REPORT R415

<u>Cellular Permutation</u>

<u>Networks : A Survey</u>

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Cellular Permutation Networks : A Survey

§1. Introduction.

A permutation network P on n bits is a switching circuit with n data input terminals I_0, I_1, ..., I_{n-1}, p control input terminals C_0, C_1,...,C_{p-1} and n data output terminals O_0, O_1,..., O_{n-1}, which can realize the n! following input-output behaviours :

$$P(\pi) : \text{For } i=0,1,\dots,n-1, \quad [I_i] = [O_{i\pi}] \; , \; (*)$$

where π is an arbitrary permutation of $\{0,1,\dots,n-1\}$ and $i\pi$ is the image of i under π. We say then that P realizes π .

We consider that the control variables are binary. Therefore $p \geqslant \log_2 (n!)$.

We will study the design of permutation networks on an arbitrary number n of bits, using as components smaller prefabricated permutation networks called cells. A cell on m bits is called an m-cell.

We are specially interested in designs using only 2-cells (also called β-elements [10]).

Two important problems confronting designers are the minimization of the cost and delay.

(*) If T is a terminal, then [T] is the signal on T.

We will describe and compare several designs and study their minimization (with respect to cost and delay). In some cases, we will give an algorithm for the control of the network.

§II. Notations and definitions.

Lower-case Latin letters will be used to denote numbers, and in particular positive integers.

Capital Latin letters (except Z, C, I and O) will denote switching networks. We will always write "input" for "data input terminal", "output" for "data output terminal" and "control line" for "control input terminal". If N is a switching network with n inputs, m outputs and p control lines, then we will write $I_i(N)$, $O_j(N)$ and $C_k(N)$ respectively for the ith input $(i=0,1,\dots,n-1)$, the jth output $(j=0,1,\dots,m-1)$ and the kth control line $(k=0,1,\dots,p-1)$ of N. Let $I(N) = \{I_i(N) \mid i=0,1,\dots,n-1\}$, $O(N)= \{O_j(N) \mid j=0,\dots,m-1\}$ and $C(N)= \{C_k(N) \mid k=0,\dots,p-1\}$. If $|I(N)| = |O(N)| = n$, then we can write N(n) for N.

If n is a positive integer, then let $Z_n=\{0,1,\dots,n-1\}$ and let Sym(n) be the group of all permutations of Z_n. An element of Sym(n) will be written as a lower-case Greek letter (except $\lambda,\phi,\psi,\theta,\gamma,\delta$, and σ). If $i \in Z_n$ and $\pi \in$ Sym(n), then $i\pi$ will be the image of i by π.

Let n be a positive integer and let Π be a subset of Sym(n). A partial permutation network $P(n,\Pi)$ on n bits is a switching circuit P with n inputs, n outputs and p control lines (where $p \geqslant \log_2(|\Pi|)$), which can realize the $|\Pi|$ input-output behaviours :

$$P(\pi) : \text{For } i=0,1,\ldots,n-1, \quad [I_i(P)] = [O_{i\pi}(P)],$$

where π is an arbitrary element of Π. By analogy with mechanical switching devices, one says that $O_{i\pi}(P)$ is connected with $I_i(P)$.

Given k partial permutation networks $P_i = P_i(n_i)$ (where $i=0,1,\ldots,k-1$) a set $I = \{I_0,\ldots,I_{n-1}\}$ of terminals called inputs, a set $O = \{O_0,\ldots, O_{n-1}\}$ of terminals called outputs (where $n \geqslant n_i$ for each $i=0,1,\ldots,k-1$) and a set Γ of connections between the terminals I_i, O_j, $I_r(P_u)$, $O_s(P_v)$, let us call $N(I,O; P_0,\ldots,P_{k-1}; \Gamma)$ the resulting network. We suppose that Γ consists only of connections of the type $(I_i, I_r(P_u))$, $(O_s(P_v), I_r(P_u))$, $(O_s(P_v), O_j)$ and (I_i,O_j). Then $N(I,O; P_0,\ldots,P_{k-1}; \Gamma)$ is a partial permutation network if :

(i) All connections are one-to-one.

(ii) The circuit contains no loop, in other words there is an order ε on Z_k such that if $(O_s(P_v), I_r(P_u)) \in \Gamma$, then $v \in u$.

The first condition ensures that the input-output behaviours are one-to-one. The second condition is introduced in order to avoid asynchronous sequential networks.

An m-cell is usually represented by a square or a vertical segment, with the inputs on the left and the outputs on the right (this is illustrated in Fig.1(a) for m=5. A 2-cell and its states are shown in Fig.1(b). The designs of a 2-cell using multiplexers or logical gates are shown in Fig. 2(a) and (b) respectively.

