§ VII. Surrounding

In page 336 of [19] and in Section 3 of [18], the concept of "surrounding" is defined.

Let S and T be two subsets of the grid. Then we say that T k-surrounds S if $S \cap T = \emptyset$ and every k-path from a pel in S to a pel in FG intersects T.

It is clear that "8-surrounds" imples "4-surrounds". An example of k-surrounding is the following : If P is a simple closed k'-path, then O(P) surrounds P and I(P), and P surrounds I(P).

The relation "k-surrounds" is a strict partial order relation on disjoint subsets. In other words, the following hold for any $R,S,T \subseteq G$:

a) If T k-surrounds S, then S does not k-surround T

b) If T k-surrounds S, if S k-surrounds R and if $R \cap T = \emptyset$, then T k-surrounds R.

We have also the following two properties :

c) If S k-surrounds R, if $S \subseteq T$ and $T \cap R = \emptyset$, then T k-surrounds R.

d) If R k-surrounds S and if $T \subseteq S$, then R k-surrounds T.

It follows then that the largest subset of G surrounding R is $G \setminus R$ and that the smallest subset of G surrounded by R is $\emptyset$.

Finally we have the following :

e) If R surrounds both S and T, then R surrounds $S \cup T$.

We deduce that there is a maximal k-surrounded set of R, also called the k-inside of R and written $I_k(R)$, which has the following properties :

(i) R k-surrounds $I_k(R)$

(ii) If R k-surrounds S, then $S \subseteq I_k(R)$.

The set $I_k(R)$ is simply the union of all sets k-surrounded by R, or the set of all pels k-surrounded by R. Clearly $I_8(R) \subseteq I_4(R)$.

Note that if P is a simple closed k'-path, then $I_k(P) = I(P)$.

The dual concept does not exist. There is no "minimal k-surrounding set" of R, because if S and T k-surround R, then $S \cap T$ does not always surround R (for example if $S \cap T = \emptyset$).

We will make the following definition instead :

We say that S k-<u>surrounds</u> T <u>primitively</u> if S k-surrounds T and for any $S' \subset S$, S' does not k-surround T.

The following holds :

<u>Theorem 9</u>. S k-surrounds T primitively if and only if S is a simple closed k'-path and $T \subseteq I(S)$. (One can compare this theorem to Theorem 4.4 of [17]).

The proof of this theorem and of the other results of this section can be found in Appendix 4.

The following consequence of Theorem 9 is immediate:

<u>Corollary 10</u>. For any $S \subseteq G$, $I_k(S) = \emptyset$ if and only if for any simple closed k'-path $P \subseteq S$, $I(P) \subseteq S$.

<u>Corollary 11</u>. If P is a closed k-path and if $I_{k'}(P) \neq \emptyset$, then P contains a simple closed k-path.

Now we can define the k-<u>outside</u> of a set R as the set of all pels of $G \setminus R$ which are not k-surrounded by R. We write it $O_k(R)$. Then clearly $O_k(R) = G \setminus (R \cup I_k(R))$ and $O_4(R) \subseteq O_8(R)$.

Note that if P is a simple closed k'-path, then $O_k(P) = O(P)$.

We will now give two elementary related results.

<u>Lemma 12</u>. Let $X \subseteq G$, $y \in G \setminus X$ and let Y be the k-connected component of $G \setminus X$ containing y. If X k-surrounds y, then X k-surrounds Y.

<u>Corollary 13</u>. Let $X, Y \subseteq G$ such that $X \cap Y = \emptyset$. If $X \subseteq I_4(Y)$ and Y is k-connected, then $Y \subseteq O_k(X)$.

Now let us apply the concept of surrounding to a figure and its background. We suppose that F and B satisfy the restricted frame assumption (RFA). The following result is a generalization of Theorem 3 of [18], whose proof inspired ours :

Theorem 14. Let X be a k-connected component of F and let $Y_1,\dots,Y_m$ be the neighbouring k'-connected components of B. Then for any i = 1,...,m, either :

(a) X k'-surrounds Y_i and $X \subseteq O_k(Y)$;

or (b) Y_i k-surrounds X and $Y_i \subseteq O_{k'}(X)$.

Moreover, there is at most one i such that (b) holds, and this happens if and only if $X \cap FG = \emptyset$.

Note. As the RFA is symmetrical between F and B, we can intervert F and B in the statement of the theorem.

Let us now consider the case where X is a k-connected component of F and Y is a neighbouring k-connected component of X. For k=4, it is possible that neither X 4-surrounds Y nor Y 4-surrounds X (we can for example take X and Y as in Figure 1-28). For k=8 we have the following :

Proposition 15. Let X be an 8-connected component of F and let $Y_1,\dots,Y_m$ be the neighbouring 8-connected components of B. Then for any i=1,...,m, one of the following holds :

(a) X 8-surrounds Y_i and $X \subseteq O_8(Y_i)$; moreover, there is a 4-connected component X^* of X which neighbours Y_i and 8-surrounds it.

(b) Y_i 8-surrounds X and $Y_i \subseteq O_8(X)$; moreover, there is a 4-connected component Y_i^* of Y_i which neighbours X and 8-surrounds it.

Moreover, there is at most one $i \in \{1,\dots,m\}$ such that (b) holds, and this happens if and only if $X \cap FG = \emptyset$.

There is also an hybrid result between Theorem 14 and Proposition 15. Let us call an 8-connected half-component of a set W an 8-connected union of 4-connected components of W. We have the following

Proposition 16. Let X be an 8-connected half-component of F and let $Y_1,\ldots,Y_m$ be the neighbouring 8-connected components of B. Then for any $i=1,\ldots,m$, one of the following holds :

(a) X 8-surrounds Y_i and $X \subseteq O_8(Y_i)$; moreover there is a 4-connected component X^* of X which neighbour Y_i and 8-surrounds it.

(b) Y_i 4-surrounds X and $Y_i \subseteq O_8(X)$.

Moreover, there is at most one i such that (b) holds, and this happen only if $X \cap FG = \emptyset$.

Note. As the RFA is symmetrical between F and B, the last two results are still true if we intervert F and B.

Note that the RFA is essential in Theorem 14 and Proposition 15. If we do not assume it, then it is possible that for $(k_1,k_2) = (4,8),(8,4)$ or $(8,8)$, for a k_1-connected component X of F and a neighbouring k_2-connected component Y of B, neither X 4-surrounds Y nor Y 4-surrounds X. For example, if we take X to be the upper half of the grid and $Y = G \setminus X$.

Given two sets X and Y, it is possible to have $Y \subseteq O_8(X)$ but $Y \not\subseteq O_4(X)$. An example is given in Figure 1-29.

Now let us define for any set $X \subseteq G$ the k-outer edge $\varepsilon^{O_k}(X)$ and the k-inner edge $\varepsilon^{I_k}(X)$:

$$\varepsilon^{O_k}(X) = \varepsilon(X, O_k(X)). \tag{67}$$

$$\varepsilon^{I_k}(X) = \varepsilon(X, I_k(X)). \tag{68}$$

Then it is clear that : $\varepsilon^{I_k}(X) \cap \varepsilon^{O_k}(X) = \emptyset$ (69)

and $\varepsilon^{I_k}(X) \cup \varepsilon^{O_k}(X) = \varepsilon(X)$. (70)

We can also define the k-<u>outer</u> k^*-<u>border</u> $\delta_{k^*}^{O_k}(X)$ and the k-<u>inner</u> k^*-<u>border</u> $\delta_{k^*}^{I_k}(X)$ of X (where $k^* = 4$ or 8) as :

$$\delta_{k^*}^{O_k}(X) = \delta_{k^*}(X, O_k(X)). \tag{71}$$

$$\delta_{k^*}^{I_k}(X) = \delta_{k^*}(X, I_k(X)). \tag{72}$$

Then clearly $\delta_{k^*}(X) = \delta_{k^*}^{I_k}(X) \cup \delta_{k^*}^{O_k}(X)$. (73)

Now from Theorem 5, Proposition 6, Theorem 14 and Proposition 15 we deduce the following :

<u>Proposition 17</u>. Let X be an 8-connected set such that $X \cap FG = \emptyset$. Then $\varepsilon^{O_k}(X)$ forms a single cycle.

This is due to the fact that $\varepsilon^{O_k}(X) = \varepsilon(X, Y_i)$, where Y_i is the neighbouring k-connected component of $G \setminus X$ which satisfies (b).

Now we will study the case where a connected component surrounds another one which it does not neighbour. We have obtained the following 3 results (we still assume the RFA) :

<u>Proposition 18</u>. Let X and Y be each an 8-connected component of F or of B. If X 8-surrounds Y, then there is a sequence $X = Z_0, \ldots, Z_n = Y$ of subsets of G such that :

(i) The sets Z_j $(j=0,\ldots,n)$ are alternately 8-connected components of B and F.

(ii) For every $i=0,\ldots,n-1$, Z_i neighbours Z_{i+1} and 8-surrounds it.

Proposition 19. Let X be an 8-connected component of F or B and let Y be an 8-connected half-component of F or B. If X 4-surrounds Y, then there is a sequence $Z_0,\ldots,Z_n=Y$ of subsets of G such that:

(i) $Z_0 \subseteq X$

(ii) The sets Z_j are pairwise disjoint.

(iii) The sets Z_j $(j=0,\ldots,n)$ are 8-connected half-components of alternately F and B.

(iv) For every $i=0,\ldots,n-1$, Z_i neighbours Z_{i+1} and 4-surrounds it. Moreover $(Z_i,Z_{i+1}) = \varepsilon^{0_4}(Z_{i+1})$ and $Z_i \subseteq 0_4(Z_{i+1})$.

Proposition 20. Let X and Y be each a k-connected component of F or a k'-connected component of B. If X 8-surrounds Y, then there is a sequence $X = Z_0,\ldots,Z_n = Y$ of subsets of G such that :

(i) The sets Z_j $(j=0,\ldots,n)$ are alternately k-connected components of F and k'-connected components of B.

