MBLE Research Laboratory
2 av. Van Becelaere
1170 - Brussels - Belgium

REPORT R456

Addendum to R454
("Digital processing of binary
images on a square grid, I :
Elementary topology and geometry")

C. Ronse

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Abstract. This report is a complement to R454. We study first the turns made by a simple closed path or by an edge between two connected components. Then we define a discrete correspondent to the topological isomorphism (called homeomorphism) between two 2-tone images, and show several of its properties.

Erratum to R456

- page 13 : - lines 5,6 : interchange "k" and "k'".
  - line 11 : "α_{ii}" instead of "α_{ij}" .
  - lines 17,18 : "d_k" instead of "d".

Addendum to R456

- Let I and I' be two images satisfying the RFA and having isomorphic oriented (k,k') - neighbourhood trees T and T' . Let ψ be a graph isomorphism $T \to T'$. Define the relation θ between G and G' by :

$$x \; \theta \; x' \text{ if } x \in C \; , \; x' \in C', \text{ when } C\psi = C'.$$

Then θ is a topological isomorphism for the adjacency matrix :

$$\begin{pmatrix} k' & \{4,8\} \\ \{4,8\} & k \end{pmatrix}$$

The proof of this result is immediate and is left to the reader .

- It is also easy to see that for an image-preserving relation θ on G (i.e. satisfying (5°)) such that $x\theta x$ for every $x \in G$, then θ is a topological isomorphism if and only if the condition (35) in (6°) on the connectivity of $\rho(x)$ and $\lambda(x)$ is satisfied.

Erratum to R. 454

- Page 1-36 : Theorem 3, point (v).

  Interchange "$\varepsilon=1$" and "$\varepsilon=-1$" in the statement.

- Page 1-60 : line 11.

  Replace "is the set of" by "contains"

- Page 1-61 : line 2.

  Replace "$(v,w-1) \in B$" by "$(v,w-1) \in Y$" .

§I. Turns along a simple closed k-path.

Consider a simple closed k-path $P=(x_0,\ldots,x_{n-1})$. On each pel x_i, P makes a turn determined as follows :

Let P_j be the center of the pel x_j as a world object, and let D_j be the directed line joining P_j and $P_{j\oplus 1}$ $(j=0,\ldots,n-1)$ (see Figure A-1(a)). Then the turn $\tau(x_i)$ on x_i is the oriented angle that D_i makes with $D_{i\ominus 1}$ $(-\pi \leqslant \tau(x_i) \leqslant \pi)$ (see Figure A-1(b)).

It is easy to see that the only values that $\tau(x_i)$ can take are :

$$\begin{aligned} \tau(x_i) &= -\pi/2,\ 0,\ +\pi/2 \quad \text{for } k=4 \ . \\ &= -\pi/2,\ -\pi/4,\ 0,\ \pi/4,\ \pi/2 \quad \text{for } k=8. \end{aligned} \tag{1}$$

Let us write :

$$\tau^*(x_i) = \frac{2}{\pi}\,\tau(x_i) \tag{2}$$

Thus $\tau^*(x_i)$ is the number of quarter turns made by P on x_i.

We have :

$$\begin{aligned} \tau^*(x_i) &= -1,\ 0,\ +1 \text{ for } k=4 \ . \\ &= -1,\ 0,\ +\tfrac{1}{2},\ +1 \text{ for } k=8 \ . \end{aligned} \tag{3}$$

The different possibilities (up to a rotation) for the configuration $(x_{i\ominus 1}, x_i, x_{i\oplus 1})$ are shown in Figure A-2, together with the corresponding value of $\tau^*(x_i)$.

We have the following result :

Proposition 1. Let $P=(x_0,\dots,x_{n-1})$ be a simple closed k-path. We have :

$$\tau^{*}(P) = \sum_{i=0}^{n-1} \tau^{*}(x_i) = 4\varepsilon(P) \quad ,$$

where $\varepsilon(P)=-1$ if P leaves O(P) on its left and $\varepsilon(P)=1$ if P leaves O(P) on its right.

If we refer to Theorem 3 of R454, then $\tau^{*}(P)=4\alpha(y)$ for $y \in I(P)$.

Note that this result does not require the FA.

Proof. Suppose that the result is false and that P is counterexample such that n is minimal. If k=8, we assume further that the number d of pairs $\{x_i, x_{i\oplus 1}\}$ such that $x_{i\oplus 1} \in N_4(x_i)$ is minimal.

Consider the configuration B(m) (m=1,2,...) shown in Figure A-3. If P contains no such configuration (up to a rotation), then P has the shape of a stair (see Figure A-4) and so P cannot be closed. Thus P contains such a configuration B=B(m), and we can choose B such that m is minimal. In Figure A-5, we show B and some neighbouring pels. It is clear that $s_1,\dots,s_m$ belong all to the background B of P. Consider the set $T=\{t_1,\dots,t_m\}$ (see Figure A-5). Suppose that $T \subseteq B$. If k=4, then we build the path P' from P as follows : delete B and replace it by $\{a,s_1,\dots,s_m,b\}$. Then n is replaced by n'=n-2 in P'. If k=8, then P' is built as follows : delete B and replace it by $\{s_1,\dots,s_m\} \cup R$, where $R \subseteq \{a,b\}$ and :

(i) $a \in R$ iff $t_0 \in P$.

(ii) $b \in R$ iff $t_1 \in P$.

Then (n,d) is replaced by (n',d') in P', where $n'=n-1+|R|$ and $d'=d-|R|$.

Now for both $k=4$ or $k=8$ it is easily seen that P' is a simple closed k-path, that $\varepsilon(P')=\varepsilon(P)$ and $\tau^*(P')=\tau^*(P)$. As P was a minimal counterexample, the result holds for P', and so it must also hold for P. Thus P is not a counterexample in this case. Therefore $T \cap P \neq \emptyset$

$$\begin{aligned} \text{Let } T' &= T \text{ if } k=8 \\ &= T \cup \{t_0, t_{m+1}\} \text{ if } k=4 \end{aligned}$$

Consider a maximal connected segment S of $T' \cap P$ ($S = \{t_u \mid i \leqslant u \leqslant j\}$, where $i \leqslant j$), such that $S \cap T \neq \emptyset$ (we know that such a segment exists by the preceding argument). Suppose that $d_k(S,B) > 1$, in other words :

$$\begin{aligned} 0 < i \leqslant j < m+1 &\quad \text{if } k=4 \\ 1 < i \leqslant j < m &\quad \text{if } k=8 \ . \end{aligned}$$

Let a' and b' be the two pels preceding and following S in P. Then $S \cup \{a',b'\}$ is a configuration of the type $B(m')$, where $m'=i-j-1$ if $k=4$ and $m'=i-j+1$ if $k=8$ (see Figure A-6). Then $m' < m$, which contradicts the minimality of m.

Thus $d_k(S,B)=1$. We can suppose that $d_k(S,a)=1$ (in other words $i=0$ for $k=4$ and $i=1$ for $k=8$). Otherwise we intervert the roles of i and j, a and b, etc... We know that $j \geqslant 1$ (since $S \cap T \neq \emptyset$). The situation is illustrated in Figure A-7. In cases (c) and (f), $B \cup S$ is closed and so $P= B \cup S$. But then it is easily seen that the result holds then. In the other cases, we make the following transformation to P.

- In case (a), we replace $(t_1, t_0, a, 0, 1)$ by $(t_1, s_1, 1, 2)$.
- In case (b), we replace $(t_1, t_0, a, 0, 1, 2, b)$ by (t_1, s_1, b).
- In case (d), we replace $(t_1, a, 1, 2)$ by $(t_1, s_1, 2)$.
- In case (e), we replace $(t_2, t_1, a, 1, 2)$ by $(t_2, s_1, 2)$.

Then P is replaced by P', where P' is also a simple closed k-path. Now the length n' of P' is smaller than n and so the result holds for P'. Morevoer we have $\varepsilon(P') = \varepsilon(P)$ and $\tau^*(P') = \tau^*(P)$, as it can easily be checked case by case. Hence the result holds also for P and so P cannot be a counterexample.

Note. The method used to prove this result could also be used for the proof of point (v) of Theorem 3 of R454.

§II. Turns in edges and Euler numbers.

Consider a figure F with background B. Let X be a k-connected component of F and Y a neighbouring k'-connected component of B. Let $\varepsilon = \varepsilon^*(X,Y)$. We can write $\varepsilon = \{\varepsilon_0, \ldots, \varepsilon_{n-1}\}$, where $\varepsilon_{i\oplus 1}$ is the direct follower of ε_i for $i = 1, \ldots\ n-1$ (we assume here the RFA).

