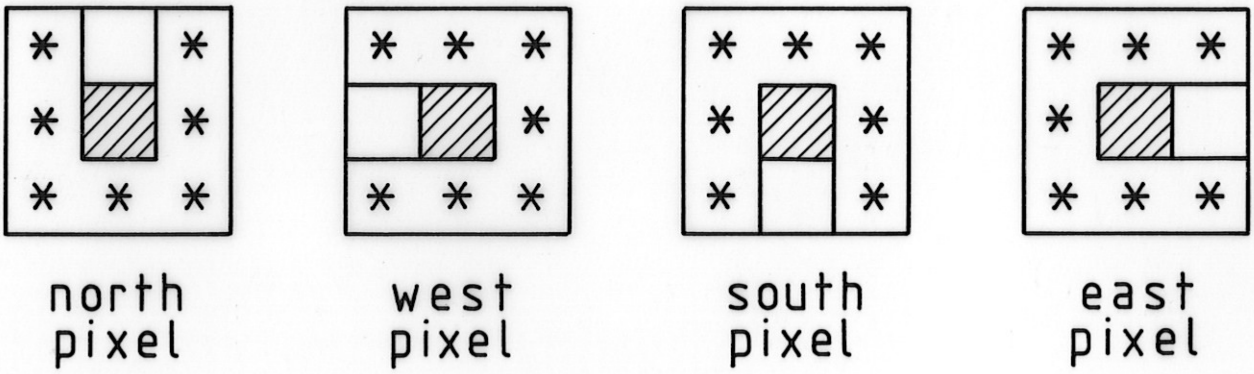


FIG. 22

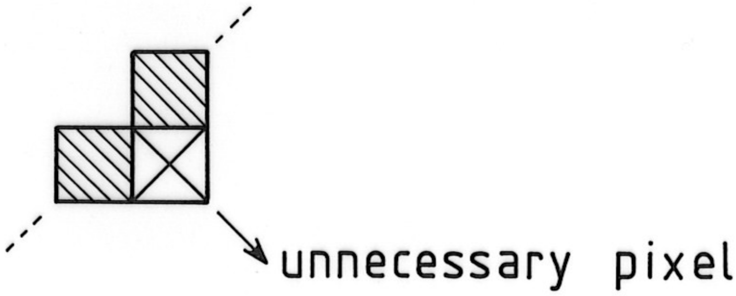


FIG. 23

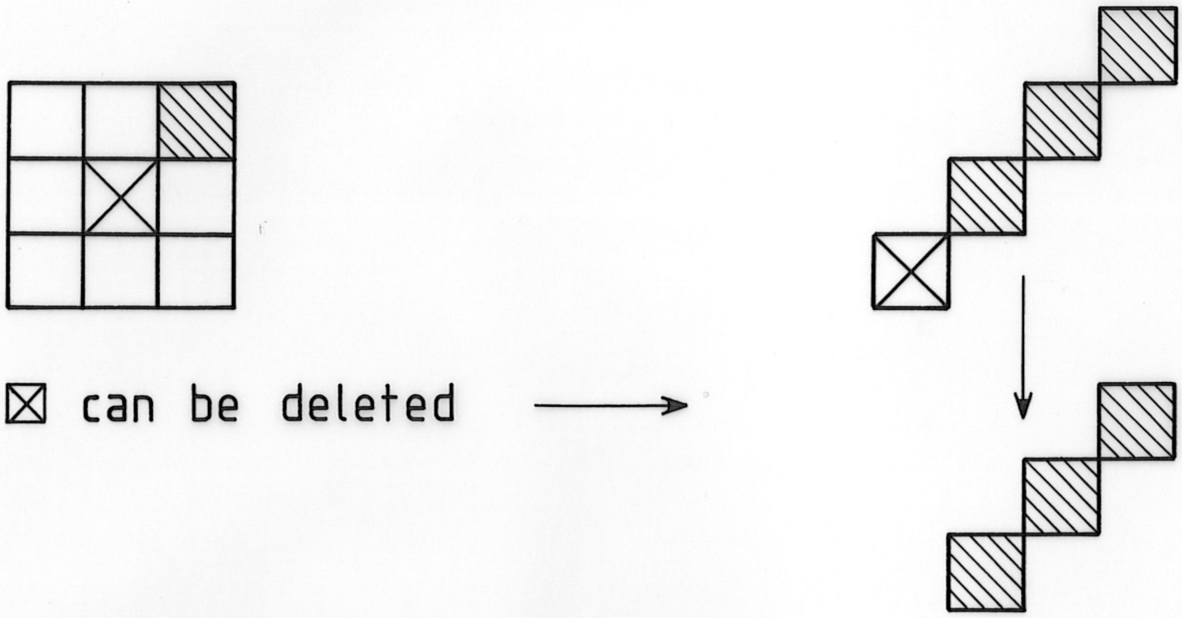


FIG. 24

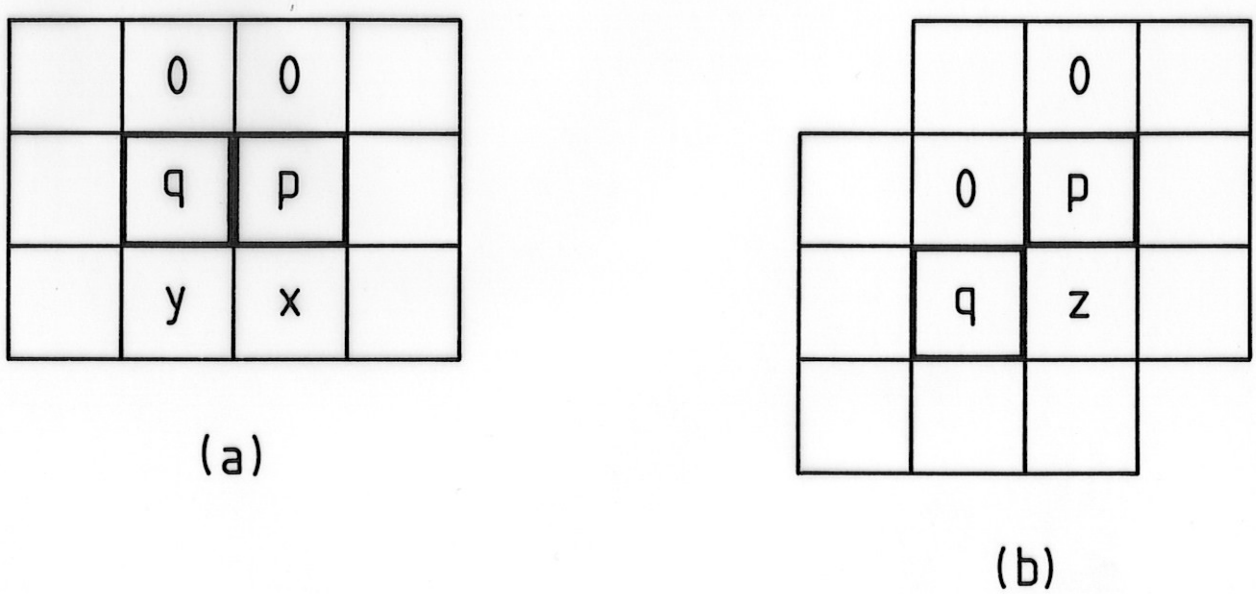


FIG. 25