(ii) For every $i=0,\ldots,n-1$, Z_i neighbours Z_{i+1} and 4-surrounds it.

(iii) For every $i=1,\ldots,n$, X 8-surrounds Z_i.

Remarks. 1) In Proposition 19, we added that the sets Z_j are pairwise disjoint and that for each $i < n$ $\varepsilon(Z_i,Z_{i+1}) = \varepsilon^{0_4}(Z_{i+1})$, while we did not mention such things in Propositions 18 and 20. Indeed, in the latter case, it is obvious that the Z_j's are pairwise disjoint and that $\varepsilon(Z_i,Z_{i+1}) = \varepsilon^{0_4}(Z_{i+1})$ or $\varepsilon^{0_8}(Z_{i+1})$ (according to whether Z_i is 4-connected or 8-connected, see Theorem 14 and Proposition 15).

2) No result like Proposition 20 can be proved if we simply assume that X 4-surrounds Y. Indeed, in Figure 1-30, the 4-connected component X of F 4-surrounds the 4-connected component Y of F and the 8-connected component Y' of B, but the 8-connected component Z of B which separates X and Y is not 4-surrounded by X; on the contrary it 8-surrounds it.

Now we will deal with the neighbourhood tree. In [16] the author defines the adjacency tree as the graph whose vertices are the k-connected components of F and the k'-connected components of B, and whose edges join vertices corresponding to neighbouring components; this graph is a tree (i.e. it is connected and has no cycles). We will call the (k_1,k_2)-neighbourhood tree (where k_1, $k_2 \in \{4,8\}$ and $(k_1,k_2) \neq (4,4)$) the graph whose vertices are the k_1-connected components of F and the k_2-connected components of B and whose edges join vertices corresponding to neighbouring components. The fact that this graph is a tree follows from Theorem 14 and Proposition 15. Indeed, if one represents the graph in such a way that the vertices are placed in different levels, where a component neighbouring and 4-surrounding another component stands in higher level, then by Theorem 14 and Proposition 15 the graph takes the form displayed in Figure 1-31.

The neighbourhood tree can be oriented : If X neighbours Y, then we orient the edge from X to Y if X 4-surrounds Y. Then the graph takes the form of Figure 1-32.

Given the oriented (k_1,k_2) neighbourhood tree, then the relation of 8-surrounding between its vertices can easily be verified. Indeed, Propositions 18 and 20 imply that if X and Y are two vertices of that tree and if X 8-surrounds Y, then there is a sequence $X = Z_0,\ldots,Z_n = Y$ of vertices of that tree, such that each (Z_j, Z_{j+1}) $(j=0,\ldots,n-1)$ is an arrow (i.e. oriented edge) of that tree. Conversely, if such a sequence exists, then X 4-surrounds Y. Moreover, X 8-surrounds Y if $(k_1,k_2) = (8,8)$. Thus in this case the existence of such a sequence is a necessary and sufficient condition for X to 8-surround Y.

However, the relation of 4-surrounding between the vertices of that tree cannot be deduced from the edges of that tree, as we explained in relation to Figure 1-30.

§ VIII. Algorithms for border following and surrounding

In the two preceding sections, we studied the properties of the borders and of the property of surrounding. However we left out of them the relevant algorithms, in order not to render these section too heavy.

Let us start first with border-following. Of course the 4-border and the 8-border can be determined at the same time as the edge by the use of the edge-following algorithm. In fact, it is the procedure underlying the proof of Proposition 7. However, one would like to have algorithms computing the border independently of the edge. Such an algorithm may be more complicated than the edge-following algorithm, but it requires fewer steps (according to [16]).

First there are evident parallel algorithms to find the borders $\delta_4(X)$ and $\delta_8(X)$ of a set X. If X is given in matrix form by a truth function $\phi_X = \phi$, where :

$$\begin{aligned} \phi(i,j) &= 1 \quad \text{if} \quad (i,j) \in X \quad , \\ &= 0 \quad \text{if} \quad (i,j) \in G \setminus X, \end{aligned} \tag{74}$$

then $\delta_4(X)$ and $\delta_8(X)$ are given in matrix form by the truth functions $\phi_4 = \phi_{\delta_4(X)}$ and $\phi_8 = \phi_{\delta_8(X)}$ defined as follows :

$$\phi_4(i,j) = \phi(i,j) \wedge (\bigvee_{(a,b) \in U_4} \overline{\phi(i+a,j+b)}), \tag{75}$$

$$\phi_8(i,j) = \phi(i,j) \wedge (\bigvee_{(a,b) \in U_8} \overline{\phi(i+a,j+b)}), \tag{76}$$

where $\qquad U_4 = \{(0,1),(1,0),(0,-1),(-1,0)\}$ $\qquad (77)$

and $\qquad U_8 = \{(a,b) \mid a,b \in \{0,1,-1\}\}.$ $\qquad (78)$

Suppose now that X is given by runs [A,b] on each row. Write :

$$[a,b]_u = \{(u,a),\ldots,(u,b)\} \tag{79}$$

(where $a \leqslant b$). In other words, $[a,b]_u$ is a run between a and b on row u. Now, if $(i,j) \in F$, then $(i,j) \in \delta_4(X)$ if and only if one of the following holds :

- There is a run $[j,n]_i$ $(n \geqslant j)$ in X.
- There is a run $[m,j]_i$ $(m \leqslant j)$ in X.
- There is no run $[m,n]_{i-1}$ $(m \leqslant j \leqslant n)$ in X.
- There is no run $[m,n]_{i+1}$ $(m \leqslant j \leqslant n)$ in X. (80)

Similarly $(i,j) \in \delta_8(X)$ if and only if one of the following holds :

- There is a run $[j,n]_i$ $(n \geqslant j)$ in X.
- There is a run $[m,j]_i$ $(m \leqslant j)$ in X.
- There is no run $[m,n]_{i-1}$ $(m < j < n)$ in X.
- There is no run $[m,n]_{i+1}$ $(m < j < n)$ in X. (81)

If in addition to the representation of X as a union of horizontal runs, we give also the representation of X as a union of vertical runs, then $\delta_4(X)$ is the set of pels which are beginnings or ends of vertical or horizontal runs.

One can also represent X as a union of diagonal runs, and this can be done in the two directions. Now if we know the 4 representations of X as a union of runs (vertical, horizontal and two diagonal), then $\delta_8(X)$ is the set of pels which are beginnings or ends of vertical, horizontal or diagonal runs.

Now let us consider the case where X is a k-connected component of F and Y is a k'-connected component of B. We wish to compute $\delta_4(X,Y)$ and $\delta_8(X,Y)$. Then we will use a sequential algorithm, as for $\varepsilon(X,Y)$.

In [16] an algorithm is given for the following of $\delta_4(X,Y)$ for both k=4 and k=8. We will also give the corresponding algorithm for the following of $\delta_8(X,Y)$. This makes thus 4 algorithms. Write $\Delta = \delta_u(X,Y)$ (u=4,8). For any x,y such that $d_8(x,y) = 1$, define $\rho_i(x,y)$ (where i=0,...,7) the element z of $N_8(x)$ which gets the number i if we number the elements of $N_8(x)$ 0,...,7, starting with y. (In particular $y = \rho_0(x,y)$). This definition is illustrated in Figure 1-33. Now our 4 algorithms are the following :

k = 4 u = 4

<u>ALGORITHM 6</u>

<u>begin</u>

$n \leftarrow 0$;

$(x_0,y_0) \leftarrow$ <u>any from</u> $\varepsilon^+(X,Y)$;

<u>if</u> every ρ_{2j} (x_0,y_0) (j=1,2,3) $\in$ B

<u>then</u> <u>goto</u> 7 <u>else</u> <u>goto</u> 1;

1 : $j \leftarrow$ smallest $w \in \{1,2,3\}$ such that ρ_{2w} $(x_n,y_n) \in F$

goto 2;

2 : if $\rho_{2j-1}(x_n,y_n) \in F$ then goto 3 else goto 4

3 : $u_n \leftarrow \rho_{2j-1}$ (x_n,y_n)

$v_n \leftarrow \rho_{2j-2}$ (x_n,y_n)

<u>goto</u> 5;

4 : $u_n \leftarrow \rho_{2j}$ (x_n,y_n)

$v_n \leftarrow \rho_{2j-1}$ (x_n,y_n)

<u>goto</u> 5;

5 : <u>if</u> $(x_n,u_n) = (x_0,x_1)$ <u>then</u> <u>goto</u> 7 <u>else</u> <u>goto</u> 6;

6 : $(x_{n+1}, y_{n+1}) \leftarrow (u_n, v_n)$

    $n \leftarrow n+1$

    goto 1;

7 : $\Delta \leftarrow \{x_0, \ldots, x_n\}$

    write Δ

end

$k = 4 \quad u = 8$

ALGORITHM 7

It is the same as Algorithm 6, except that we replace the statement 3 by the following one :

3 : $u_n \leftarrow \rho_{2j}(x_n, y_n)$

    $v_n \leftarrow x_n$

    goto 5;

$k = 8 \quad u = 4$

ALGORITHM 8

begin

$n \leftarrow 0;$

$(x_0, y_0) \leftarrow$ any from $\varepsilon^+(X,Y)$;

if every $\rho_j\ (x_0, y_0)\ \ (j=1,\ldots,7) \in B$

then goto 4 else goto 1;

1 : $j \leftarrow$ smallest $w \in \{1,\ldots,7\}$ such that $\rho_w(x_n, y_n) \in F$

    goto 2;

2 : if $(x_n, \rho_j\ (x_n, y_n)) = (x_0, x_1)$

    then goto 4 else goto 3;

3 : $x_{n+1} \leftarrow \rho_j\ (x_n,y_n)$

$y_{n+1} \leftarrow \rho_{j-1}\ (x_n,y_n)$

$n \leftarrow n+1$

<u>goto</u> 1;