As in simple closed paths, we can define turns in edges. Let $\tau(\varepsilon_i, \varepsilon_{i\oplus 1})$ be the angle that $\varepsilon_{i\oplus 1}$ makes with ε_i. We have

$$\tau(\varepsilon_i, \varepsilon_{i\oplus 1}) = -\pi/2,\ 0,\ \pi/2 \quad . \tag{4}$$

Let us write :

$$\tau^*(\varepsilon_i, \varepsilon_{i\oplus 1}) = \frac{2}{\pi}\, \tau(\varepsilon_i, \varepsilon_{i\oplus 1}) \tag{5}$$

Thus $\tau^*(\varepsilon_i,\varepsilon_{i\oplus 1})$ is the number of quarter turns made by ε between ε_i and $\varepsilon_{i\oplus 1}$. We have :

$$\tau^*(\varepsilon_i,\varepsilon_{i\oplus 1}) = -1,0,+1 \ . \tag{6}$$

We show on Figure A-8 the different possibilities for the configuration $(\varepsilon_i,\varepsilon_{i\oplus 1})$ (up to a rotation) and the corresponding values of $\tau^*(\varepsilon_i,\varepsilon_{i\oplus 1})$.

The following result corresponds to Proposition 1:

<u>Proposition 2</u>. We have for $\varepsilon=\varepsilon^+(X,Y)$:

$$\tau^*(\varepsilon) = \sum_{i=0}^{n-1} \tau^*(\varepsilon_i,\varepsilon_{i\oplus 1}) = 4\lambda(X,Y) \ ,$$

where $\lambda(X,Y)=1$ if Y 4-surrounds X and $\lambda(X,Y)= -1$ if X 4-surrounds Y.

<u>Proof</u>. Suppose that k=4. As in the proof of Theorem 5 of R454, we subdivide each pel into 9 pels (see Figure 1-43). For any subset Z of G, let Z^* be corresponding subset of the new grid. Let $P^*=\delta_8(X^*,Y^*)$. Then P^* is a simple closed 4-path as shown in that theorem. Let $\varepsilon^*=\varepsilon^+(X^*,Y^*)$. We can write $\varepsilon^*=\{\varepsilon^*_{0,0},\varepsilon^*_{0,1},\varepsilon^*_{0,2},\ldots,\varepsilon^*_{i,0},\varepsilon^*_{i,1},\varepsilon^*_{i,2},\ldots$ $\varepsilon^+_{n-1,0},\varepsilon^*_{n-1,1},\ \varepsilon^*_{n-1,2}\}$, where for $i=0,\ldots,n-1$, $\varepsilon^*_{i,0},\varepsilon^*_{i,1}$ and $\varepsilon^*_{i,2}$ are the three consecutive elements of ε^* corresponding to ε_i (see Figure A-9). Then we see that $\tau^*(\varepsilon^*_{i,j},\varepsilon^*_{i,j+1})=0$ $(j=0,1)$ and that $\tau^*(\varepsilon_i,\varepsilon_{i\oplus 1})=\tau^*(\varepsilon^*_{i,2},$ $\varepsilon^*_{i\oplus 1,0})$.

Now from Figure A-10 one deduces that $\tau^*(\varepsilon^*_{i,2},\varepsilon^*_{i\oplus 1,0})= \sum_{p^*\in P^*_i} \tau^*(p^*)$, where P^*_i is the set of pels of P^* along the edges $\varepsilon^*_{i,1},\varepsilon^*_{i,2},\varepsilon^*_{i\oplus 1,0}$.

Therefore $\tau^*(\varepsilon) = \tau^*(P^*)$.

By Proposition 1, $\tau^*(P^*) = -4$ if P^* leaves $O(P^*)$ on its left (i.e. if $\delta_4(Y^*,X^*) \not\subseteq O(P^*)$ and $\tau^*(P^*) = +4$ if P^* leaves $O(P^*)$ on its right (i.e. if $\delta_4(Y^*,X^*) \subseteq O(P^*)$). Now it is clear that if X 4-surrounds Y, then X^* 4-surrounds Y^* and so $\delta_4(Y^*,X^*) \subseteq Y^* \subseteq G^*\backslash O(P^*)$; it is also clear that if Y 4-surrounds X, then X^* does not 8-surround Y^* and so that $\delta_4(Y^*,X^*) \subseteq Y^* \subseteq O(P^*)$.

Thus $\tau^*(\varepsilon) = \tau^*(P^*) = -4$ if X 4-surrounds Y and $\tau^*(\varepsilon) = \tau^*(P^*) = +4$ if Y 4-surrounds X. Thus the result holds in this case.

Now suppose that k=8. We can apply the result to $\varepsilon' = \varepsilon^-(X,Y) = \varepsilon^+(Y,X)$, interchanging the role of F and B. We get thus :

$\tau^*(\varepsilon) = 4\lambda(Y,X)$.

Now $\lambda(Y,X) = -\lambda(X,Y)$ and it is easily checked that $\tau^*(\varepsilon)) = -\tau^*(\varepsilon)$. Hence the result holds for ε in this case.

Now consider the edge $\varepsilon = \varepsilon^+(F)$ of a figure F (we still assume the RFA). We define $\tau_k^*(\varepsilon)$ as the sum of all $\tau^*(\varepsilon^+(X,Y))$, where X is a k-connected component of X and Y is a neighbouring k'-connected component of B. Let $N(k,F)$ be the set of all such pairs (X,Y). By Proposition 2 we have :

$$\frac{1}{4}\tau_k^*(\varepsilon) = \sum_{(X,Y) \in N(k,F)} \lambda(X,Y) \; . \qquad (7)$$

Now we define

$C(k,F) = \{$k-connected components of F$\}$.

$C'(k,F) = \{x \in C(k,F) \mid FG \subseteq X\}$.

$C(k',B) = \{$k'-connected components of B$\}$.

$C'(k',B') = \{Y \in C(k',B) \mid FG \subseteq Y\}$.

$$\mathcal{M}(k,F) = C(k,F) \cup C(k',B) \ .$$

$$\mathcal{M}'(k,F) = C'(k,F) \cup C'(k',B) \ . \tag{8}$$

Now there is a bijection π: $\mathcal{M}'(k,F) \to \mathcal{N}(k,F)$ defined as follows :

- For $x \in C'(k,F)$, $\pi(X)=(X,Y)$, where $(X,Y) \in \mathcal{N}(k,F)$ and Y 4-surrounds X.
- For $Y \in C'(k',B)$, $\pi(Y)=(X,Y)$, where $(X,Y) \in \mathcal{N}(k,F)$ and X 4-surrounds Y.

(9)

The fact that π is well-defined and is a bijection follows from Theorem 14 of T454.

Now by definition of $\lambda(X,Y)$ and by (7) and (9), we have :

$$\frac{1}{4}\tau_k^*(\varepsilon) = \sum_{X \in C_k'(F)} \lambda(\pi(X)) + \sum_{Y \in C_{k'}'(Y)} \lambda(\pi(Y))$$

$$= |C_k'(F)| - |C_{k'}'(B)| \tag{10}$$

But we have two cases in the RFA :

(i) $FG \subseteq F$ and so :

$$|C_k(F)| = |C_k'(F)| + 1$$
$$\text{and } C_{k'}(B) = C_{k'}'(B) \tag{11}$$

(ii) $FG \subseteq B$ and so :

$$|C_{k'}(B)| = |C_{k'}'(B)| + 1$$
$$\text{and } C_k(F) = C_k'(F) \tag{12}$$

Thus, combining with (10), we have :

- If $FG \subseteq F$, then :

$$\frac{1}{4}\tau_k^*(\varepsilon) = |C_k(F)| - |C_{k'}(B)| - 1 \tag{13}$$

- If $FG \subseteq B$, then :

$$\frac{1}{4}\,\tau_k^*(\varepsilon) = |C_k(F)| - |C_{k'}(B)| + 1 \tag{14}$$

We note that in (14) (the case where the FA holds), the right-hand side of the equality is in fact $g_{(k,k')}(F)$ (see §IX of R454). Therefore :

$$g_{(k,k')}(F) = \frac{1}{4}\,\tau_k^*(\varepsilon) \text{ if } FG \subseteq B\ . \tag{15}$$

In § IX of R454, we defined the configurations Q, T^* and D^*. We show them in Figure A-11.