A <u>cellular partial permutation network</u> is a partial permutation network P of the form $N(I,O; P_0,\ldots,P_{k-1};\Gamma)$, where $P_0,\ldots,P_{k-1}$

are cells. If P is a permutation network, then we say that it is a <u>cellular permutation network</u>.

Consider two cellular partial permutation networks on n bits having both the same number k of cells, say $P=N(I,O;\ P_0,\dots,P_{k-1};\ \Gamma)$ and $Q=N(I',O';\ Q_0,\ \dots,\ Q_{k-1};\ \Delta)$.

(1) P and Q are <u>isomorphic</u> and write $P \cong Q$ if there is a map ϕ: $\{P_0,\dots,P_{k-1}\} \to \{Q_0,\dots,Q_{k-1}\}$ such that :

(i) For $i=0,\dots,k-1$, P_i and $P_i\phi$ have the same number of inputs (or outputs)

(ii) The map ϕ' induced by ϕ on Γ , defined by

$$(I_i,O_j)\phi' = (I_i',\ O_j')$$
$$(I_i,\ I_r(P_u))\phi' = (I_i',\ I_r(P_u\phi))$$
$$(O_s(P_v),\ I_r(P_u))\phi' = (O_s(P_v\phi),\ I_r(P_u\phi))$$
$$(O_s(P_v),\ O_j)\phi' = (O_s(P_v\phi),\ O_j')$$

is a bijection $\Gamma \to \Delta$.

(2) P and Q are <u>equivalent</u> and write $P \simeq Q$ if P is isomorphic to Q up to a relabelling of the inputs and outputs of Q.

(3) P and Q are <u>quasiisomorphic</u> and write $P \approx Q$ if P and Q are isomorphic up to a relabelling of the inputs and outputs of each Q_i.

(4) P and Q are <u>quasiequivalent</u> and write $P \sim Q$ if P and Q are equivalent up to a relabelling of the inputs and outputs of each Q_i.

These concepts are illustrated in Fig.3(a), (b), (c) and (d). These 4 relations are equivalence relations.

Lastly, let S be the simple connection between 1 input and 1 output.

§III. Shuffles and generalized shuffles.

This section is a summary of [16] .

When making connections between different stages of cells, one often uses permutations called generalized shuffles. To define them, one needs first to define mixed radix number representation systems.

Let $b_0,\ldots,b_{n-1}$ be integers bigger than 1 ; let $q=b_0.\ldots.b_{n-1}$. Then any integer comprised between 0 and q-1 can be represented in a unique way as a vector $[a_{n-1},\ldots,a_0]$ with $a_i \in Z_{b_i}$ for i=0,...,n-1, by the following rule

$$[a_{n-1},\ldots,a_0] = a_{n-1}b_{n-2}.\ldots.b_0+a_{n-2}b_{n-3}.\ldots.b_0+\ldots a_1b_0+a_0 \;.$$

This representation of the elements of Z_q is called the mixed radix representation with respect to the basis $[b_{n-1},\ldots,b_0]$ (See [4]).

If has the following property : Suppose that for i=0,...,n-1, $b_i=b_{i,0} \ldots b_{i,m(i)-1}$. Let $m \in Z_q$. If m has $[a_{n-1},\ldots,a_0]$ as mixed radix representation with respect to the basis $[b_{n-1},\ldots,b_0]$ and if for i=0,...,n-1, a_i has $[a_{i,m(i)-1},\ldots,a_{i,0}]$ as mixed radix representation with respect to the basis $[b_{i,m(i)-1},\ldots,b_0]$, then m has

$$[a_{n-1,m(n-1)-1},\ldots,a_{n-1,0},\ldots,a_{i,m(i)-1},\ldots,a_{i,0},\ldots,a_{0,m(0)-1},\ldots,a_{0,0}]$$

as mixed radix representation with respect to the basis

$$[b_{n-1,m(n-1)-1},\ldots,b_{n-1,0},\ldots,b_{i,m(i)-1},\ldots,b_{i,0},\ldots,b_{0,m(0)-1},\ldots,b_{0,0}] .$$

Let us now define the perfect shuffle. Let q=ab. Any element m of Z_q can be written as ib+j (with $i \in Z_a$ and $j \in Z_b$) or as j'a+i' (with $i' \in Z_a$ and $j' \in Z_b$). The (a,b)-shuffle on Z_q is the permutation $\sigma(a,b)$ of Z_q defined as follows (see [4]) :

$$\sigma(a,b): m = ib+j \rightarrow m\sigma(a,b) = ja+i \qquad (i \in Z_a,\ j \in Z_b).$$

Thus $\sigma(a,b)$ maps [i,j] (in the basis [a,b]) onto [j,i] (in the basis [b,a])

Note that $\sigma(a,b)$ fixes 0 and q-1 and that $\sigma(b,a)$ is the inverse of $\sigma(a,b)$.