4 : $\Delta \leftarrow \{x_0,\dots,x_n\}$

<u>write</u> Δ

<u>end</u>

$k = 8 \quad u = 8$

<u>ALGORITHM 9</u>

<u>begin</u>

$n \leftarrow 0$

$(x_0,y_0) \leftarrow$ <u>any</u> <u>from</u> $\varepsilon^+(X,Y)$

<u>if</u> every $\rho_j(x_0,y_0)\ (j=1,\dots,7) \in B$

<u>then</u> <u>goto</u> 7 <u>else</u> <u>goto</u> 1;

1 : $j \leftarrow$ smallest $w \in \{1,\dots,7\}$ such that $\rho_w\ (x_n,y_n) \in F$

<u>goto</u> 2;

2 : <u>if</u> $j \in \{1,3,5\}$ and $\rho_{j+1}\ (x_n,y_n) \in F$

<u>then</u> <u>goto</u> 3 <u>else</u> <u>goto</u> 4;

3 : $u_n \leftarrow \rho_{j+1}\ (x_n,y_n)$

$v_n \leftarrow x_n$

<u>goto</u> 5;

4 : $u_n \leftarrow \rho_j\ (x_n,y_n)$

$v_n \leftarrow \rho_{j-1}\ (x_n,y_n)$

<u>goto</u> 5;

5 : <u>if</u> $(x_n,u_n) = (x_0,x_1)$ <u>then</u> <u>goto</u> 7 <u>else</u> <u>goto</u> 6

6 : $(x_{n+1}, y_{n+1}) \leftarrow (u_n, v_n)$

$n \leftarrow n+1$

goto 1;

7 : $\Delta \leftarrow \{x_0, \ldots, x_n\}$

write Δ

end

In [16] the validity of Algorithm 6 is proved. The same can easily be done for the 3 other algorithms. We leave it to the reader. The algorithms end when we get $(x_n, u_n) = (x_0, x_1)$. We can also end them when we get $(u_n, v_n) = (x_1, y_1)$. This choice is equivalent to the original one. In the [16], the author gives another algorithm which is very simple, but has several defects : 1) it takes many extra steps to follow even a simple border, 2) it visits some elements twice although they have only consecutive edges, 3) sometimes it keeps to a 4-connected component, sometimes to an 8-connected component. The same algorithm is presented in pages 290-293 of [5] and in pages 50-51 of [14].

In [17,19] a variant of Algorithm 8 (k=8, u=4) is presented. It is called BF. The basic idea is to associate to each pel a number : 1, 3, 4 for pels in X and 0, 2 for pels in Y. Starting with the original image (with pels of X marked 1 and pels of Y marked 0), we start by associating 3 to x_0 and 2 to y_0. Then we mark the successive pels $x_1, \ldots, x_n$ as 4, until we find the pair $(x_0, y_0) = (3,2)$, which we change to (4,0). Then Δ is the set of pels labelled 4. Then we have 4 algorithms of this type, corresponding to the 4 cases considered above. We call them Algorithms 6', 7', 8' and 9'. Each Algorithm n' (n=6,7,8,9) is the same as Algorithm n, but with the following changes :

1) In the beginning, we label (x_0,y_0) as (3,2).

2) The test "$(x_n,u_n) = (x_0,x_1)$ (or $(x_n,\rho_\gamma\ (x_n,y_n)) = (x_0,x_1)$ in Algorithm 8)" is replaced by "x_n has label 3 and there is some $i < j$ such that ρ_i (x_n,y_n) has label 2".

3) When we define a new (x_n,y_n), we change the label of x_n from 1 to 4.

4) Then Δ is the set of pels labelled 4.

Some authors have designed border-following algorithms using a moving 2×2 window. The one presented in Exercise 17 of [19] is equivalent to the extraction by the border using the edge-following algorithm. There is a similar algorithm in pages 52-53 of [14], which can also be found in [23]. Here the window moves by one step along the x-axis or the y-axis (this gives 4 possible movements : INCX, DECX, INCY, DECY) according to the disposition of white and black pels in the window. One also defines a "tangent vector" which is in fact the element of $\varepsilon^+(X,Y)$ found in that window. The algorithm stops when one returns to the initial window. We reproduce in Figure 1-34 the figure of [14] giving the movement of the window and the "tangent vector" in function of the content of the window.

Now we will give an algorithm for the detection of all distinct borders between the components of a figure and those of the background. It comes from pages 344-345 of [19], but we have modified its form.

We remark first that if X is a k-connected component of F and Y is a k'-connected component of B, then $\varepsilon^+(X,Y)$ has an element (x,y) such that y is the left neighbour of x. Indeed, such an element corresponds to an arrow segment of the type $\downarrow$ (see Figure 1-22), and as $\varepsilon^+(X,Y)$ forms a single cycle and does not contain only horizontal elements (of type $\rightarrow$ and $\leftarrow$), it must contain a positive number w of vertical elements ($\downarrow$ or $\uparrow$), of whose w/2 are of type $\downarrow$.

So we need only to detect the pairs $(x,y) \in \varepsilon^+(F)$ such that y is the left neighbour of x, and for each such pair to determine the border $\delta_u(X,Y)$ to which it belongs. This procedure is easier than the comparison of rows of (80) and (81), and it has the advantage to separate the different borders.

Let us write :

$$Z = \{(i,j) \in F \mid j > 0 \text{ and } (i,j-1) \in B\} \tag{82}$$

We assume the RFA (In [19] , the authors do not assume it, but they consider that every $(i,0) \in F$ belongs to Z). Clearly $Z \leqslant \delta_u(F)$.

We now present the algorithm. Here $\mathcal{B}$ is the set of borders $\delta_u(X,Y)$ that have been detected, and $\mathcal{C}$ is the set of elements of these borders which belong to an edge element of type $\downarrow$. For any (i,j) such that $i=0,\ldots,M-1$, $j=0,\ldots,N-1$ and $(i,j) \neq (M-1,N-1)$, write $(i,j)\,\phi$ for the follower of (i,j), i.e. the pel (i',j') such that $i'N+j' = iN+j+1$.

ALGORITHM 10

<u>begin</u>

$\mathcal{B} \leftarrow \emptyset$;

$\mathcal{C} \leftarrow \emptyset$;

<u>for</u> $(i,j) \leftarrow (0,0)$ <u>step</u> ϕ <u>until</u> $(M-1,N-1)$

<u>do</u> <u>if</u> $(i,j) \in Z \setminus \mathcal{C}$ <u>then</u>

   <u>begin</u>

   <u>comment</u> $(i,j) \in \delta_u(X,Y)$ for some k-connected component X of F and some neighbouring k'-connected component Y of B;

$\Delta \leftarrow \delta_u(X,Y)$;

$\Gamma \leftarrow \{(v,w) \in \Delta \mid (v,w-1) \in B\}$;

$\mathcal{B} \leftarrow \mathcal{B} \cup \{\Delta\}$;

$\mathcal{C} \leftarrow \mathcal{C} \cup \Gamma$

<u>end</u>;

<u>comment</u> $\mathcal{B}$ is the set of n-borders $\delta_u(X,Y)$ (with X a k-connected component of F and Y a k'-connected component of Y)

<u>end</u>

We will now consider the problem of reconstructing a figure or a component from its borders.

First, suppose that we know that X is a k-connected component of F, that Y is a neighbouring k'-connected component of B, that $\delta_u(X,Y)$ is known, and that we know one element $(x,y) \in \varepsilon^+(X,Y)$. Then it is easily shown that we can reconstruct from it the edge $\varepsilon^+(X,Y)$, except if $(u,k) = (4,4)$.

Indeed, if $(u,k) \neq (4,4)$, then for any found element $(x_k, y_k) \in \varepsilon^+(X,Y)$, we consider the 2×2 square $\begin{matrix} u & v \\ x & y \end{matrix}$, and we find the following edge element (x_{k+1}, y_{k+1}) from whether u or v belongs to $\delta_u(X,Y)$. This procedure is illustrated in Figure 1-35.

Now for $(u,k) = (4,4)$, case (4) of Figure 1-35 fails. We give an example in Figure 1-36, where for two distinct images, we can get the same $\delta_u(X,Y)$, but distinct $\varepsilon^+(X,Y)$.

If we know only $\delta_u(X,Y)$ (with $(u,k) \neq (4,4)$), then we cannot reconstruct $\varepsilon^+(X,Y)$. We need a second information, as :

(1) an element of $\varepsilon^+(X,Y)$; or

(2) which of X and Y 4-surrounds the other; or

(3) a pel y in Y.

Indeed, from (2) we can get (1), because if x is the rightmost pel of $\delta_u(X,Y)$ and if y_ℓ and y_r are the left and right 4-neighbours of x, then $(x,y_\ell) \in \varepsilon^+(X,Y)$ if X 4-surrounds Y and $(x,y_r) \in \varepsilon^+(X,Y)$ if Y 4-surrounds X. From (3) we can get (2), becuase X 4-surrounds Y if and only if $\delta_u(X,Y)$ 4-surrounds y.

In [19] , pages 345-346, the authors give a method for constructing X when for any neighbouring Y, $\delta_u(X,Y)$ and an element of $\varepsilon^+(X,Y)$ are given (they suppose there that $(k,u) = (8,4)$).

Note that if $\varepsilon^+(X)$ is given, then it is trivial to reconstruct X, because X is the union of horizontal runs [a,b] such that a is bordered on the left by an edge element $\downarrow$, b is bordered on the right by an edge element $\uparrow$, and there is no vertical edge element of $\varepsilon^+(X)$ between pels of [a,b].

Let us now consider the problem of k-surrounding.

Given a set X, $0_k(X)$ is the k-connected component of $G \setminus X$ containing $FG \setminus X$. This k-connected component can be found by Algorithm 2 ("propagation") for example.