Now a pair of consecutive edge elements $(\varepsilon_i,\varepsilon_{i\oplus 1})$ in $\varepsilon^+(X,Y)$ can be found only in configuration which are up to a rotation equal to Q, T^* or D^*. For Q and T^*, the configuration contains only one such pair, while for D^*, it contains two pairs, which depend on k. We illustrate these pairs, together with the value of $\tau^*(\varepsilon_i,\varepsilon_{i\oplus 1})$ in Figure A-12. If we write q^*,t^* and d^* for the number of configurations of type Q, T^* and D^* respectively (up to a rotation), then we get from the values of $\tau^*(\varepsilon_i,\varepsilon_{i\oplus 1})$ in Figure A-12 the following equalities :

$$\tau_4^*(F) = q-t^*+2d^* \tag{16}$$

$$\tau_8^*(F) = q-t^*-2d^*\ . \tag{17}$$

Combining with (15), we deduce from (16) and (17) the formulas (95) and (96) of R454 for the genus ($g_{(4,8)}(F)$ and $g_{(8,4)}(F)$) of F when the FA holds. These formulas were proved using a topological argument in the real plane. The proof presented here uses the properties of the grid G.

Note that the formulas (90), (91) and (92) of R454 can be derived from (95) and (96). Thus all arguments on the genus of a grid figure do not depend anymore on the topology of the real plane, but only on the properties of the grid G.

§ III. Topological isomorphisms between grid images.

In the real plane, a topological isomorphism (called *homeomorphism*) is defined as a permutation of that plane which induces a permutation of the set of open subsets of that plane. However two grid images can be considered as topologically equivalent, although there may be no bijection realizing that equivalence. For example, in Figure A-13, I and I' are topologically equivalent, but there may not be a bijection mapping F onto F'.

Thus we need to define a topological isomorphism in another way, by making a correspondence between parts of I and parts of I'. Such a correspondence must have suitable properties. We will find them by analogy with homeomorphisms in real planes.

Consider two grids G and G' on which we define the images I and I' respectively. We will not restrict ourselves to binary images. We suppose thus that for some set $K=\{0,\ldots,m-1\}$ $(m \geq 1)$ I is a map from G onto K and I' is a map from G' onto K.

If I and I' are topologically equivalent as real images, then there is a homeomorphism ψ of the real plane mapping the rectangle G onto the rectangle G' and such that $n \in K$, $\psi(I^{-1}(n))=I'^{-1}(n)$.

Now any set V of pels of G corresponds to a surface in the real plane, which is mapped by ψ onto a surface V^* inside G'. If we quantizise V^* by square-box quantization, we get a set V' of pels of G', and V' is the set corresponding to M. We write $V'=\rho(V)$. For a pel p, we write $\rho(p)$ for $\rho(\{p\})$.

Now for a set W' of pels of G', we can construct in the same way, but this time using ψ^{-1}, a set W of pels of G, and we write $W=\lambda(W')$. For a pel p', we write $\lambda(p')$ for $\lambda(\{p'\})$.

Now we define a relation θ between G and G', in other word a subset of $G\times G$, in the following way :

For $p \in G$ and $p' \in G$, $p\ \theta\ p'$ (i.e. $(p,p') \in \theta$) if and only if $\psi(p) \cap p' \neq \emptyset$ (where p and p' are considered here as subsets of the real plane). Note that this is equivalent to $p \cap \psi^{-1}(p') \neq \emptyset$.

We will now give the properties of θ, λ and ρ. Let $p \in G$, $p' \in G'$, $V \subseteq G$ and $V' \subseteq G'$. We have the following features :

(1°) Reciprocity :

$$p \in \lambda(p') \text{ iff } p\ \theta\ p' \text{ iff } p' \in \rho(p) \ . \tag{18}$$

Thus :

$$\rho(p) = \{q' \in G' \mid p\ \theta\ q'\} \ . \tag{19}$$

$$\lambda(p') = \{q \in G \mid q\ \theta\ p'\} \ . \tag{20}$$

(2°) Additivity :

$$\rho(V) = \bigcup_{q \in V} \rho(q) = \{q' \in G' \mid \exists q \in V,\ q\ \theta\ q'\} \ . \tag{21}$$

$$\lambda(V') = \bigcup_{q' \in V'} \lambda(q') = \{q \in G \mid \exists q' \in V',\ q\ \theta\ q'\} \ . \tag{22}$$

(3°) Totality :

$$\rho(p) \neq \emptyset \text{ and } \lambda(p') \neq \emptyset \ . \tag{23}$$

(4°) Frame preservation :

$$\text{If } V \cap FG \neq \emptyset, \text{ then } \rho(V) \cap FG' \neq \emptyset \ . \tag{24}$$

$$\text{If } V' \cap FG' \neq \emptyset, \text{ then } \lambda(V') \cap FG \neq \emptyset \ . \tag{25}$$

Note that (23) and (24) are respectively equivalent to the following two statements :

$$\text{If} \quad p \in FG\ , \text{ then } \rho(p) \cap FG' \neq \emptyset \quad . \tag{26}$$

$$\text{If} \quad p' \in FG', \text{ then } \lambda(p') \cap FG \neq \emptyset \quad . \tag{27}$$

For an infinite grid, the frame is the infinity and the condition is the same, but we replace for $X \subseteq G$ (resp. $X' \subseteq G'$) the statement "$X \cap FG \neq \emptyset$" (resp. "$X' \cap FG' \neq \emptyset$") by "$|X| = \infty$" (resp. "$|X'| = \infty$"). Note that we can have an isomorphism between a finite image and an infinite one.

(5°) Image preservation

$$\text{If} \qquad p\ \theta\ p', \text{ then } I(p) = I'(p') \quad . \tag{28}$$

Although θ was defined from the homeomorphism of the real plane, we will now attempt to characterize the isomorphism purely in terms of grid features. We start from a relation θ between F and G' and we define ρ and λ by the properties (1°) and (2°) of reciprocity and additivity, and we suppose that θ,λ and ρ satisfy the properties (3°), (4°) and (5°) of totality, frame preservation and image preservation.

For any $n \in K$, write $A_n = I^{-1}(n)$ and $A'(n) = I'^{-1}(n)$. Suppose that θ, λ and ρ are defined by (1°) and (2°) and that they satisfy (3°), (4°) and (5°). We have the following results :

- If $V \subseteq W \subseteq G$, then $\rho(V) \subseteq \rho(W)$. (29)

- If $V' \subseteq W' \subseteq G'$, then $\lambda(V) \subseteq \lambda(W)$. (30)

- If $V \subseteq G$, then $V \subseteq \lambda(\rho(V))$. (31)

- If $V' \subseteq G'$, then $V' \subseteq \rho(\lambda(V))$. (32)

<u>Lemma 3</u>. For any $n \in K$, $\rho(A_n) = A'_n$ and $\lambda(A'_n) = A_n$.

<u>Proof</u>. By (5°) we have $\rho(A_n) \subseteq A'_n$ and $\lambda(A'_n) \subseteq A_n$. Now if we apply the four preceding results, we get

$$A_n \subseteq \lambda(\rho(A_n)) \subseteq \lambda(A'_n) \text{ and}$$

$$A'_n \subseteq \rho(\lambda(A'_n)) \subseteq \rho(A_n) .$$

Hence the result holds.

As we defined it now, our isomorphism is purely set-theoretic, it does not take in account topological features, which are based on adjacencies. If $K=\{0,\ldots,m-1\}$, we define the <u>adjacency matrix</u> of I, written A(I). It is a symmetric m×m matrix

$$\begin{pmatrix} \alpha_{0,0} & \cdots & \alpha_{0,m-1} \\ \vdots & & \vdots \\ \alpha_{m-1,0} & \cdots & \alpha_{m-1,m-1} \end{pmatrix} \qquad (33)$$

where each $\alpha_{i,j}$ is a subset of $\{4,8\}$. Here $\alpha_{i,i}$ is the set of adjacencies which are taken in account inside A_i, while for $i \neq j$ $\alpha_{i,j}=\alpha_{j,i}$ is the set of adjacencies between A_i and A_j which we take in account. For example, in a binary image, we take generally the matrix :

$$\begin{pmatrix} k & 4 \\ 4 & k' \end{pmatrix} \tag{34}$$

We will now define the adjacency condition for the isomorphism between two images. Suppose that I and I' have the same adjacency matrix. Then we add to (1°-5°) the following condition

(6°) <u>Adjacency preservation</u>:

For any $i \in K$, $p, q \in A_i$, $p', q' \in A_i'$ and $k \in \alpha_{i,j}$,

- $\rho(p)$ and $\lambda(p')$ are k-connected . (35)

- if $d_k(p,q) \leqslant 1$, then $d_k(\rho(p),\rho(q)) \leqslant 1$. (36)

- if $d_k(p',q') \leqslant 1$, then $d_k(\lambda(p'),\lambda(q')) \leqslant 1$. (37)

For any $i \neq j \in K$, $p \in A_i$, $q \in A_j$, $p' \in A_i'$, $q' \in A_j'$ and $k \in \alpha_{i,j}$,

- if $d(p,q) = 1$, then $d(\rho(p),\rho(q)) = 1$. (38)

- if $d(p',q')=1$, then $d(\lambda(p'),\lambda(q')) = 1$. (39)

The conditions (35,36,37) ensure that θ preserves the k-connectivity inside A_i and A_i', while (38,39) are for the preservation of neighbourhood relations between A_i and A_j and between A_i' and A_j'.