We now define a generalization of the perfect shuffle, called the <u>generalized shuffle</u>. Let q be an integer bigger than 1 and suppose that $q=b_{n-1}\cdot\ldots\cdot b_0$, where each b_i is an integer bigger than 1. Let $m \in Z_q$. If $\pi \in Sym(n)$, then we can write m in the mixed radix representation with respect to the basis $[b_{(n-1)\pi}, \ldots, b_{0\pi}]$:

$$m = a_{(n-1)\pi}\, b_{(n-2)\pi} \cdots b_{0\pi} + \ldots + a_{1\pi}\, b_{0\pi} + a_{0\pi}$$

$$= \sum_{i=0}^{n-1} \left(a_{i\pi} \prod_{j=0}^{i-1} b_{j\pi}\right), \text{ where } a_i \in Z_{b_i} \text{ for } i \in Z_n .$$

Now, if ρ is another permutation of Z_q, then $a_{(n-1)\rho}\, b_{(n-2)\rho} \cdots b_{0\rho}+\ldots+a_{1\rho} b_{0\rho}+ a_{0\rho} = \sum_{i=0}^{n-1} (a_{i\rho} \prod_{j=0}^{i-1} b_{j\rho})$ is the mixed radix representation of a number $m' \in Z_q$ with respect to the basis $[b_{(n-1)\rho},\ldots,b_{0\rho}]$.

We define the $\binom{(n-1)\pi,\ldots,\ 0\pi}{(n-1)\rho,\ldots,\ 0\rho}$-shuffle on Z_q as the following permutation of Z_q :

$$\sigma\binom{(n-1)\pi,\ldots,0\pi}{(n-1)\rho,\ldots,0\rho} : \sum_{i=0}^{n-1}\left(a_{i\pi}\prod_{j=0}^{i-1} b_{j\pi}\right) \to \sum_{i=0}^{n-1}\left(a_{i\rho}\prod_{j=0}^{i-1} b_{j\rho}\right) .$$

It corresponds to the following change of basis in a mixed radix representation of Z_N :

$$[b_{(n-1)\pi},\ldots, b_{0\pi}] \to [b_{(n-1)\rho},\ldots,b_{0\rho}] .$$

If n=2, then $\sigma(b_1,b_0)= \sigma\binom{1,0}{0,1}$ with respect to the basis $[b_1,b_0]$.

If n=3, then we will write $b_2\ \sigma(b_1,b_0)$ for $\sigma\binom{2,1,0}{2,0,1}$ and $\sigma(b_2,b_1)b_0$ for $\sigma\binom{2,1,0}{1,2,0}$. It is easily seen that $b_2\ \sigma(b_1,b_0)$ is the union of b_2 copies of $\sigma(b_1,b_0)$, while $\sigma(b_2,b_1)b_0$ induces $\sigma(b_2,b_1)$ on b_2b_1 sets of size b_0.

The generalized shuffles have the following two properties :

(1) : If $\pi,\rho,\tau \in \mathrm{Sym}(n)$, then we have :

$$\sigma\binom{(n-1)\pi,\ldots,\ 0\pi}{(n-1)\rho,\ldots,\ 0\rho}\ \sigma\binom{(n-1)\rho,\ldots,\ 0\rho}{(n-1)\tau,\ldots,\ 0\tau} = \sigma\binom{(n-1)\pi,\ldots,\ 0\pi}{(n-1)\tau,\ldots,\ 0\tau}$$

In particular, $\sigma\binom{(n-1)\rho,\ldots,\ 0\rho}{(n-1)\pi,\ldots,\ 0\pi}$ is the inverse of $\sigma\binom{(n-1)\pi,\ldots,\ 0\pi}{(n-1)\rho,\ldots,\ 0\rho}$.

(2) : If for $i \in Z_n$, $b_i=b_{i,0}\cdot\ldots\cdot b_{i,m(i)-1}$, then $\sigma\binom{(n-1)\pi,\ldots,\ 0\pi}{(n-1)\rho,\ldots,\ 0\rho}$ (with respect to the basis $[b_{n-1},\ldots,b_0]$)

$$= \sigma\binom{((n-1)\pi,m((n-1)\pi)-1),\ldots,((n-1)\pi,0),\ldots,(0\pi,m(0\pi)-1),\ldots,(0\pi,0)}{((n-1)\rho,m((n-1)\rho)-1),\ldots,((n-1)\rho,0),\ldots,\ (0\rho,m(0\rho)-1),\ldots,(0\rho,0)}$$

(with respect to the basis $[b_{n-1,m(n-1)-1},\ldots,b_{n-1,0},\ldots,b_{0,m(0)-1},\ldots,b_{0,0}]$).

<u>Example</u>. If n=4, then $\sigma\binom{3,2,1,0}{1,0,3,2}= \sigma(b_3b_2,b_1,b_0)$.