If X is a k_1-component of F and Y is a neighbouring k_2-component of B, where $(k_1,k_2) \neq (4,4)$, then the two following tests determine which of X and Y surrounds the other (see Theorem 14 and Proposition 15) :

Test 1. Let $x = (i,j)$ be a pel of X such that j is maximal. Then we have the following :

(a) If $j=N-1$ or if $j \leqslant N-2$ and $y = (i,j+1) \notin Y$, then X k_2-surrounds Y

(b) If $j \leqslant N-2$ and $y = (i,j+1) \in Y$, then Y k_1-surrounds X.

Test 2. Let $x = (i,j)$ be a pel of $\delta_u(X,Y)$ (where u = 8 or 4) such that j is maximal. Then either :

(a) X k_2-surrounds Y, $j \geqslant 1$ and $y_\ell = (i,j-1) \in Y$; or

(b) Y k_1-surrounds X, $j \leqslant N-2$ and $y_r = (i,j+1) \in Y$.

The proof is elementary and follows from Theorem 14 and Proposition 15.

If we know the set $\mathcal{C}$ of k_1-connected components of F and of k_2-connected components of B, if we know which element of $\mathcal{C}$ contains FG (we assume the RFA) and what is the neighbourhood relation on $\mathcal{C}$, then we can construct the oriented (k_1,k_2)-neighbourhood tree (see Figure 1-32). Indeed, we start with the element Z of $\mathcal{C}$ containing FG, and we put an arrow (Z,Y) for any $y \in \mathcal{C}$ neighbouring Z. Then for any such Y, we put an arrow (Y,X) for any $X \in \mathcal{C}$ neighbouring Y such that $X \neq Z$. Then we continue in the same way; whenever we have an arrow (U,V), we mark an arrow (V,W) for any W neighbouring V such that W has not yet been considered. We stop when there are no more elements of $\mathcal{C}$ left.

If the image is given as a union of black and horizontal runs, if for every run we know the number of runs of the same colour in the next row that are adjacent to it (we use k-adjacency for black runs and k'-adjacency for white ones), then one can derive from that information the (k,k')-neighbourhood tree (see [1] and § 5 of [18]).

§ IX. The Euler number (or genus) of a figure

The concept of the Euler number (or genus) is defined for surfaces in a space over the real numbers. Indeed, let S be such a surface. We draw on it a graph G such that two edges of G may intersect only in a vertex of G, and that S is decomposed by G into simply connected surfaces called faces. One often chooses G in such a way that all faces are triangular; then this decomposition is called a triangulation. Let v be the number of vertices, e the number of edges and f the number of faces of G. Then the number

$$g(S) = v-e+f \tag{83}$$

is independent of the choice of G. This number is called the genus or Euler number of that surface.

It is well-know that the genus of the Euclidean plane (or the sphere) is equal to 2.

Consider now a bounded and connected surface S of the plane having h holes and let us decompose it by a planar graph as explained above (see Figure 1-37), obtaining v vertices, e edges and f faces. Now this decomposition is also a decomposition of the plane, but the plane has h+1 more faces : the h holes of S and the portion of the plane surrounding S. We get then :

$$2=g\ (\text{plane})= v-e+(f+1+h) = g(S)+h+1, \tag{84}$$

in other words we have :

$$g(S) = 1-h. \tag{85}$$

Now if S is not connected and has c connected components $S_1,\ldots,S_c$ having respectively $h_1,\ldots,h_c$ holes, then S has $h = h_1+\ldots+h_c$ holes and we get :

$$g(S) = g(S_1)+\ldots+g(S_c). \tag{86}$$
$$= (1-h_1)+\ldots+(1-h_c.$$
$$= c-h. \tag{87}$$

Thus the genus of a bounded surface is equal to its number of components minus its number of holes.

Now let us turn to grid surfaces. If F is a figure, then we want to define the genus of F in such a way that (87) holds. But then the definition depends on the choice of the adjacency relation for F and B. If we take the k-adjacency for F and the k'-adjacency for B, if we call a k'-<u>hole</u> a k'-connected component of B neighbouring a k-connected component of F and 4-surrounded by it, then we define the (k,k')-<u>genus</u> of F as the number of k-connected components of F minus the number of k'-holes of B. We write it $g_{(k,k')}(F)$.

As we assumed above that the surface S is bounded, we make the <u>frame assumption</u> of F.

Now we want to find a decomposition of the figure F -or of a world surface F^* similar to F- such that we can apply (83).

In fact, we will choose as graph G a graph defined on the dual grid, whose vertices are the pels of F and whose edges are some pairs of 8-adjacent pels. The surface F^* will be the portion of F (as world surface) enclosed in G.

Let us define the numbers v,e,d,t,s,d^* and t^* as the number of configurations in F which are -up to a rotation of $k\,\frac{\pi}{2}$ (k=0,1,2,3)- equal to the configurations V,E,D,T,S,D^* and T^* shown in Figure 1-38. It is easily seen that we have :

$$t^{*} = t-4s \tag{88}$$

(by counting the number of pairs (X,Y), where $X \subseteq Y$, X is of type T and Y is a square),

and
$$d^{*} = d-2s-t^{*}$$
$$= d-t+2s. \tag{89}$$

(by counting the number of pairs (X,Y), where $X \subseteq Y$, X is of type D and Y is a square).

Let us now consider the case where k=4. We define the graph G on the dual grid G^{*} as follows : the vertices of G are the pels of F and the edges of G are the pairs of 4-adjacent pels in F. Let F^{*} be the world surface equal to the portion of F (as world surface) enclosed inside G. It is clear that under this decomposition of F^{*}, the faces of G are the condifurations of type S in F.

We have illustrated this construction in Figure 1-39. It is easily seen that there is a 1 to 1 correspondence between the 4-connected components of F and the connected components of F^{*}, and between the 8-holes of F and the holes of F^{*}. Hence $g_{(4,8)}(F) = g(F^{*})$ by (87). By (83) we get :

$$g_{(4,8)}(F) = v-e+s \tag{90}$$

If k=8, we do the same thing, except that we add to G the following edges : the pairs of diagonally adjacent pels in configurations of type D^{*} and T^{*}. Then the faces of G are the configurations of type S and T^{*} in F.

We illustrate this construction in Figure 1-40. We have algo $g_{(8,4)}(F) = g(F^{*})$, for an argument similar to the one used above. By (83) and (89) we get :

$$g_{(8,4)}(F) = v-(e+d^*+t^*)+(s+t^*)$$

$$= v-e-d^*+s. \tag{91}$$

$$= v-e-d+t-s. \tag{92}$$

The formulas (90) and (92) can be found in page 349 of [19].

We will now give another formula, which comes from [8] and can also be found in page 349 of [19].

Let Q be the configuration $\begin{matrix}1 & 0\\ 0 & 0\end{matrix}$ and R the configuration $\begin{matrix}1 & 1\\ 0 & 0\end{matrix}$. Let q and r be the number of configurations in F which are -up to a rotation of $k\,\frac{\pi}{2}$ (k=0,1,2,3)- equal to Q and R respectively.

By counting the number of pairs (X,Y) such that $X \subseteq Y$, X is of type E and Y is a square, we get :

$$2e = r+2t^*+4s$$

$$= r+2t-4s \qquad \text{(using (88))}$$

and hence :

$$r = 2e-2t+4s. \tag{93}$$

By counting the number of pairs (X,Y) such that $X \subseteq Y$, X is of type V and Y is a square, we get :

$$4v = q+2r+2d^*+3t^*+4s$$

$$= q+4e-4t+8s+2d-2t+4s+3t-12s+4s$$

$$= q+4e-3t+2d+4s \qquad \text{(using (88),(89) and (93))}$$

and hence

$$q = 4v-4e+3t-2d-4s. \tag{94}$$

Then it is easily checked that by (88), (89), (90) and (91) we have :

$$\frac{1}{4}\,(q-t^*+2d^*) = v-e+s = g_{(4,8)}(F) \tag{95}$$

and :

$$\frac{1}{4}\,(q-t^{*}-2d^{*}) = v-e+s-d^{*} = g_{(8,4)}(F) \tag{96}$$

Note that in [19], (95) and (96) are interpretated as follows : the genus of F is one fourth of the number of convex corners minus the number of concave corners.

Other formulas for the genus of a figure can be found in [8].

Note : Elementary topology of the hexagonal grid

It seems that most of the topics of this chapter are relevant to the hexagonal grid. Thus one can speak of paths (instead of k-path), of the hexagonal distance d (corresponding to d_4 and d_8), of simple closed paths, of surrounding (instead of k-surrounding), of the border $\delta(X,Y)$ (instead of the k-border $\delta_k(X,Y)$), of the edge-following algorithm (which is then unique, since it does not depend on k or k'). Moreover, it is likely that all important results of this chapter can be proved in the hexagonal case, but they become simplified by the uniqueness of the adjacency relation.

Appendix 1. A sequential algorithm for computing distances to a set

Let U be a set of pels. Suppose that the pels of G are ordered; we label them $x_0,\ldots,x_{NM-1}$ (we suppose that the grid is finite). Let d be a regular distance. Assume the following :

If $\{x_b,x_c\} \subseteq N_d(x_a)$ for $a < b,c$, then either $\{x_b,x_c\} \subseteq N_d(x_e)$ for some $e > b,c$, or $\{x_b,x_c\} \subseteq U$. (97)

If $x_a \notin U$, then there exist $b < a$ such that

$$x_b \in N_d(x_a). \tag{98}$$

Now if d is isotropic, then for two pels $x = (i,j)$ and $y = (i',j')$, $y \in N_d(x)$ if and only if $(i'-i,\ j'-j) \in I$ for some set I invariant under the transformations $(u,v) \to (\pm u,\ \pm v)$ and $(u,v) \to (\pm v,\pm u)$. Let $q = |I|$ and let F_d be the set of pels x such that $|N_d(x)| < q$. For example, if $d = d_4$ or d_8, then F_d is the frame FG.