We will now prove some more results.

Proposition 4. Let $i \in K$, $k \in \alpha_{i,i}$, and let $A_{i,1},\dots,A_{i,t}$ be the k-connected components of A_i. Then A_i' has t k-connected components $A'_{i,1},\dots,A'_{i,t}$, such that for $j=1,\dots,t$, we have :

$$\rho(A_{i,j}) = A'_{i,j} \quad .$$

$$\text{and} \quad \lambda(A'_{i,j}) = A_{i,j} \quad .$$

Proof. Let $A'_{i,j} = \rho(A_{i,j})$ for any $j=1,\dots,t$. Then each $A'_{i,j} \subseteq A_i'$ by Lemma 3 and is k-connected by (35) and (36). If we had $d_k(A'_{i,j}, A'_{i,j}) \leqslant 1$, then we would have $d_k(A_{i,j}, A'_{i,j'}) \leqslant 1$ by (37). Thus the k-connected components of A_i' are $A'_{i,1},\dots,A'_{i,t}$. Now it is clear from (35) and (37) that for each j $\lambda(A'_{i,j})$ is k-connected. But $A_{i,j} \subseteq \lambda(A'_{i,j})$ by (31), and as $A_{i,j}$ is a k-connected component of A_i and $\lambda(A'_{i,j}) \subseteq A_i$, $A_{i,j} = \lambda(A'_{i,j})$.

Note that by (38) and (39), if $k \in \alpha_{i,r}$, then a k_i-connected component of A_i ($k_i \in \alpha_{i,i}$) is k-adjacent to a k_j-connected component of A_j ($k_j \in \alpha_{j,j}$) if and only if the corresponding k_i-connected component of A_i' is k-adjacent to the corresponding k_j-connected component of A_j'.

Before going further, let us define :

$$\xi = \{X \subseteq G \mid \lambda(\rho(X)) = X\} \tag{40}$$

$$\xi' = \{X' \subseteq G \mid \rho(\lambda(X)) = X'\} \tag{41}$$

If we consider the restriction $\rho|_\xi$ of ρ to ξ and the restriction $\lambda|_{\xi'}$ of λ to ξ', then $\rho|_\xi$ is a map $\xi \to \xi'$, $\lambda|_{\xi'}$ is a map $\xi' \to \xi$ and we have

$$\lambda|_{\xi'} \cdot \rho|_\xi = 1_\xi \tag{42}$$

$$\rho|_\xi \cdot \lambda|_\xi = 1_{\xi'}$$

Thus $\lambda|_{\xi'}$ and $\rho|_{\xi}$ constitute a bijection $\xi' \to \xi$ and its inverse.

Now we state the second result, which is on k-surrounding. As the property of k-surrounding is based on k-paths in G, we need to take k such that k-adjacency is taken in account between any two pels, in other words $k \in \alpha_{r,s}$ for any $r,s \in K$.

Proposition 5. Suppose that $k \in \alpha_{r,s}$ for any $r,s \in K$. Let $A,B \in \xi$. Then A k-surrounds B if and only if $\rho(A)$ k-surrounds $\rho(B)$.

Proof. We need only to prove the "if" part, because the "only if" part uses the same argument, interverting I and I', λ and ρ, ξ and ξ', A and $\rho(A)$, B and $\rho(B)$, etc...

Suppose that $\rho(A)$ k-surrounds $\rho(B)$. Then $\rho(A) \cap \rho(B) = \emptyset$. As $\rho(A \cap B) \subseteq \rho(A) \cap \rho(B)$, we must have also $A \cap B = \emptyset$. Let $p=(x_0,\dots,x_n)$ be a k-path from B to FG. For $i=0,\dots,n$, write $X_i=\rho(x_i)$. By (4°) and (6°) there exist $y_i,z_i \in X_i$ $(i=0,\dots,n)$ such that $z_n \in FG'$ and for $0 \leqslant i < n$, $d_k\,(z_i, y_{i+1}) \leqslant 1$; moreover each X_i is k-connected. Thus there is a k-path $R=(y_0,\dots,z_0, y_1,\dots,z_1,\dots,y_{n-1},\dots,z_{n-1}, y_n,\dots,z_n)$ in $X_0 \cup \dots \cup X_n$. As $x_0 \in B$, $y_0 \in \rho(B)$. As $\rho(A)$ k-surrounds $\rho(B)$, R intersects $\rho(A)$. Thus $\rho(A) \cap X_i \neq \emptyset$ for some i. As $X_i=\rho(x_i)$, $x_i \in \lambda(p)$ for some $p \in \rho(A) \cap X_i$ and so $x_i \in \lambda(\rho(A))$. Now by definition $A=\lambda(\rho(A))$ and so P intersects A in x_i. As P was arbitrary, it follows that A k-surrounds B.

Now let us define the following four sets :

$$C = \{X \subseteq G \mid \exists i \in K,\ \exists k \in \alpha_{i,i},\ X \text{ is a k-connected component of } A_i\} \ . \tag{44}$$

$$C' = \{X' \subseteq G' \mid \exists i \in K,\ \exists k \in \alpha_{i,i},\ X \text{ is a k'-connected component of } A'_i\} \ . \tag{45}$$

$$D = \{\text{unions of elements of } C\}. \tag{46}$$

$$D' = \{\text{unions of elements of } C'\}. \tag{47}$$

Then by Proposition 4, $C \subseteq \xi$, $C' \subseteq \xi'$, and ρ and λ interchange C and C'. By additivity, $D \subseteq \xi$, $D' \subseteq \xi$, and ρ and λ interchange D and D'.

Therefore Proposition 5 is true for $A,B \in C$ or $A,B \in D$.

Now let us consider the general case, where A and B do not necessarily belong to ξ:

Proposition 6. Suppose that $k \in \alpha_{r,s}$ for any $r,s \in K$ Let $A,B \subseteq G$. If A k-surrounds B and $\rho(A) \cap \rho(B) = \emptyset$, then $\rho(A)$ k-surrounds $\rho(B)$.

Proof. Let $P(x_0,\ldots,x_r)$ be a k-path from $\rho(B)$ to FG'. As in the proof of Proposition 5, we build a k-path $R=(y_0,\ldots,y_m)$ in $\lambda(x_0) \cup \ldots \cup \lambda(x_n)$ such that $y_0 \in B$ and $y_m \in FG$. As a k-surrounds B, $y_i \in A$ for some j. Now $y_j \in \lambda(x_i)$ for some i. Thus $x_i \in \rho(y_i) \subseteq \rho(A)$. As P was arbitrary, $\rho(A)$ k-surrounds $\rho(B)$.

Note. There is of course a dual version of Proposition 6, where we take $A', B' \subseteq G'$ and λ instead of ρ.

Now let us consider the problem of the composition of two isomorphisms θ and θ'.

Suppose that we have three images

$$I : G \to K,$$

$$I' : G' \to K,$$

$$I'' : G'' \to K, \tag{48}$$

an isomorphism θ between I and I' and an isomorphism θ' between I' and I''. How to define the composition $\theta\theta'$ of θ by θ', which would be an isomorphism between I and I'?