§IV. Operations on partial permutation networks.

We will define here ten operations on partial permutation networks.

(1) Dual. This operation is defined for cellular partial permutation networks only. Let $P=N(I,O;\ P_0,\dots,P_{k-1};\Gamma)$, where $P_0,\dots,P_{k-1}$ are cells. Then the dual P^* of P is built as follows : $P^*=N(I,O;\ P_0,\dots,P_{k-1};\Delta)$, where $\Delta=\{(Y^*,X^*)\ |\ (X,Y)\in\Gamma)$, with $I_i^*=O_i$, $O_j^*=I_j$, $I_r(P_u)^* = O_r(P_u)$ and $O_s(P_v)^*=I_s(P_v)$.

Thus P^* is built from P by inverting inputs and outputs in all cells and all connections. This construction is illustrated in Fig.4.

Clearly, if P realizes the set Π of permutation, then P^* realizes $\Pi^{-1}=\{x^{-1}\ |\ \pi\in\Pi\}$.

(2) Union. Let $A_0,\dots,A_{n-1}$ be partial permutation networks. Then we define the partial permutation network $A_1\cup\dots\cup A_n$ by taking pairwise disjoint copies of $A_1,\dots,A_n$, taking $I(A_1)\cup\dots\cup I(A_n)$ as set of inputs and $O(A_1)\cup\dots\cup O(A_n)$ as set of outputs and considering the resulting network.

(3) Left scalar multiplication. Let m be a positive integer and A a partial permutation network on n bits. Let $A^{(0)},\dots,A^{(m-1)}$ be m disjoint copies of A. Label the ith input/output of $A^{(j)}$ $(i\in Z_n,\ j\in Z_m)$ nj+i. Then $mA=A^{(0)}\cup\dots\cup A^{(m-1)}$ with this labelling.

(4) Right scalar multiplication. We do like in (3), but label the ith input/output of $A^{(j)}$ mi+j. Then we get the netwok Am. Note that mA and Am are equivalent.

(5) <u>Composition</u>. Let $A_0,\dots,A_{m-1}$ be partial permutation networks on n bits. For i=0,...,m-2 and j=0,1,...,n-1, connect $O_j(A_i)$ with $I_j(A_{i+1})$. Take $I(A_0)$ as set of inputs and $O(A_{m-1})$ as set of outputs. Then the resulting network is $A_0.A_1 \dots A_{m-1}$.

In the five preceding operations, one can replace a partial permutation network by a permutation (which can be considered as a cellular permutation network without cell and without control line). If π and ρ are permutations, then $\pi^*=\pi^{-1}$ and the composition $\pi.\rho$ is the group-theoretic product of π by ρ . Note that the definitions of $m\sigma(a,b)$ and $\sigma(a,b)m$ given in §III are identical to the ones given in (3) and (4) of this section.

Let us now define five more constructions using the perfect shuffle :

(6) <u>Product</u> [15] . Let A and B be partial permutation networks on a and b bits respectively. Then the product A×B is the partial permutation network $bA.\sigma(b,a).aB$.

(7) <u>Extended product</u> [15] . Take A and B as in (6). Suppose that A is cellular. Then the extended product A××B is the partial permutation network $bA.\sigma(b,a).aB.\sigma(a,b).bA^*$.

If A and B are permutation networks, then A××B is a permutation network by the theorem of Slepian and Duquid [5,17] .

(8) <u>The Goldstein-Leibholtz construction</u> [6] .

Let A and B be as in (7).

Then the Goldstein-Leibholz construction A ∧ B is built as follows : Take the extended product A××B, delete the first copy of A^* in the third stage and replace it by aS, where S is a simple connection.

If A and B are permutation networks, then $A \wedge B$ is a permutation network by Theorem 1 of [6] . We prove this result in the Appendix.

(9) A construction of Benes [1,2 (p. 114, Theorem 3.10)] .

Let $A_0, \ldots, A_{n-1}$ be cellular partial permutation networks on $a_0, \ldots, a_{n-1}$ bits respectively. Let $q = a_0 \cdot \ldots \cdot a_{n-1}$. Then we define $F(A_0, \ldots, A_{n-1}) =$

$$\prod_{i=0}^{n-2} \left(\frac{q}{a_i} A_i \, \sigma\left(a_{i+1}, \frac{q}{a_i+1}\right)\right) \frac{q}{a_{n-1}} A_{n-1} \cdot \prod_{i=n-2}^{0} \left(\sigma\left(\frac{q}{a_{i+1}}, a_{i+1}\right) \frac{q}{a_i} A_i^*\right) .$$

Benes showed that if $A_0, \ldots, A_{n-1}$ are permutation networks , then $F(A_0, \ldots, A_{n-1})$ is a permutation network. We will show later that $F(A_0, \ldots, A_{n-1})$ is equivalent to $A_0 \times\times (A_1 \times\times (\ldots \times\times (A_{n-2} \times\times A_{n-1}) \ldots))$ (which generalizes Benes'result).