If we order the pels in the lexicographical order, then for an isotropic distance d, conditions (97) and (98) are implied by the following :

$$F_d \subseteq U. \tag{99}$$

Indeed if in (97) $x_a = (i_a,\ j_a)$, $x_b = (i_b,\ j_b)$ and $x_c = (i_c,\ j_c)$, then if $x_b \notin F_d$ we have only to take $x_e = (i_b+i_c-i_a,\ j_b+j_c-j_a)$ (in other words x_a,x_b,x_e and x_c forming a parallelogram) and then x_e has the required properties. On the other hand, if $x_a = (i_a,j_a)$ in (98), then we take $x_b = (i_a+i,i_b+j)$, where $i < 0$ or $i = 0$ and $j < 0$.

Let us now give a consequence of (97) :

For any $x_a \in G$, if $d(x_A,U) = r > 0$, then there is some $s \in \{0,\ldots,r\}$ and a sequence $x_{a_0},x_{a_1},\ldots,x_{a_{r-1}},x_{a_r}$ such that :

(i) $x_{a_0} = x_a$ and $x_{a_r} \in U$

(ii) $x_{a_{i+1}} \in N_d\,(x_{a_i})$ for $i=0,\dots,r$.

(iii) $a_i < a_{i+1}$ for $0 \leqslant i \leqslant s-1$.

$a_i > a_{i+1}$ for $s \leqslant i \leqslant r-1$. (100)

Indeed, by the regularity of d, we know that there is a sequence satisfying (i) and (ii). For any such sequence, we write

$$\lambda_i = 1 \quad \text{if} \quad a_i < a_{i+1}$$
$$= 0 \quad \text{if} \quad a_i > a_{i+1} \quad (0 \leqslant i \leqslant r-1)$$

and $$\lambda = \sum_{i=0}^{r-1} i\lambda_i .$$

If we take such a sequence with minimum λ and if (iii) is not satisfied, then there is some $i \in \{0,\dots,r-2\}$ such that $a_i > a_{i+1} < a_{i+2}$, in other words $\lambda_i = 0$ and $\lambda_{i+1} = 1$. By (97) there is some a'_{i+1} such that $x_{a_i}, x_{a_{i+2}} \in N_d(x_{a'_{i+1}})$ and $a_i < a'_{i+1} > a_{i+2}$. By replacing a_{i+1} by a'_{i+1} in the sequence, we get $\lambda_i = 1$ and $\lambda_{i+1} = 0$, and so λ is decreased by 1, which is a contradiction. Thus (iii) holds.

Now the algorithm can be performed in two passes over the pels x_a, the first one the direct order, and the second one in the reverse order. Indeed we define sequentially :

$$g(x_a) = 0 \quad \text{if} \quad x_a \in U,$$
$$= 1 + \min\{g(x_b) \mid x_b \in N_d(a) \text{ and } b < a\}$$
$$\text{if } x_a \notin U, \qquad (a=0,\dots,NM-1), \tag{101}$$

and then :

$$h(x_a) = \min\{g(x_a), 1+h(x_b) \mid x_b \in N_d(a) \text{ and } b > a\},$$
$$(a=NM-1,\ldots,0). \qquad (102)$$

Now we claim that $h(x_a) = d(x_a,U)$. First we show that $g(x_a) \geqslant d(x_a,U)$ for any $a=0,\ldots,NM-1$. We use induction on a. If a=0, then it is true by (98). If $a > 0$ and the result is true for $b < a$, then either $a \in U$ and so $g(a) = 0 = d(x_a,U)$, or there exists some $b < a$ such that $x_b \in N_d(x_a)$. For any such b we have :

$$g(x_b) \geqslant d(x_b,U) \qquad \text{(by induction hypothesis)}$$

and $$d(x_b,U) \geqslant d(x_a,U)-1 \qquad \text{(by triangularity)}$$

Therefore, we get :

$$1+g(x_b) \geqslant d(x_a,U).$$

Using (101), it follows that $g(x_a) \geqslant d(x_a,U)$. By induction hypothesis, the result is thus true for any a.

We now show that $h(x_a) \geqslant d(x_a,U)$ for any a. If $a = NM-1$, then $h(x_a) = g(x_a)$ and so the result holds. Suppose that $a < NM-1$ and that the result holds for $b > a$. If there is some $b > a$ such that $x_b \in N_d(x_a)$, then we have for any such b :

$$h(x_b) \geqslant d(x_b,U) \qquad \text{(by induction hypothesis}$$

and $$d(x_b,U) \geqslant d(x_a,U)-1 \qquad \text{(by triangularity)}$$

Hence we get :

$$1+h(x_b) \geqslant d(x_a,U).$$

Using (102) and the fact that $g(x_a) \geqslant d(x_a,U)$, we obtain that $h(x_a) \geqslant d(x_a,U)$. By induction hypothesis, the result is thus true for any a.

Finally we show that $h(x_a) \leqslant d(x_a,U)$ for any a. We use induction on $r = d(x_a,U)$. If $r=0$, then $g(x_a) = 0$ and $h(x_a) = 0$ by (101) and (102). Suppose that $r > 0$ and that the result is true for any x_b such that $d(x_b,U) < r$.

Consider the sequence $x_a = x_{a_0}, x_{a_1}, \ldots, x_{a_r} \in U$ described in (100). It is clear that $d(x_{a_i},U) = r-i$ for $i=0,\ldots,r$. We have two cases : $s>0$ or $s=0$.

If $s > 0$, then $a_1 > a$ and we have thus by (102) and the induction hypothesis :

$$h(x_a) \leqslant 1+h(x_{a_1}) \leqslant 1+d(x_{a_1},U) = r.$$

If $s=0$, then $a_{i+1} < a_i$ for $i=0,\ldots,r-1$ and (101) implies that for $i=0,\ldots,r-1$:

$$g(x_{a_i}) \leqslant g(x_{a_{i+1}})+1$$

As $g(x_{a_r}) = 0$, it follows that $g(x_a) = g(x_{a_0}) = r$ and so (102) implies that $h(x_a) \leqslant r$. Thus the result holds for a in any of the two cases. By induction hypothesis, it holds for any a.

This algorithm is presented in page 356 of [19] in the case where $d=d_4$, and a short proof is given in [20].

Appendix 2. Proofs of propositions concerning simple closed paths

Proof of Proposition 1. From the center of any pel $z \in B$ we can draw a half-line L in any direction and a perpendicular line M; then M and L determine two quarters of the plane, Π_1 on the right of L and Π_0 on the left (see Figure 1-41). It is clear that for any $v=0,\ldots,n-1$, $|\alpha_v(z)| \leqslant \frac{\pi}{2}$. Thus if P crosses L, then it passes from a pel in Π_1 (including the right side of M but not L) to a pel in Π_0 (including L), or conversely. Now $\alpha_p(z)$ is the number of times that P turns around z. Thus it is equal to the number of $u \in \{0,\ldots,n-1\}$ such that $x_u \in \Pi_1$ and $x_{u \oplus 1} \in \Pi_0$, minus the number of $v \in \{0,\ldots,n-1\}$ such that $x_v \in \Pi_0$ and $x_{v \oplus 1} \in \Pi_1$. If one has two pels z and $z' \in B$, one can make the construction of Figure 1-42. It is clear that P cannot go from Π_B to Π_D or from Π_A to Π_C. Therefore $\alpha_p(z)-\alpha_p(z')$ is the number of $u \in \{0,\ldots,n-1\}$ such that $x_u \in \Pi_A$ and $x_{u \oplus 1} \in \Pi_D$, minus the number of $v \in \{0,\ldots,n-1\}$ such that $x_v \in \Pi_D$ and $x_{v \oplus 1} \in \Pi_A$. Applying this to y_0 and y_1, the result follows.

Proof of Theorem 3. As said in (45) for any $t=0,\ldots,n-1$, $B \cap S_8(x_t)$ has two k'-connected components. Now it is easily seen from Figure 1-15 that $S_8(x_{t \oplus 1})$ intersects both of them.

Let us now show that $S_8(Q) \cap B$ has at most 2 k'-connected components (here Q is the set defined in point (iii) of the Theorem). We use induction on s. If $s = 0$, then the result holds by our preceding remark. Suppose that $s > 0$ and that the result holds for s-1. Write $Q' = \{x_r,\ldots, x_{r \oplus (s-1)}\}$. Now $B \cap S_8(Q) = (B \cap S_8(Q')) \cup (B \cap S_8(x_{r \oplus s}))$, both $B \cap S_8(Q')$ and $B \cap S_8(x_{r \oplus s})$ have at most two k'-components, and by our preceding remark (with $t=r \oplus s$), both k'-connected components of $B \cap S_8(x_{r \oplus s})$

intersect $S_8(x_{r \oplus (s-1)})$; thus they intersect $B \cap S_8 (x_{r \oplus (s-1)}) \subseteq B \cap S_8(Q')$, and it follows that $B \cap S_8(Q)$ has at most two k'-connected components, and the result holds for s. Thus it holds for any value of s, and in particular for s=n-1; in other words, $B \cap S_8(P)$ has at most two k'-connected components.

Now it is clear that any pel $y \in B$ is 4-connected in B to a pel $z \in B \cap S_8(P)$;indeed there is a 4-path from y to any $x_r \in P$, and the pel of that path preceding the first pel in P can be chosen as z.

Therefore B has at most two k'-connected components, containing each all the pels which are 4-connected in B to a same k'-connected component of $S_8(P) \cap B$.

If $y \in FG$, then $\alpha_p(y) = 0$, because P cannot pass beyond the side of FG to which y belongs. (See also the proof of Proposition 1 and Figure 1-41).

If we take y and z in two distinct k'-connected components of $S_8(x_r) \cap B$, then it is easily seen from Proposition 1 and Figure 1-15 that $|\alpha_p(y)-\alpha_p(z)| = 1$. More precisely, we have $\alpha_p(z) = \alpha_p(y)-1$ if $(x_{r \ominus 1}, x_r, x_{r \oplus 1})$ leaves z on its right, while we have $\alpha_p(z) = \alpha_p(y) + 1$ if it leaves z on its right.