We will again use the analogy with the real case. Thus suppose that there are two homeomorphisms ψ: $(G,I) \to (G',I')$ and ψ' : $(G',I') \to (G'',I'')$ determining θ and θ', in other words, for any $p \in G$, $p' \in G'$ and $p'' \in G''$:

$$p \ \theta \ p' \text{ iff } \psi(p) \cap p' \neq \emptyset \qquad (49)$$

$$p' \ \theta \ p'' \text{ iff } \psi'(p') \cap p'' \neq \emptyset \ . \qquad (50)$$

Now if $\psi'' = \psi' . \psi$, $\theta''=\theta.\theta'$ is defined by :

$$p \ \theta'' \ p'' \text{ iff } \psi''(p) \cap p'' \neq \emptyset \ . \qquad (51)$$

Now (51) is equivalent to the following statements :

$$p \ \theta'' \ p'' \text{ iff } \psi(p) \cap \psi'^{-1} (p'') \neq \emptyset \ . \qquad (52)$$

$$p \ \theta'' \ p'' \text{ iff there is some } q' \in G' \text{ such that}$$

$$\emptyset \neq \psi(p) \cap \psi'^{-1}(p'') \cap q' \quad . \qquad (53)$$

Now they imply the following :

If $p \ \theta'' \ p'$, then there is some $q' \in G'$ such that $\psi(p) \cap q' \neq \emptyset$ and $q' \cap \psi'^{-1}(p'') \neq \emptyset$, in other words $P \ \theta \ q'$ and $q' \ \theta \ p''$. (54)

Now we can again forget ψ, ψ and ψ'' and define $\theta'' = \theta . \theta'$ by

$$p \ \theta'' \ p'' \text{ iff } \rho(p) \cap \lambda'(p'') \neq \emptyset \quad . \qquad (55)$$

(Indeed, $\rho(p) \cap \lambda'(p'') \neq \emptyset$ means that there is some $q' \in G'$ such that $p \ \theta \ q'$ and $q' \ \theta' \ p''$). (Here λ' and ρ' correspond to θ').

Now λ'' and ρ'' are defined by reciprocity and additivity and we get for any $V \subseteq G$ and $V'' \subseteq G''$:

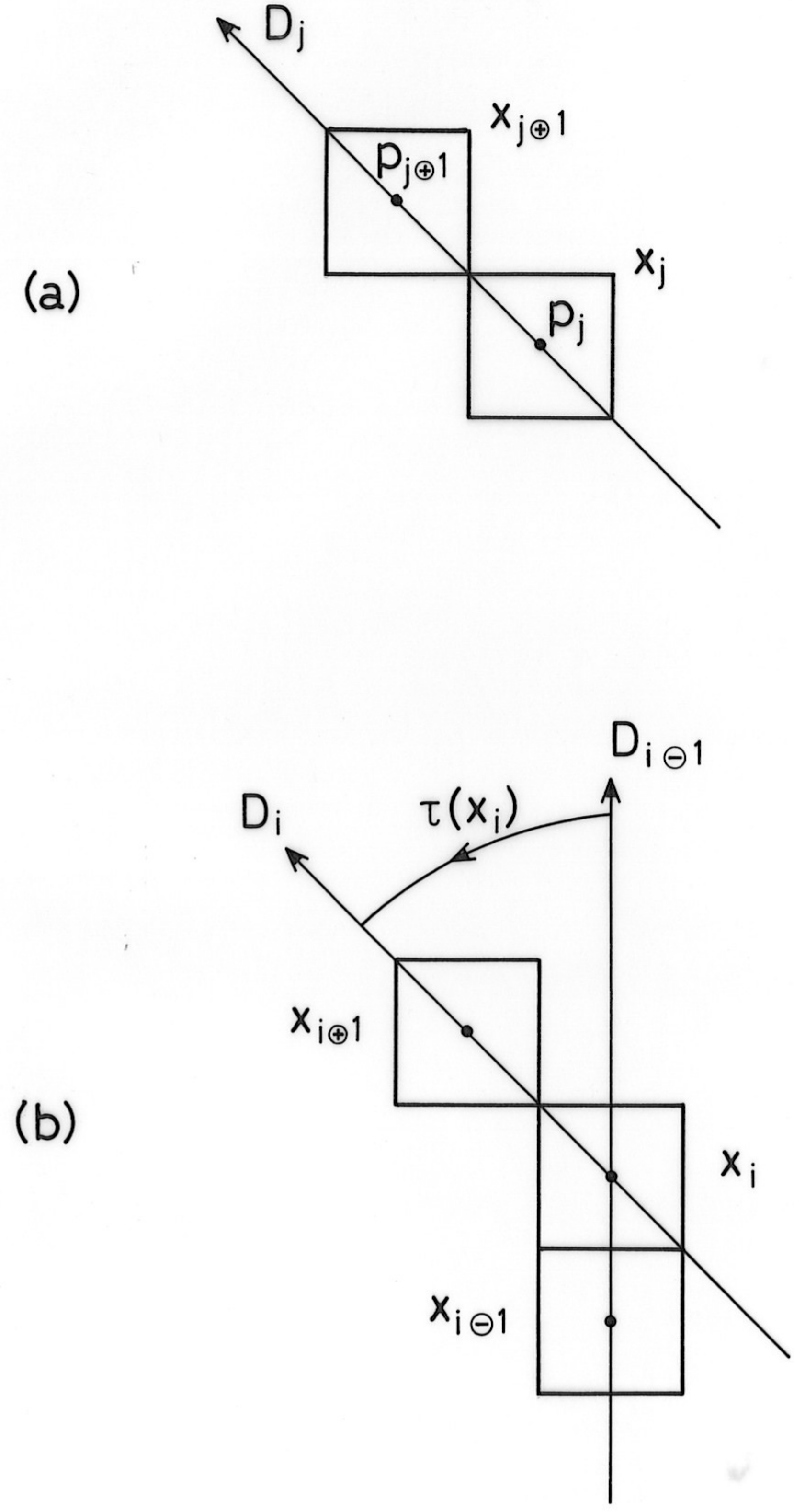
$$\rho''(V) = \rho'(\rho(V)) \quad . \tag{56}$$

$$\lambda''(V'') = \lambda(\lambda'(V'')) \quad .$$

It is easily checked that θ'', ρ'' and λ'' verify the properties of totality, frame preservation, image preservation and adjacency preservation.

This shows that (55) is a valid definition for the composition $\theta.\theta'$. Moreover, it corresponds to the natural definition of the composition of two relations.

If we say that two binary images (on a square grid) are topologically isomorphic if there is an isomorphism θ between them (having the above properties), then the relation of topological isomorphism is clearly an equivalence relation (i.e. it is reflexive, symmetric and transitive).



(a)
D_j
$x_{j\oplus 1}$
$p_{j\oplus 1}$
x_j
p_j
(b)
$D_{i\ominus 1}$
D_i
$\tau(x_i)$
$x_{i\oplus 1}$
x_i
$x_{i\ominus 1}$

Figure A-1.