(10) The truncation method. This method, designed by several authors [7,11,12,13] , can be used to build permutation networks on m bits when m is of the form rn-k, with $k \in Z_n$, $n > 1$ and $r > 1$.

Indeed, let r and n be integers larger than 1 and let $k \in Z_n$. Let A, A', B and B' be partial permutation networks on n,n-k,r and r-1 bits respectively (a partial permutation network on 1 bit is the simple connection S). Suppose that A is cellular.

Now (A,A',B,B') is defined as follows : Take $A \wedge B$. Replace the first copy of A in the first stage by $kS \cup A'$. Replace the k first copies of B in the second stage by k copies of $S \cup B'$. Then I_i is connected to O_i for $i \in Z_k$. Remove these k inputs, outputs and all interconnections between them.

There remains a partial permutation network on rn-k bits, written (A,A',B,B').

We will show later that if A,A',B and B' are permutation networks, then (A,A',B,B') is a permutation network. Note that for k=0, we have $(A,A,B,B')=A \wedge B$.

The constructions (6), (7), (8) and (10) are illustrated on Fig. 5,6,7 and 8 respectively.

Let us now prove the two announced results. We need first the following :

LEMMA 1. If B is a partial permutation network on n bits and if $\pi \in \mathrm{Sym}(m)$, then $\pi n.mB.(\pi n)^{-1} \cong mB$.

The proof is elementary and is left to the reader.

PROPOSITION 2. Let $A_0,\ldots,A_{n-1}$ be cellular partial permutation networks. Then $F(A_0,\ldots,A_{n-1})$ is equivalent to $A_0 \times\times (A_1 \times\times (\ldots \times\times (A_{n-2} \times\times A_{n-1})\ldots))$.

Proof. We can suppose that each A_i is on a_i bits. Let $q=a_0\ldots a_{n-1}$. Then we can write $F(A_0,\ldots,A_{n-1})$ as :

$$\prod_{i=0}^{n-2} \left(\frac{q}{a_i} A_i\ \beta(i,i+1)\right).\ \frac{q}{a_{n-1}}\ A_{n-1}\ .\ \prod_{i=n-2}^{0} \left(\beta(i+1,i)\ \frac{q}{a_i}\ A_i^*\right),$$

where $\beta(i,i+1)=\sigma(a_{i+1}, \frac{q}{a_{i+1}})$ and $\beta(i+1,i)=\beta(i,i+1)^{-1}$.

By induction, we verify that $A_0 \times\times (A_1 \times\times (\ldots \times (A_{n-2} \times A_{n-1})\ldots))$ can be written as :

$$\prod_{i=0}^{n-2} \left(\frac{q}{a_i} A_i . \pi(i,i+1)\right).\ \frac{q}{a_{n-1}} A_{n-1}\ .\ \prod_{i=n-2}^{0} \left(\pi(i+1,i).\frac{q}{a_i} A_i^*\right),$$

where $\pi(i,i+1)=a_0,\ldots,a_{i-1}\ \sigma(a_{i+1},\ldots,a_{n-1},a_i)$ and $\pi(i+1,i)=\pi(i,i+1)^{-1}$.

With respect to the basis $[a_{n-1},\ldots,a_0]$, we can write for $i=0,\ldots,n-2$:

$$\pi(i,i+1) = \sigma\begin{pmatrix}0,\ldots,i-1, & n-1, & \ldots\ldots, & i\\ 0,\ldots\ldots, & i, & n-1,\ldots, & i+1\end{pmatrix}$$

and $$\beta(i,i+1) = \sigma\begin{pmatrix}i+1,\ldots & \ldots\ldots, & n-1,0, & \ldots, & i\\ i+2,\ldots, & n-1, & 0, & \ldots\ \ldots, & i+1\end{pmatrix}.$$

For $i=0,\ldots,n-2$, define :

$$\psi(i) = \sigma\begin{pmatrix}0, & \ldots, & i-1, & n-1, & \ldots, & i\\ i+1, & \ldots, & n-1, & 0, & \ldots, & i\end{pmatrix}.$$

Let $\psi(n-1)$ be the identity. Then we can easily check that for $i=0,\ldots,n-2$, we have :

$$\psi(i).\beta(i,i+1) = \pi(i,i+1).\psi(i+1).$$

Thus we get :

$$\beta(i,i+1) = \psi(i)^{-1}.\pi(i,i+1).\psi(i+1) \text{ and}$$

$$\beta(i+1,i) = \psi(i+1)^{-1}.\pi(i+1,i).\psi(i).$$

Hence $F(A_0,\ldots,A_{n-1})$

$$= \prod_{i=0}^{n-2}\left(\frac{q}{a_i}A_i.\psi(i)^{-1}\pi(i,i+1)\psi(i+1)\right).\frac{q}{a_{n-1}}A_{n-1}.\prod_{i=n-2}^{2}\left(\psi(i+1)^{-1}\pi(i+1,i)\psi(i).\frac{q}{a_i}A_i^*\right)$$

$$= \psi(0)^{-1}\prod_{i=0}^{n-2}(B_i.\pi(i,i+1)).B_{n-1}.\prod_{i=n-2}^{0}(\pi(i+1,i).B_i^*).\psi(0),$$

where $B_i=\psi(i).\frac{q}{a_i}A_i\ \psi(i)^{-1}$ for $i\in Z_n$.