It follows thus from Corollary 2 (iii) that y and z do not belong to the same k'-connected component and so B has at least two k'-connected components. By what we say above, it follows that B has two k'-connected components C_0 and C_1, where C_0 contains a pel $u \in FG$. By Corollary 2 (iii), $\alpha_p(v) = \alpha_p(u) = 0$ for any $v \in C_0$, while $\alpha_p(v) = \varepsilon$ for any $v \in C_1$, where ε is a constant equal to 1 if P leaves C_1 on its left and -1 if P leaves C_1 on its right (this follows from the preceding paragraph).

Clearly $C_0 \cap S_8(Q)$ and $C_1 \cap S_8(Q)$ are k'-disconnected in B. As $B \cap S_8(Q)$ has at most two k'-connected components, it follows that $C_0 \cap S_8(Q)$ and $C_1 \cap S_8(Q)$ are the two k'-connected components of $B \cap S_8(Q)$.

Consider $x \in B$ and the set $H(x)$ defined before Theorem 3. When $H(x)$ intersects P in a run R, let y be the pel in H(x) preceding R and z be the one following it. Clearly $y,z \in S_8(R)$, and it follows from the definition of touching and crossing that if H(x) touches P in R, then y and z belong to the same k'-connected component of $S_8(R)$, while the contrary holds if H(x) crosses P in R. As these two connected components are $C_0 \cap S_8(R)$ and $C_1 \cap S_1(R)$, there is a touching if and only if y and z belong to the same k'-connected component C_i (i=0,1). If u is the last pel in H(x), then $u \in FG \subseteq C_0$. If $x \in I(P)$, then there is an odd number of crossings and so $x \in C_1$. If $x \in O(P)$, then there is an even number of crossings and so $x \in C_0$.

Therefore $C_0 = O(P)$ and $C_1 = I(P)$.

Hence the points (i), (ii), (iii) and (v) of the Theorem hold. Now point (iv) follows from the fact that for any such symmetrical y and z, they belong to distinct k'-connected components of $S_8(x_r) \cap B$, as can be seen in Figure 1-15.

Proof of Proposition 4. Define the closed k-path $P' = (x'_0,\dots,x'_m)$ as follows :

(i) $x'_0 = x_0$

(ii) If for some $i \geqslant 0$, $x'_i = x_{n-1}$, then set m=i

(iii) For $i \geqslant 0$, if $x'_i \neq x_{n-1}$, then let $j = \max\{r \in \{0,\dots,n-1\} | n_{i'} \in N_k (x_r)\}$ and set $x'_{i+1} = x_j$.

Then it is easily checked that P' is a simple closed k-path and that $x'_m = x_{n-1}$ and $x'_1 = x_1$, in other words that x_{n-1}, x_0 and x_1 are successive pels in P'.

Now $S_8(x_0) \cap B$ must by hypothesis be one of the configurations of Figure 1-15, except that in the second one for k=4, the condition may not hold. Then one easily verifies that $S_8(x_0) \cap B$ has two k'-connected components, which are included each in one of the two connected components of $S_8(x_0) \cap B'$ (where B' is the background of P'), in other words in I(P') and O(P').

Appendix 3. Proofs of propositions concerning edges

Proof of Theorem 5. We have only to prove the result for $\varepsilon(X_4,Y_8)$, because the result for $\varepsilon(X_8,Y_4)$ follows then by interchanging F and B. We subdivide each pel of G in 9 as shown in Figure 1-43, and we get then a new grid G^*. For any set V of pels, write V^* for its image by this transformation.

Now this transformation respects the RFA, the property of k-connectedness and the relation of neighbouring between sets. Thus X_4^* is a 4-connected component of F, Y_8^* is a neighbouring 8-connected component of B and $\varepsilon(X_4,Y_8)$ becomes $\varepsilon(X_4^*,Y_8^*)$.

Let C^* be a cycle in $\varepsilon(X_4^*,Y_8^*)$. Let P^* be the portion of $\delta_8(X_4^*,Y_8^*)$ along C^*. It is easily seen from Figure 1-44 that for every pel $x^* \in P^*$, $|N_4(x^*) \cap P^*| = 2$. As C^* is a cycle, P^* is a simple closed 4-path.

Let R^* be the background of P^*. By theorem 3(i), R^* has two 8-connected components, $I(P^*)$ and $O(P^*)$. As Y_8^* is 8-connected and $Y_8^* \subseteq R^*$, we have $Y_8^* \subseteq I(P^*)$ or $Y_8^* \subseteq O(P^*)$. We can suppose that $Y_8^* \subseteq O(P^*)$ (if $Y_8^* \subseteq I(P^*)$, then we interchange $I(P^*)$ and $O(P^*)$ in the following argument). Now $P^* \subseteq X_4^*$. For any three successive edge elements in C^* arising from a single edge element of $\varepsilon(X_4,Y_8)$, we get the configuration of Figure 1-45. Following Theorem 3 (iv), it follows that the three pels in the bottom row of that square belong to P^* or $I(P^*)$. In particular, the one in the middle of that row belongs to $I(P^*)$. Thus $I(P^*) \cap X_4^* \neq \emptyset$. Now X_4^* is 4-connected. Let $x^* \in X_4^* \cap I(P^*)$ and let y^* be any other element of X_4^*; then there is a 4-path $Q^* \subseteq X_4^*$ joining x^* and y^*. If Q^* intersects $O(P^*)$, then the first pel z^* of $Q^* \cap O(P^*)$ belongs to Y_8^*, since it belongs to $\delta_4(O(P^*),P^*)$, a set which is included in Y_8^*, as can be seen from Figure 1-45. But this is impossible, since $Q^* \subseteq X_4^*$. Thus $Q^* \subseteq P^* \cup I(P^*)$ and so $y^* \in P^* \cup I(P^*)$. As y^*

was an arbitrary element of X_4^*, $X_4^* \subseteq P^* \cup I(P^*)$. If $u \in X_4^*$, $v \in Y_8^*$ and $\{u,v\} \in \varepsilon(X_4^*,Y_8^*)$, then $v \in O(P^*)$, $u \in P^* \cup I(P^*)$ and $d_4(u,v) = 1$. It follows then (by Theorem 3(i)) that $u \in P^*$ and so that $\{u,v\} \in C^*$. Thus C^* is the only cycle of $\varepsilon(X_4^*,Y_8^*)$ and so $\varepsilon(X_4,Y_8)$ has only one cycle.

Proof of Proposition 6. We will in fact show the following result : If Y is an 8-connected component of B, if X is 8-connected and is a union of 4-connected components of F, and if X and Y are neighbouring, then $\varepsilon^+(X,Y)$ contains only one cycle. This result contains Proposition 6, because X_8 is a union of 4-connected components of F.

We use induction on the number t of 4-connected components of F contained in X. If t=1, then the result holds by Theorem 5.

Suppose now that $t > 1$ and that the result is true for $1,\dots,t-1$. Write $X = Z_1 \cup \dots \cup Z_t$, where $Z_1,\dots,Z_t$ are 4-connected components of F. At least one of the sets Z_i must be neighbouring Y. We may suppose that it is Z_1. Let $U_1,\dots,U_s$ be the 8-connected components of $X \setminus Z_1$; each U_i is a union of Z_j's. As X is 8-connected, Z_1 is 8-connected to each U_i.

We make the same transformation from G to G^* as in the proof of Theorem 5. Then taking $P^* = \delta_8(Z_1^*,Y^*)$, P^* is a simple closed 4-path and we have either $Y^* \subseteq O(P^*)$ and $Z_1^* \subseteq P^* \cup I(P^*)$, or $Y^* \subseteq I(P^*)$ and $Z_1^* \subseteq P^* \cup O(P^*)$. We can assume that the first holds (otherwise we interchange $I(P^*)$ and $O(P^*)$ in the following argument).

For $i=1,\dots,s$, either U_i is not neighbouring Y or U_i satisfies the same hypothesis as X, but with a smaller number of 4-connected components. In that case $\varepsilon^+(U_i,Y)$ forms a single cycle. As $P^* \subseteq Z_1^*$, $U_i^* \subseteq I(P^*) \cup O(P^*)$; as U_i^* is 8-connected, we have $U_i^* \subseteq I(P^*)$ or $U_i^* \subseteq O(P^*)$. If $U_i^* \subseteq I(P^*)$, then U_i^* may not be neighbouring Y^* (since $Y^* \subseteq O(P^*)$), and so U_i is not neighbouring Y.

Hence if U_i is neighbouring Y, then $U_i^* \subseteq O(P^*)$, and U_i^* is 8-connected to $Z_1^* \subseteq I(P^*) \cup P^*$. As $O(P^*)$ is not 8-connected to $I(P^*)$, U_i^* is 8-connected to P^*. Thus there is a 2×2 rectangle $\begin{matrix} a & u \\ z & b \end{matrix}$ such that $z \in P^*$ and $u \in U_i^*$, but $a,b \notin F^*$ (since U_i is not 4-connected to Z_1). Thus $a,b \in O(P^*)$ (since $a,b \notin P^*$ and $O(P^*)$ is 8-connected), and so $a,b \in \delta_4(O(P^*),P^*)$. Now $\delta_4(O(P^*), P^*) \subseteq Y^*$ (this is explained in the preceding proof with Figure 1-45). Thus $a,b \in Y^*$ and the square $\begin{matrix} a & u \\ z & b \end{matrix}$ in G^* as in the center of a 2×2 square $\begin{matrix} a_i & u_i \\ z_i & b_i \end{matrix}$ in G, with $a_i,b_i \in Y$, $z_i \in Z_1$ and $u_i \in U_i$.

Now $\varepsilon^+(U_i,Y)$ contains the two successive elements (u_i,a_i) and (u_i,b_i), while $\varepsilon^+(Z_1,Y)$ contains the two successive elements (z_i,b_i) and (z_i,a_i). Now we know that $\varepsilon^+(Z_1,Y)$ and $\varepsilon^+(U_i,Y)$ form each a single cycle. Now we change the ordering of the edge elements of $\varepsilon^+(X,Y)$ in the following way :

- (z_i,b_i) is followed by (u_i,b_i) $(i=1,\dots,s)$
- (u_i,a_i) is followed by (z_i,a_i) $(i=1,\dots,s)$

Then the cycles of $\varepsilon^+(X,Y)$ are merged in a single cycle and so the result holds.