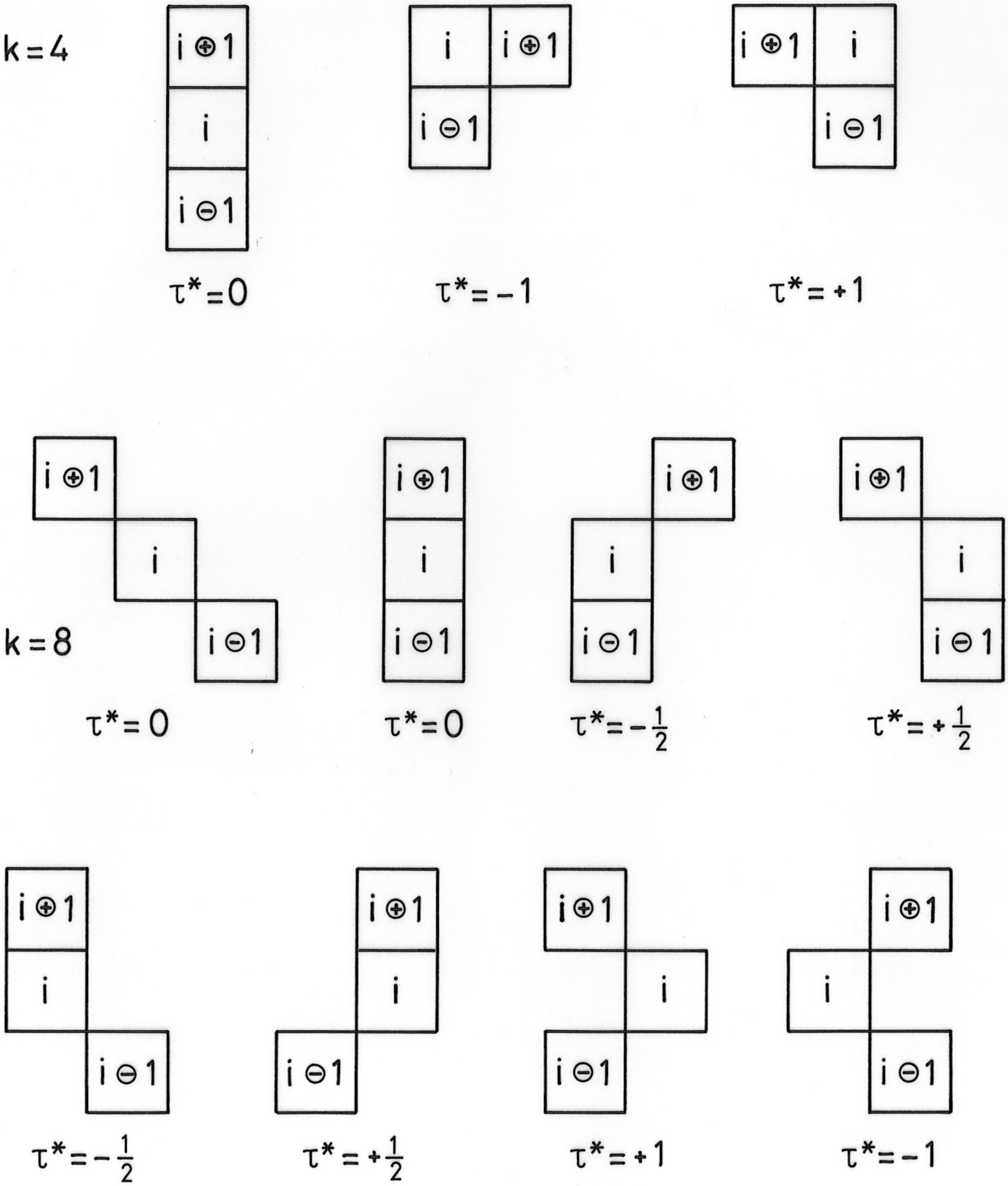


Figure A-2. $\tau^* = \tau^*(x_i)$

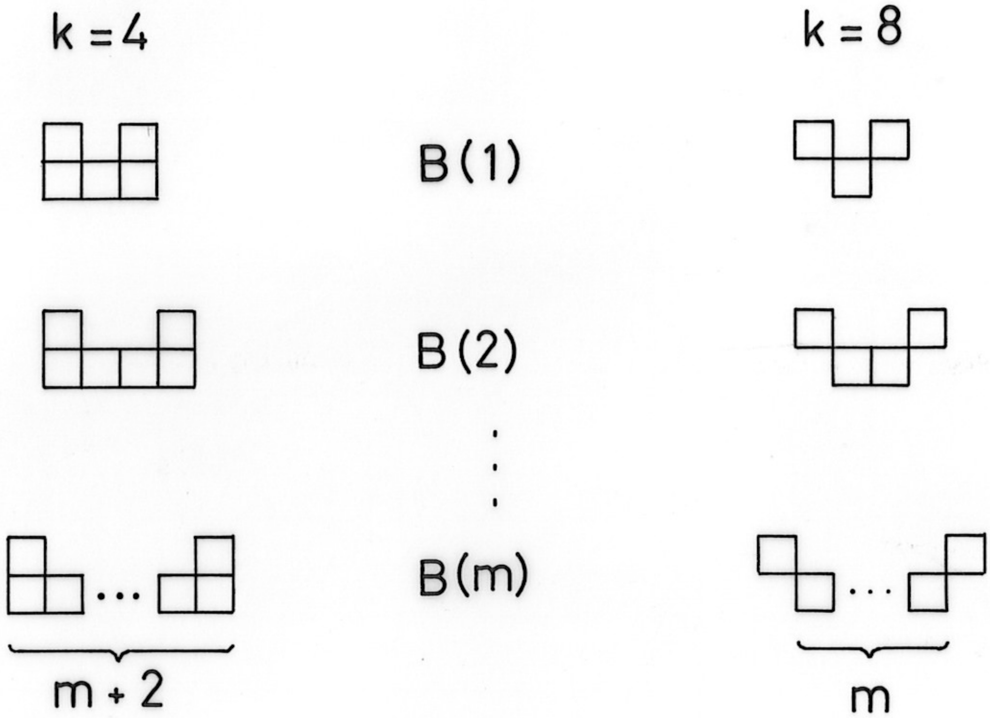
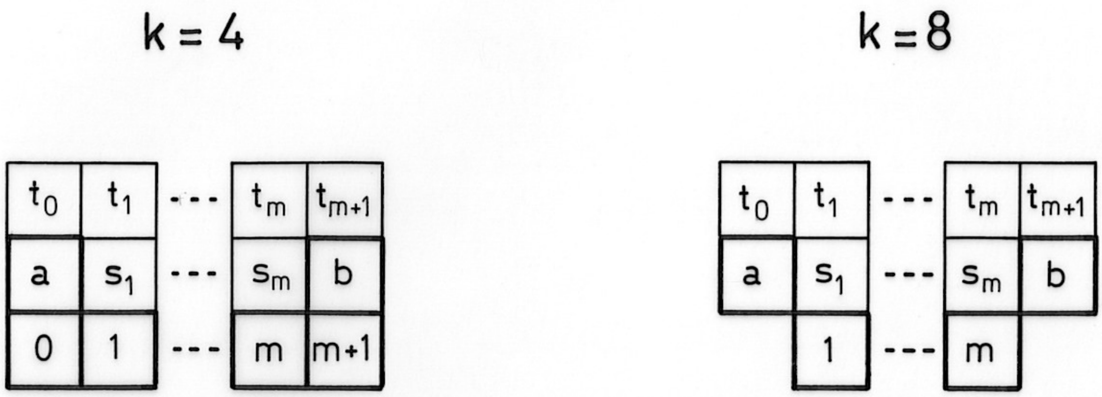


Figure A-3 : B(m)

Figure A-4 : A stair



B= {a,0,1,...,m,m+1,b}

B= {a,1,...,m,b}

Figure A-5

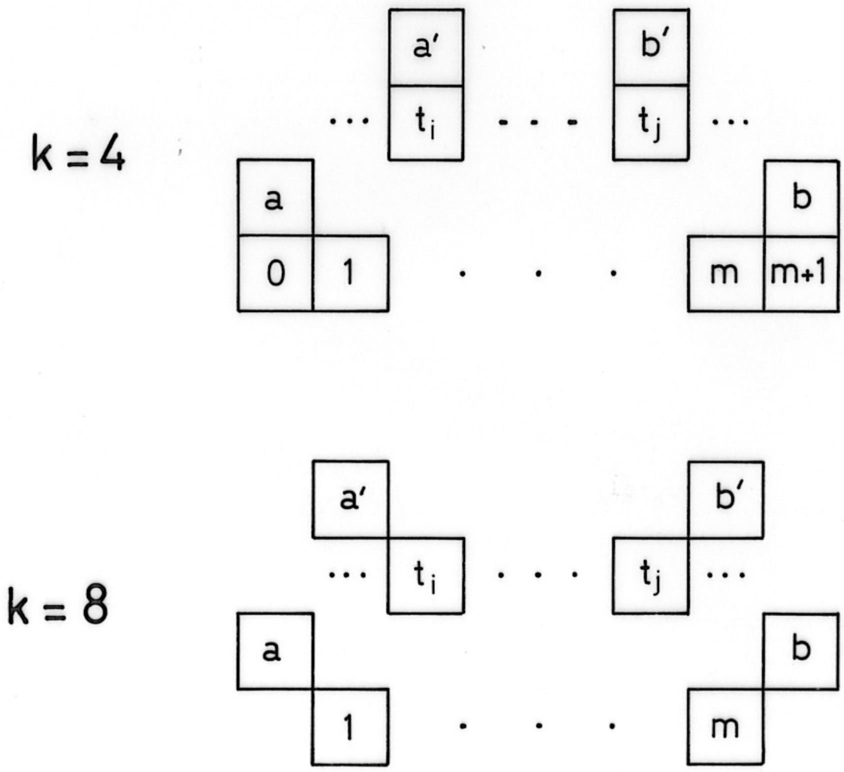


Figure A-6.