Now for $i\in Z_n$, $\psi(i)=\phi(i)a_i$ for some $\phi(i)\in \mathrm{Sym}(\frac{q}{a_i})$.

By Lemma 1, it follows that $B_i \cong \frac{q}{a_i} A_i$. Thus $F(A_0,\ldots,A_{n-1})$

$$\cong \psi(0)^{-1} \prod_{i=0}^{n-2} \left(\frac{q}{a_i} A_i\ \pi(i,i+1)\right) \frac{q}{a_{n-1}} A_{n-1} \prod_{i=n-2}^{0} \left(\pi(i+1,i)\ \frac{q}{a_i} A_i^*\right).\psi(0)$$

$$\cong \psi(0)^{-1}.\ (A_0\times\times(A_1\times\times(\ldots(A_{n-2}\times A_{n-1})\ldots))).\psi(0).$$

Therefore the proposition follows.

Let us now prove our second announced result:

PROPOSITION 3. Let A,A',B and B' be the permutation networks on n, $n-k$, r and $r-1$ bits respectively, where r and n are integers bigger than 1 and $k \in Z_n$. Then (A,A',B,B') is a permutation network.

Proof. Consider $A \wedge B$. It is a permutation network. It can therefore realize all permutations of $\Pi=\{\pi \in \mathrm{Sym}(rn) \mid i\pi=i \text{ for } i \in Z_k\}$. Let $\pi \in \Pi$. If $A \wedge B$ is in a state realizing π, then $I_i(A \wedge B)$ must be connected to $O_i(A \wedge B)$ for $i \in Z_k$. Now $I_i(A \wedge B)$ is connected by $A^{(0)}$ to some $O_j(A^{(0)})$, which is connected to $I_0(B^{(j)})$, where $j \in Z_n$. Now $O_i(A \wedge B)$ is connected to $O_0(B^{(i)})$. As $I_0(B^{(j)})$ must be connected to $O_0(B^{(i)})$, we have $i=j$. Thus for $i \in Z_k$, $I_i(A \wedge B)=I_i(A^{(0)})$ is connected to $O_i(A^{(0)})$ and $I_0(B^{(i)})$ is connected to $O_0(B^{(i)})$. Thus, if we replace $A^{(0)}$ by $kS \cup A'$ and each $B^{(i)}$ $(i \in Z_k)$ by a copy of $S \cup B'$, then the resulting network realizes Π. If we delete for each $i \in Z_k$ $I_i(A \wedge B)$, $O_i(A \wedge B)$ and the connections between them, then the resulting network, which is (A,A',B,B'), can realize every permutation of $\mathrm{Sym}(rn-k)$, and so it is a permutation network.

The following result links the different operations studied up to now. The proof is elementary and is omitted.

PROPOSITION 4. For any partial permutation networks A and B on a and b bits respectively, for every positive integers m and n, we have :

(i) $(A \cup B)^* = A^* \cup B^*$.

(ii) $(mA)^* = m(A^*)$.

(iii) $(Am)^* = (A^*)m$.

(iv) $(A \cdot B)^* = B^* \cdot A^*$ (when $a=b$) .

(v) $(A \times B)^* = B^* \times A^*$.

(vi) $(A{\times\times}B)^* = A{\times\times}B^*$.

(vii) $Am \cong \sigma(a,m) . mA . \sigma(m,a)$.

(viii) $m(A.B) = (mA) . (mB)$.

(ix) $(A.B)m = (Am) . (Bm)$.

(x) $m(nA) = (mn)A$.

(xi) $(Am)n = A(mn)$.

(xii) $(mA)n = m(An)$.

Note that in these equalities, one can replace A or B by a permutation.