<u>Proof of Proposition 7</u>. Write $\varepsilon^+(X,Y) = \{\varepsilon_0,\dots,\varepsilon_{n-1}\}$, where $\varepsilon_i = (x_i,y_i)$ $(x_i \in X, y_i \in Y)$ for each i. Now for each i, ε_i and ε_j (where $j \equiv i+1 \pmod{n}$) form (up to a rotation) one of the configurations of Figure 1-46.

We have $\delta_4(X,Y) = \{x_0,\dots,x_{n-1}\}$. Now it is clear from Figure 1-46 that each x_i is an 8-neighbour of x_j. Thus $\delta_8(X,Y)$ is a closed 8-path.

Now we can write $\delta_8(X,Y) = \{x_0,z_0,\dots,x_{n-1},z_{n-1}\}$, where z_i is an element of X between x_i and x_j, which is 8-adjacent to Y. Concretely we have $z_i = x_i$ in (a) and (b), while in (c) $z_i = x_i$ if $v \in Y$ and $z_i = v$ if $v \in X$.

Then it is clear that x_i and x_j are 8-neighbours of z_i and so $\delta_8(X,Y)$ is a closed 8-path. Moreover, if X is 4-connected, then $\varepsilon^+(X,Y)$ can be found by the edge-following mapping E_F^+, in other words, we have $\varepsilon_j = \varepsilon_i E_F^+$. It follows thus (see Figure 1-23) that in (c) we must have $v \in X$ and so $z_i = v$. But then it is easily checked that x_i and x_j are in (a), (b) and (c) 4-neighbours of z_i and so $\delta_8(X,Y)$ is a closed 4-path.

<u>Proof of Proposition 8</u>. It is clear that $\varepsilon(X,\bar{X}) = \varepsilon(X,B)$. Moreover, it is easily seen that $\varepsilon(X,\bar{X})$ and $\varepsilon(X,B)$ have the same cycles under the edge-following algorithm, because this algorithm depends only on the shape of X. If $Y_1,\ldots,Y_m$ are the k'-connected components of $\bar{X}$ neighbouring X, then these cycles are $\varepsilon(X,Y_1),\ldots,\varepsilon(X,Y_m)$. Moreover, B has m k'-connected components neighbouring X, say $Z_1,\ldots,Z_m$, where $\varepsilon(X,Y_i) = \varepsilon(X,Z_i)$ for each $i=1,\ldots,m$. It follows that $\delta_4(Y_i,X) = \delta_4(Z_i,X)$, $\delta_4(X,Y_i) = \delta_4(X,Z_i)$ and $\delta_8(X,Y_i) = \delta_8(X,Z_i)$ for each i. It is clear that each $Y_i \cap B$ is a union of k'-connected components of B; as Z_i and $Y_i \cap B$ have $\delta_4(Y_i,X)$ in common and as Z_i is k'-connected, $Z_i \subseteq Y_i \cap B$ and so Y_i is the k'-connected component of $\bar{X}$ containing Z_i for each i.

Appendix 4. Proofs of results concerning surrounding

Proof of Theorem 9. The "if" part follows from Theorem 3(iii). The "only if" part is proved in 4 steps :

Step 1. We may suppose that $S \cap FG = \emptyset$.

Proof. If $S \cap FG \neq \emptyset$, then we extend the grid by one pel on each side of FG, and we get a new grid G' with frame FG'. We have then $S \cap FG' = \emptyset$ and we have only to prove that S k-surrounds T primitively in G'. If $P = (x_0, \ldots, x_{m-1})$ is a k-path from a pel in T to a pel in FG', if x_n is the first pel in P such that $x_n \in FG'$, then $P' = (x_0, \ldots, x_{n-1})$ is a k-path from a pel in T to a pel in FG and so P' intersects S; thus P intersects S. If $S' \subset S$, then there is a k-path $Q = (y_0, \ldots, y_{m-1})$ joining a pel in T to a pel y_{m-1} in FG such that Q does not intersect S. Now y_{m-1} is 4-adjacent to a pel $y_m \in FG'$; hence the k-path $Q' = (y_0, \ldots, y_{m-1}, y_m)$ joins a pel of T to a pel of FG', and Q' does not intersect S. Therefore S surrounds T primitively in G'.

Step 2. Let V be the k-connected component of $G \setminus S$ containing T, and let W be the one containing FG. Then $V \neq W$ and for any $y \in S$, $N_k(y) \cap V \neq \emptyset \neq N_k(y) \cap W$.

Proof. If $V = W$, then T is k-connected to FG in $G \setminus S$, which is impossible; thus $V \neq W$. Let $y \in S$. As S k-surrounds T primitively, $S \setminus \{y\}$ does not k-surround T. Thus there is a k-path P from T to FG, which does not intersect $S \setminus \{y\}$; however P intersects S, since S surrounds T, and so $P \cap Y = \{y\}$. As P is a k-path from V to W, it follows that the pel v preceding the first occurrence of y in P belongs to V, and that the one w following the last occurrence of y in P belongs to W. As $v, w \in N_k(y)$, the result holds.

Step 3. Choose $y = (i,j) \in S$ such that $iN+j$ is maximum. Then $N_{k'}(y) \cap S = \{y',y''\}$, with $y' \notin N_{k'}(y)$, and $S_8(y) \setminus S$ has two k-connected components, namely $S_8(y) \cap V$ and $S_8(y) \cap W$.

Proof. It is clear that the pels $(i,j+1)$, $(i+1,j-1)$, $(i+1,j)$ and $(i+1,j+1)$ are in $G \setminus S$ and are 4-connected to FG in $G \setminus S$ (by the path $(x_1,\ldots,x_{n-1})$, where $x_r = (i+r,j)$ and $n=M-i$; hence they belong to W.

Now y has a k-neighbour $x \in V$. As V is not k-connected in $G \setminus S$ to pels in W, there must be two k'-neighbours y' and y'' of y such that x lies between y and y' in $S_8(y)$ (see Figure 1-47). If k=8, then the result holds. If k=4, then the result holds, except if $N_8(y)$ contains 3 elements y',y',y^* of S. This situation is illustrated in Figure 1-48. Now we get a contradiction as follows :

In a), y' has a 4-neighbour in V. Thus $u \in V$. But then we must have $z \in S$, since u may not be 4-connected to W in $G \setminus S$. But now $z=(i',j')$ with $i'N+j' > iN+j$, which contradicts the hypothesis.

In b), y^* has a 4-neighbour in W. Thus $f \in W$ and hence $c \notin V$, since V and W are not neighbouring. Now y'' has a 4-neighbour in V. As $c \notin V$, it follows that $d \in V$. As d is not 4-connected to W in $G \setminus S$, we must have $g \in S$. But then $g = (i',j')$ with $i'N+j' > iN+j$, which contradicts the hypothesis.

Step 4. The result holds.

Proof. Let Y be the k'-connected component of S containing y. Then $Y = \delta_k(Y,V)$ and by Proposition 7, Y is a closed k'-path. By Proposition 4, Y contains a simple closed k'-path P such that y', y and y'' are successive pels in P. By Theorem 3 (iii) and by Step 3, $\{I(P) \cap S_8(y), O(P) \cap S_8(y)\} = \{V \cap S_8(y), W \cap S_8(y)\}$. As I(P) and O(P) are the two k-connected components of $G \setminus P$, as $FG \subseteq W$ and as V and W are k-connected, we have $V \subseteq I(P)$ and $W \subseteq O(P)$. Thus P k-surrounds V, and as $T \subseteq V$, P k-surrounds T. As $P \subseteq S$ and S k-surrounds T primitively, $P=S$ and the result holds.

Proof of Lemma 12. Let $z \in Y \setminus \{y\}$ and let Q be a k-path from z to FG. By definition of Y, there is a k-path R from y to z such that R does not intersect X. Let P be the k-path built by concatenating R and Q. As X surrounds y and P is a k-path from y to FG, P must intersect X and so R intersects X. Thus X k-surrounds z and the result holds.

Proof of Corollary 13. If $Y \not\subseteq O_k(X)$, then there is some $y \in Y$ such that X k-surrounds y. But then X k-surrounds Y by Lemma 12, which is a contradiction.

Proof of Theorem 14. By Corollary 13, it is clear that if X 4-surrounds Y_i, then $X \subseteq O_k(Y)$, and that if Y_i 4-surrounds X, then $Y_i \subseteq O_{k'}(Y)$. Thus the results (a) and (b) reduce to :

(a) X k'-surrounds Y_i.

(b) Y_i k-surrounds X.

Suppose that for some $i = 1,\ldots,m$, result (a) does not hold. Then $FG \subseteq G \setminus X$ and Y_i is k-connected to FG in $G \setminus X$. Now by Proposition 8 the k'-connected component of $G \setminus X$ which contains Y_i and FG may not contain any other Y_j, $j \neq i$. Therefore there is at most one $i \in \{1,\ldots,m\}$ such that Y_i does not satisfy (a). Moreover, this happens if and only if $FG \subseteq G \setminus X$, i.e. $FG \cap X = \emptyset$.

Let $P = (x_0,\ldots,x_{n-1})$ be a 4-path from a pel of X to a pel of FG. Let x_u be the last pel in P which belongs to X. Then $x_{u+1} \notin F$ (since $x_{u+1} \in N_4(x_u)$ and X is a union of 4-connected components of F) and so $x_{u+1} \in Y_j$ for some $j=1,\ldots,m$. If $j \neq i$, then $Q = (x_{u+1},\ldots,x_{n-1})$ is a 4-path from a pel in Y_j to a pel in FG and Q does not intersect X; this means that X does not 4-surround Y_j, which is impossible, since Y_j satisfies (a). Hence $j=i$ (i.e. $x_{u+1} \in Y_i$) and as P was chosen arbitrarily, it follows that Y_i 4-surrounds X.