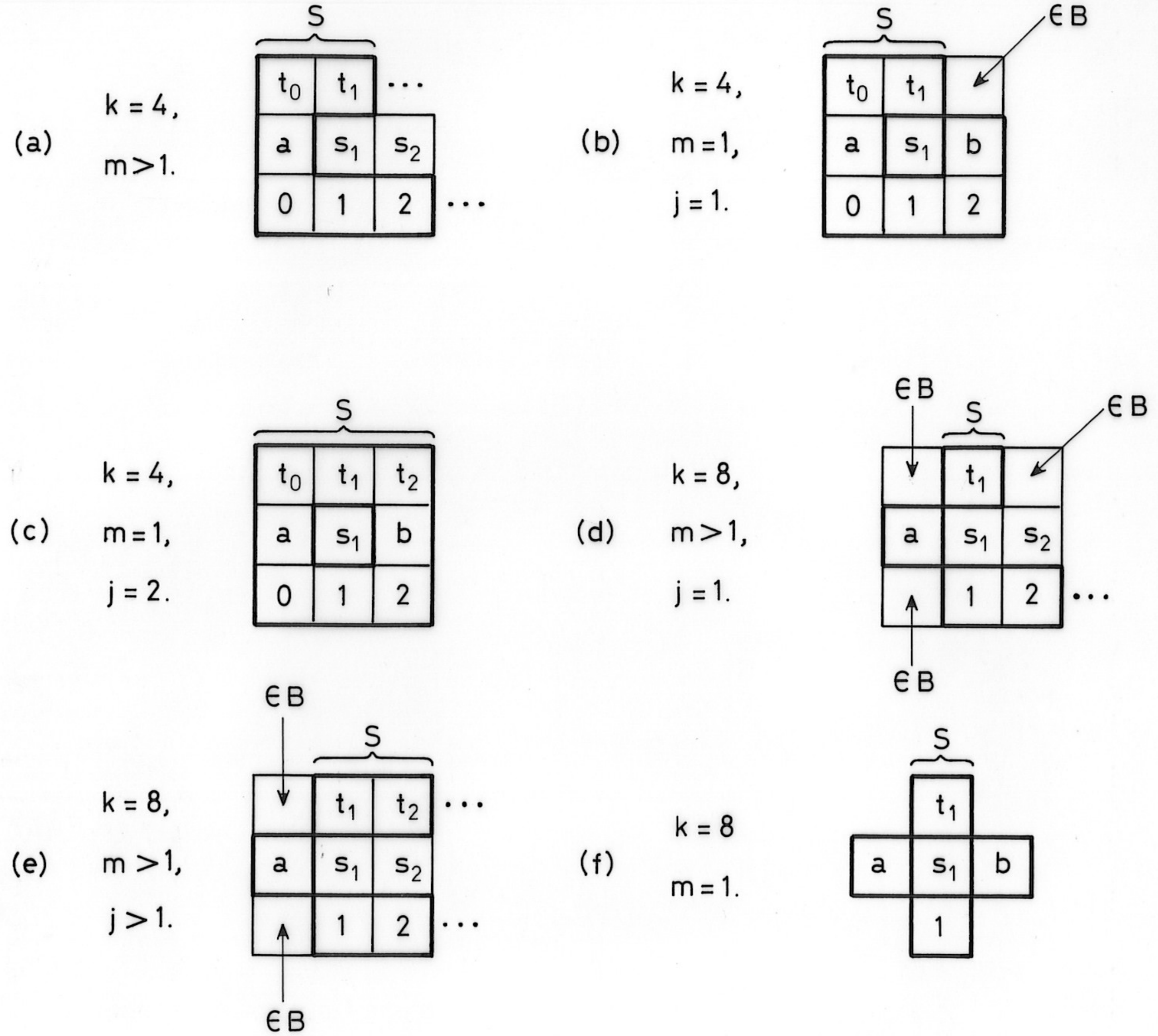


Figure A-7.

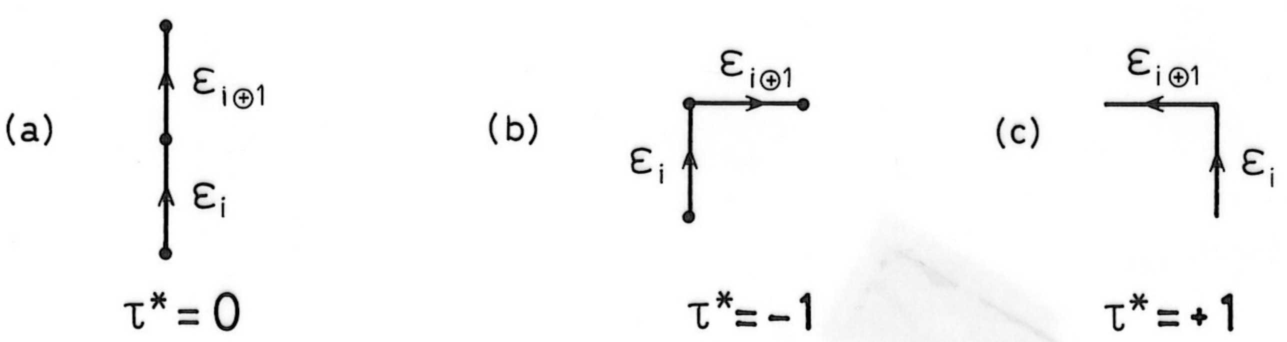


Figure A-8. $\tau^* = \tau^*(\varepsilon_i, \varepsilon_{i\oplus 1})$.

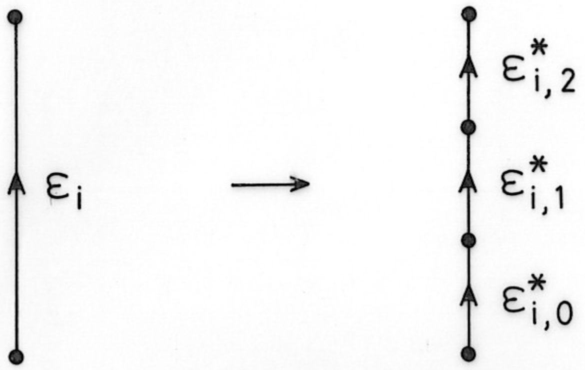


Figure A-9.

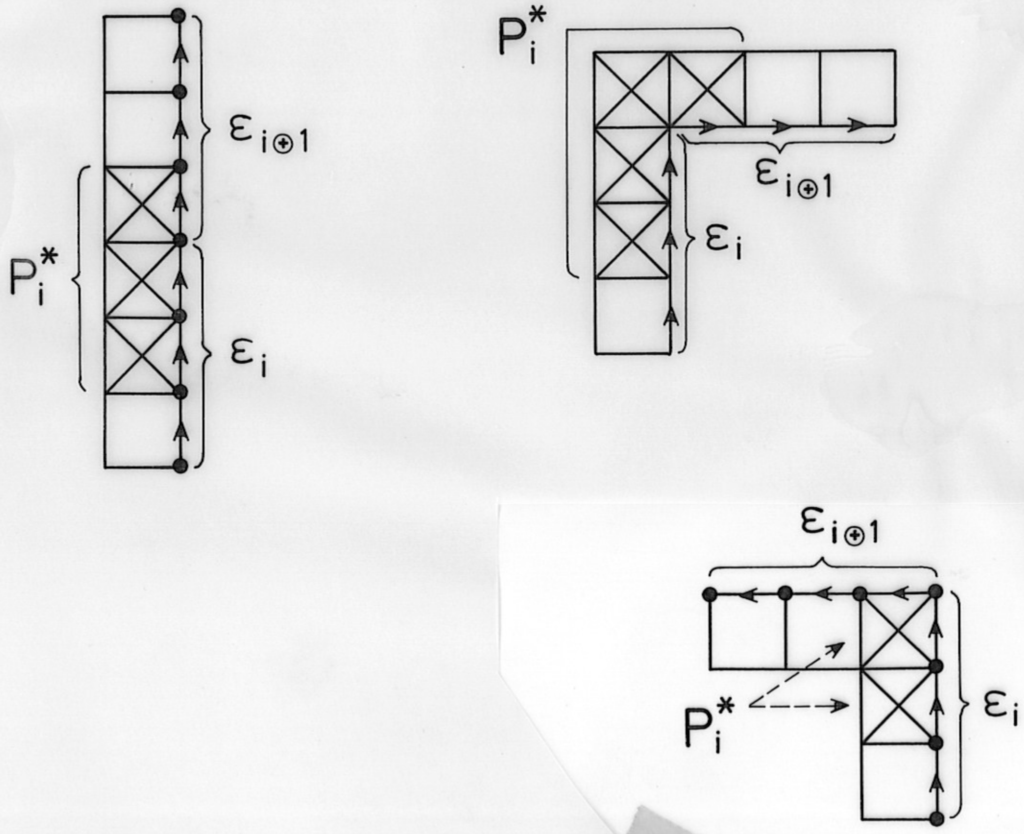


Figure A-10.