The following result is due to Pippenger [15] :

PROPOSITION 5. Let A,B and C be partial permutation networks. Then :

(i) $A\times(B \times C) \cong (A \times B) \times C$

(ii) If A and B are cellular, then $A{\times\times}(B{\times\times}C) \cong (A \times B){\times\times}C$

Proof. Suppose that A,B and C are on a,b and c bits respectively. Then it is easy to check that :

$$(A \times B)\times C = bcA.c\sigma(b,a).acB.\sigma(c,ab).abC.$$

$$A\times(B \times C) = bcA.\sigma(bc,a).caB.a\sigma(c,b).abC.$$

Now $\sigma(bc,a) = c\sigma(b,a).\sigma(c,a)b$ and

$$a\sigma(c,b) = (\sigma(c,a)b)^{-1}.\sigma(c,ab).$$

By Lemma 1, $caB \cong \sigma(c,a)b.acB.(\sigma(c,a)b)^{-1}$ and

$$\text{so } (A \times B)\times C \cong bcA.c\sigma(b,a).(\sigma(c,a)b.acB.(\sigma(c,a)b)^{-1}).\sigma(c,ab).abC$$

$$= A\times(B \times C)$$

Hence (i) follows. Now (ii) is proved in the same way.

§V. Some known designs for permutation networks built from 2-cells.

A. The networks of Benes, Waksman and Green.

Benes [2] defined a permutation network $B(2^n)$ on 2^n bits for every positive integer n.

Put : $B(2) = P(2)$, the elementary 2-cell

For $n > 1$, $B(2^n)=P(2)\times\times B(2^{n-1})$.

Waksman [20] made a similar construction with the operation $\wedge$ instead of $\times\times$. Let : $W(2)=P(2)$, and for $n=2,3,4,\ldots$, put $W(2^n)=P(2)\wedge W(2^{n-1})$.

Green [7] (see also [13]) gave a construction of a permutation network $G(m)$ for every $m > 1$. When $m=2^n$, $G(m)=W(m)$. This network is defined recursively as follows :

. $G(2) = P(2)$.

. For $a > 2$, $G(a)=P(2) \wedge G(\frac{a}{2})$ if a is even.

$=(P(2),S,G(\frac{a+1}{2}), G(\frac{a-1}{2}))$ if a is odd.

We will study the cost and delay of these networks. The cost is the number of 2-cells and the delay is the maximum number of cells through which an input can go before reaching an output.

It is easily seen that the networks $B(2^n)$ and $W(2^n)$ have delay $2n-1$.

Now the cost of $B(2^n)$ is 2^n plus twice the cost of $B(2^{n-1})$. It follows by induction that $B(2^n)$ has cost $n2^n-2^{n-1}$.

To study the cost of $G(n)$, let us define the function ψ, defined for every integer larger than 1 :

$$\psi(n) = \sum_{x=2}^{n} \lceil \log_2(x) \rceil .$$

LEMMA 6. For every integer $n \geqslant 2$, we have :

(i) $\psi(n) = n \lceil \log_2(n) \rceil - 2^{\lceil \log_2(n) \rceil} + 1$

(ii) $\psi(2n) = 2n-1+2\psi(n)$

(iii) $\psi(2n-1) = 2(n-1)+\psi(n)+\psi(n-1)$.

Proof. (i) Suppose first that $n=2^a$.

Then $\psi(n) = \sum_{x=1}^{a} 2^{x-1}.x$. We show easily by induction that $\psi(2^a)=(a-1)2^a+1$.

Hence the result holds for $n=2^a$.

Suppose now that $2^a < n < 2^{a+1}$. Then

$$\psi(n) = \psi(2^a) + \sum_{x=2^a+1}^{n} \lceil \log_2(x) \rceil$$

$$= \psi(2^a) + (n-2^a)(a+1)$$

$$= (a-1)2^a + 1 + (n-2^a)(a+1)$$

$$= n(a+1)-2^{a+1}+1 = n\lceil \log_2(n)\rceil - 2^{\lceil \log_2(n)\rceil} + 1 .$$

Hence the result holds.

(ii) $$\psi(2n) = 2n \lceil \log_2(2n)\rceil - 2^{\lceil \log_2(2n)\rceil} + 1$$

$$= 2n(\lceil \log_2(n)\rceil + 1) - 2^{\lceil \log_2(n)\rceil + 1} + 1$$

$$= 2(n\lceil \log_2(n)\rceil - 2^{\lceil \log_2(n)\rceil} + 1) + 2n - 1$$

$$= 2\psi(n) + 2n - 1 .$$

(iii) $$\psi(2n-1) = \psi(2n) - \lceil \log_2(2n)\rceil$$

$$= \psi(2n) - 1 - \lceil \log_2(n)\rceil$$

$$= 2n-1 + 2\psi(n) - 1 - \lceil \log_2(n)\rceil$$

$$= 2(n-1) + \psi(n) + (\psi(n) - \lceil \log_2(n)\rceil) .$$

$$= 2(n-1) + \psi(n) + \psi(n-1) .$$

PROPOSITION 7. $G(n)$ has cost $\psi(n)$.

Proof. Use induction on n. This is true for $n=2$. Suppose that $n > 2$ and that the result is true for $m < n$. We have two cases.