As the RFA is symmetrical between F and B, we can apply the preceding argument to Y_i, where we intervert B and F and also k and k'.

Thus if the k-connected components of F neighbouring Y_i are $X = Z_1$, $\ldots,Z_w$, then for any $j=1,\ldots,w$, Y_i k-surrounds Z_j or Z_j 4-surrounds Y_i. As Y_i 4-surrounds X, X may not 4-surround Y_i. It follows then that Y_i k-surrounds X and so Y_i satisfies result (b).

Now we prove first Proposition 16, and then Proposition 15.

<u>Proof of Proposition 16</u>. If Y_i 4-surrounds X, then $Y_i \subseteq O_8(X)$ by Corollary 13, and so (b) holds. If Y_i does not 4-surround X, then consider the 4-connected components $Z_1,\ldots,Z_u$ of X which neighbour Y_i. If some Z_j 8-surrounds Y_i, then X 8-surrounds Y_i and $X \subseteq O_8\ (Y_i)$ by Corollary 13, in other words (a) holds. If this does not happen, then Y_i 4-surrounds every Z_j by Theorem 14. As Y_i does not 4-surround X, there is some $y,z \in X$ such that Y_i does not surround y and $z \in Z_j$ for some $j=1,\ldots,u$. As X is 8-connected, there is an 8-path $P = (y_0,\ldots,y_{n-1})$ such that $y_0 = z$, $y_{n-1} = y$ and each y_s $(s=0,\ldots,n-1)$ belongs to X. Let y_v be the last pel in P which is 4-surrounded by Y_i; then $v < n-1$. Write $x_0 = y_{v+1}$; then there is a 4-path $Q = (x_0,\ldots,x_{g-1})$ connecting y_{v+1} to FG, such that Q does not intersect Y_i. If $w \in N_4(y_v) \cap N_4(y_{v+1})$, then the 4-path $R = (y_v,w,x_0,\ldots,x_{g-1})$ connects y_v to FG. As Y_i 4-surrounds y_v, R must intersect Y_i, and as Q does not intersect Y_i, we get $w \in Y_i$. Thus y_{v+1} neighbours Y_i and so $y_{v+1} \in Z_j$ for some $j=1,\ldots,u$, which is a contradiction, since Y_i 4-surrounds Z_j, but not y_{v+1}.

Thus we have proved that either (b) or (a) holds.

Clearly if $FG \subseteq X$, then (b) holds for no Y_i. Let us now show that if $X \cap FG = \emptyset$, then (b) holds for exactly one Y_i.

Let $X_1,\dots,X_t$ be the 4-connected components of X which are not 8-surrounded by other 4-connected components of X. Then $X' = X_1 \cup \dots \cup X_t$ is 8-connected. Indeed, given any $x, x' \in X'$, there is an 8-path $P = (x_0, \dots, x_h)$ such that $x_0 = x$, $x_h = x'$ and every $x_r \in X$ $(r=1,\dots,h)$. Now for any $r=1,\dots,h$, $x_r \in X'$ or $x_r \in I_8(X_q)$ for some $q=1,\dots,t$. If for some r, s (with $r \leqslant s$) we have $x_{r-1}, x_{s+1} \notin I_8(X_q)$ but $x_j \in I_8(X_q)$ for $r \leqslant j \leqslant s$, then we have $x_{r-1}, x_{s+1} \in X_q$ and so the portion $Q = (x_{r-1}, x_r, \dots, x_s, x_{s+1})$ of P can be replaced by a path Q' in X_q. Thus if we make this replacement for any such r, then P is replaced by an 8-path P' which is included in X', and so X' is 8-connected.

If for two distinct $p,q \in \{1,\dots,t\}$, X_p and X_q are 8-adjacent, then there is some $x \in X_p$ and $x' \in X_q$ such that x and x' are 8-adjacent, but not 4-adjacent. Let $y \in N_4(x) \cap N_4(x')$. Then clearly $y \in B$. As X_p does not 4-surround X_q, $x' \in O_4(X_p)$ and so $y \in O_4(X_p)$ by Lemma 12. Similarly $y \in O_4(X_q)$. If, in consequence of Theorem 14, we write V_a for the unique 8-component of B which neighbours X_a and 4-surrounds it $(a=1,\dots,t)$, then $y \in V_p \cap V_q$, in other words $V_p = V_q$. As X' is 8-connected, this means that there is an 8-component V of B such that $V_a = V$ for $a=1,\dots,t$. Then clearly if Y_i 4-surrounds X, then Y_i neighbours some X_a $(a=1,\dots,t)$ (otherwise Y_i would neighbour a 4-component of X in some $I_8(X_a)$, and we would have $Y_i \subseteq I_8(X_a)$), and so $Y_i = V$. Conversely V 8-surrounds X and neighbours it. Thus V is the only Y_i which satisfies (b).

<u>Proof of Proposition 15.</u> We use Proposition 16. As X is an 8-connected half-component of F, for any $i=1,\dots,m$, either (a) holds or we have the following :

(b') Y_i 4-surrounds X.

Now Proposition 16 can be applied interverting F and B. As each Y_i is an 8-connected half-component of B, for any $i=1,\dots,m$, either (b) holds or we have the following :

(a') X 4-surrounds Y_i.

As (a') and (b') are incompatible, (a') must imply (a) and so either (a) or (b) holds. By Proposition 16, there is at most one i such that (b) holds, and this happens only if $X \cap FG = \emptyset$.

<u>Proof of Proposition 18.</u> We use induction on $d_8(X,Y)$. If $d_8(X,Y) = 1$, then X and Y are neighbouring 8-connected components of F and B. Suppose now that $d_8(X,Y) > 1$ and that the result holds for any Y' satisfying the hypothesis such that $d_8(X,Y') < d_8(X,Y)$.

We can suppose without loss of generality that $Y \subseteq B$. Let Y' be the 8-connected component of F which neighbours Y and 8-surrounds it (Y' exists and is unique by Proposition 13). Then X 8-surrounds Y'. Indeed, if X does not 8-surround Y', then as $Y \cup Y' \subseteq G \setminus X$ and $Y \cup Y'$ is 8-connected, Lemma 12 implies that X does not 8-surround Y, which is impossible.

If $d_8(X,Y) = d$, then there is an 8-path $U = (u_0,\dots,u_d)$ such that $u_0 \in Y$ and $u_d \in X$. Then $u_1,\dots,u_{d-1} \notin Y$ and so $u_1 \in 0_8(Y)$, since $X \subseteq 0_8$ (Y). As u_1 belongs to an 8-connected component of F neighbouring Y, we must have $u_1 \in Y'$. Thus $d(X,Y') \leqslant d-1$ and so we may apply induction hypothesis : There exists a sequence $X = Z_0,\dots,Z_{n-1} = Y'$ satisfying (i) and (ii). Taking $Y = Z_n$, then the result holds.

<u>Proof of Proposition 19</u>. We use induction on $d_4(X,Y)$. If $d_4(X,Y) = 1$, then X neighbours Y and by Proposition 16, $\varepsilon(X,Y) = \varepsilon^{O_8}(Y)$. Let $X_1,\ldots,X_g$ be the 4-connected components of X such that for every $i=1,\ldots,g$, X_i neighbours Y and $X_i \subseteq O_4(Y)$. Let $X' = X_1 \cup \ldots \cup X_g$. Now as $\varepsilon^{O_4}(Y) \subseteq \varepsilon^{O_8}(Y) = \varepsilon(X,Y)$ and as $X \cap O_4(Y)$ is a union of 4-connected components of X (by Lemma 12), $\varepsilon(X',Y) = \varepsilon^{O_4}(Y)$. By Proposition 17, $\varepsilon^{O_4}(Y)$ consists of a single cycle. Thus, following the argument of Proposition 7, $\delta_4(X',Y)$ is a closed 8-path. As $\delta_4(X',Y) = \delta_4(X_1,Y) \cup \ldots \cup \delta_4(X_g,Y)$, this means that X' is 8-connected. It is clear that X' 4-surrounds Y, because any 4-path from Y to FG must cross $\varepsilon^{O_4}(Y)$ and so cross $\delta_4(X',Y) = \delta_4^{O_4}(Y)$. Taking $Z_0 = X'$ and $Z_1 = Y$, the result holds in this case.

Suppose now that $d_4(X,Y) > 1$ and that the result holds for any X^*, Y^* satisfying the hypothesis with $d_4(X^*,Y^*) < d_4(X,Y)$. We can suppose without loss of generality that $Y \subseteq B$. Let Z be the 8-connected component of F which 4-surrounds Y. As $d_4(Z,Y) = 1$, the induction hypothesis implies that Z contains an 8-connected half-component Z' such that the result holds for $Z' = Z_0$ and $Y = Z_1$. In fact Z' is (by the preceding paragraph) a union of 4-connected components of Z neighbouring Y. Thus $Z' \cap X = \emptyset$ and every pel of Z' is 4-connected in $Y \cup Z'$ to a pel in Y. Hence X 4-surrounds Z' (by Lemma 12). Now an argument similar to the one used in Proposition 18 shows that $d_4(X,Z') < d_4(X,Y)$. By induction hypothesis, there is a sequence $Z_0,\ldots,Z_{n-1} = Z'$ which verifies to the statement of the proposition. Let us take $Z_n = Y$. Then clearly for every $j=0,\ldots,n-r$, $Z_n \cap Z_j \subseteq I_4\,(Z_{j+1}) \cap O_4\,(Z_{j+1}) = \emptyset$ and so the sequence $Z_0,\ldots,Z_n$ verifies the result.

Proof of Proposition 20. It is the same as the proof of Proposition 18, except that we use induction on $d_4(X,Y)$ (instead of $d_8(X,Y)$), and that we replace 8-connected components by k- or k'-connected components.

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