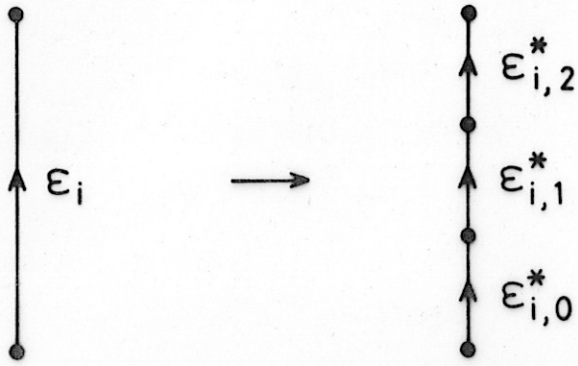


Figure A-9.

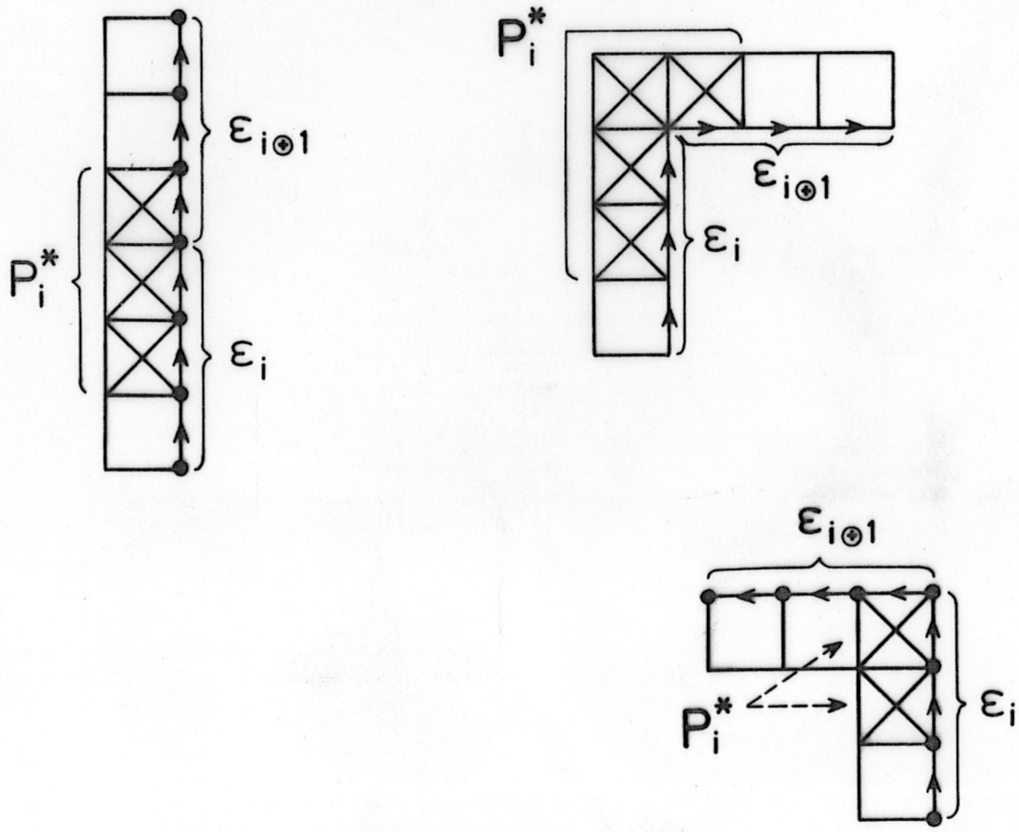


Figure A-10.

$$Q: \begin{array}{|c|c|}\hline 1 & 0 \\ \hline 0 & 0 \\ \hline \end{array} \qquad T^*: \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 1 & 0 \\ \hline \end{array} \qquad D^*: \begin{array}{|c|c|}\hline 1 & 0 \\ \hline 0 & 1 \\ \hline \end{array}$$

Figure A-11.

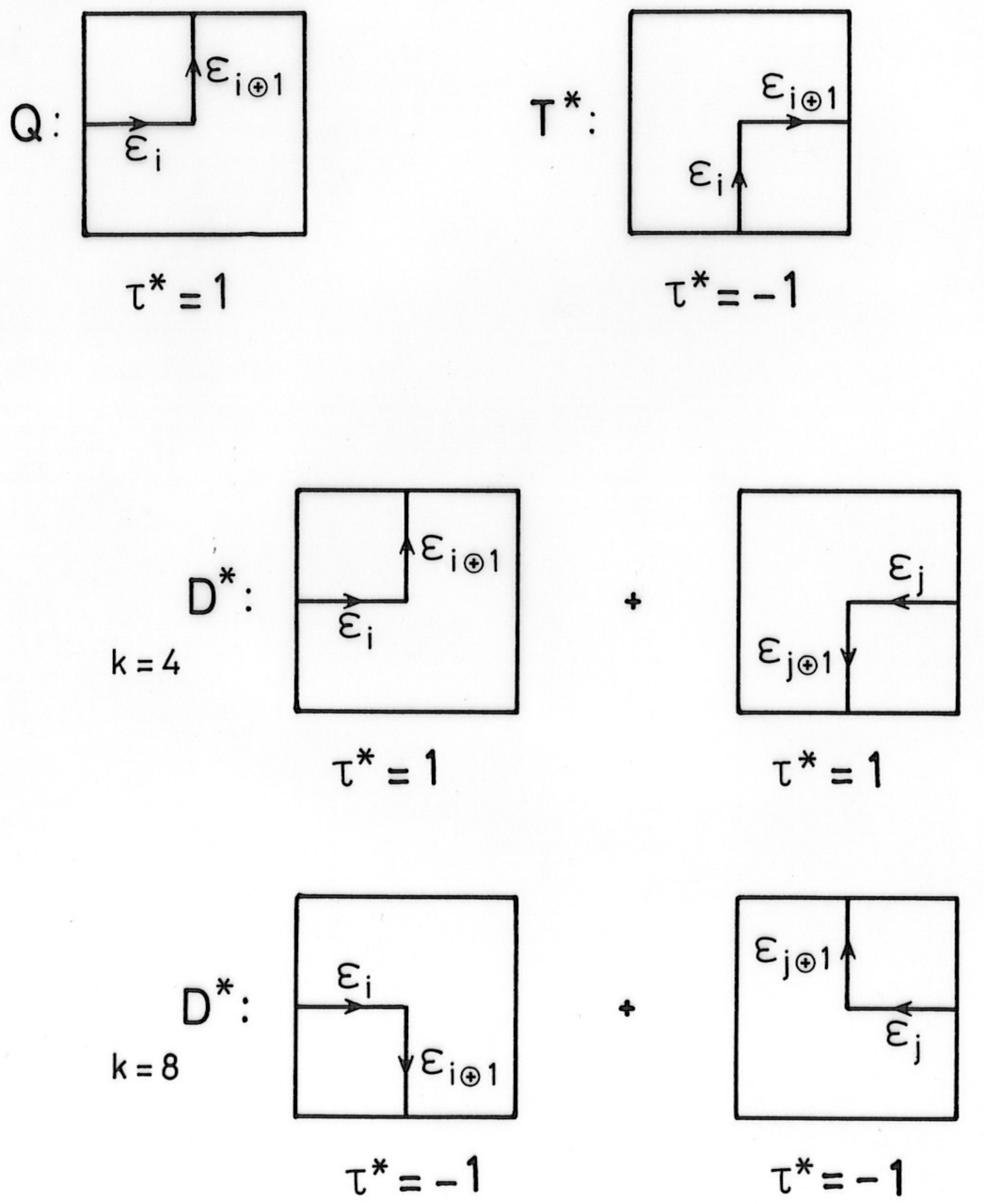


Figure A-12.

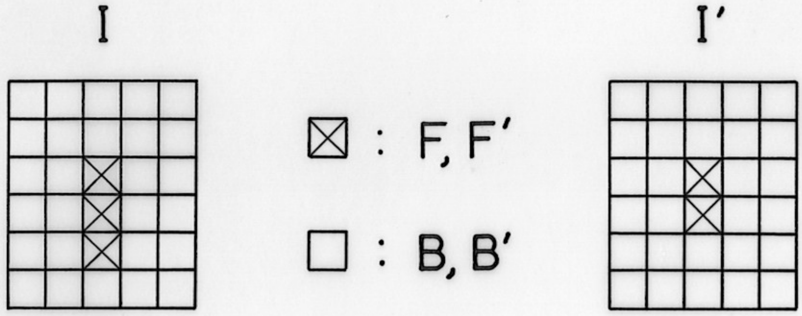


Figure A-13.