(i) n is even. Then the cost of $G(n)$ is :

$$n-1 + 2 \operatorname{cost}(G(\tfrac{n}{2})) = n-1+2\psi(\tfrac{n}{2}) = \psi(n)$$

by Lemma 6 (ii)

(ii) n is odd. Then the cost of G(n) is :

$$n-1 + \text{cost}(G(\frac{n+1}{2})) + \text{cost}(G(\frac{n-1}{2})) = n-1+\psi(\frac{n-1}{2})+\psi(\frac{n-1}{2}) = \psi(n)$$

by Lemma 6 (iii).

<u>PROPOSITION 8</u>. G(n) has delay $2\lceil \log_2(n) \rceil - 1$.

Again, this result is shown by induction on n.

<u>Remark</u>. G(n) has an inductive control algorithm, called "<u>looping</u>" (see [13,20]). This algorithm is relatively costly.

B. <u>Joel's nested tree</u> [10].

Let P(2) be the elementary 2-cell. Define Y(2)=P(2) and $Y(2^k)=Y(2^{k-1})\times P(2)$ for k=2,3,4,...

Joel builds the nested tree $T(2^k)$ ($k > 1$) as follows :

- For n=1,...,k, take a copy of $Y(2^n)$
- Take 2^k inputs $I_0,\ldots,I_{2^k-1}$ and 2^k outputs $O_0,\ldots,O_{2^k-1}$.
- Make the following connections :

(1) For n=1,...,k-1 and $m \in Z_{2^{n-1}}$ connect

$I_{2^{k-n}(2m+1)-1}$ with $I_{2m}(Y(2^n))$.

$I_{2^{k-n}(2m+1)}$ with $I_{2m+1}(Y(2^n))$.

$O_{2m}(Y(2^n))$ with $I_{2^{k-n}(2m+1)-1}(Y(2^k))$.

$O_{2m+1}(Y(2^n))$ with $I_{2^{k-n}(2m+1)}(Y(2^k))$.

(2) Connect : I_0 with $I_0(Y(2^k))$

I_{2^k-1} with $I_{2^k-1}(Y(2^k))$

and $O_j(Y(2^k))$ with O_j for every $j \in Z_{2^k}$.

This construction is illustrated in Fig. 9 for k=4.

Joel's nested tree $T(2^k)$ is not equivalent to the dual $W(2^k)^*$ of Waksman's network. This can be seen in Fig. 10 for k=2. Indeed, if all the cells are put in the 0-state, then two outputs (in both T(4) and $W(4)^*$) are reached after two stages. But in T(4), these two outputs are not connected to the same cell, while in $W(4)^*$ they are. In fact, we can prove the following :

<u>PROPOSITION 9</u>. For any $k \geqslant 2$, $T(2^k) \sim W(2^k)^*$.

<u>Proof</u>. Let us define T(2)=P(2). Then clearly $T(2)=W(2)^*$. Define Z(2)=P(2) a nd $Z(2^k)=P(2)\times Z(2^{k-1})$ for k=2,3,4,... Then $Z(2^k)\cong Y(2^k)$ for any $k \geqslant 1$ by Proposition 5(i). Thus we can replace $T(2^k)$ $(k \geqslant 1)$ by $T'(2^k)$, which is built as follows :

- T'(2)=T(2)
- If k > 1, then for n=1,...,k-1, replace $Y(2^n)$ by $Z(2^n)$ in the construction of $T(2^k)$.

Now clearly $T'(2^k)\cong T(2^k)$. The rest of the proof consists of 8 steps :

<u>Step 1.</u> The following eight maps are permutations of Z_{2^k} :

(1) $\alpha(k) = (0,2^k-1)$ (k=1,2,3,...)

(2) $\beta(k) = 2\alpha(k-1)=(0,2^{k-1}-1)(2^{k-1},2^k-1)$ (k=2,3,4,...)

(3) $\delta(k) = (1,2)\dots(2^k-3,2^k-2)$ (k=2,3,4,...)

(4) $\varepsilon(k) = (0,1)\dots(2^k-2,2^k-1)$ (k=1,2,3,...)

(5) $\tau(k) : x \to x \oplus 2^{k-1} \pmod{2^k}$ $(k=1,2,3,\ldots)$.

(6) $\pi(k)$ fixes $0, 2^{k-1}-1$, 2^{k-1}, 2^k-1 and for $n=2, \ldots, k-1$ (if $k \geqslant 3$) and $v \in Z_{2^{n-2}}$, $\pi(k)$ maps

$$2^{k-n}(4v+1)-1 \quad \text{on} \quad 2^{k-n}(2v+1)-1 \ ,$$

$$2^{k-n}(4v+3)-1 \quad \text{on} \quad 2^{k-n}(2v+1) \ ,$$

$$2^{k-n}(4v+1) \quad \text{on} \quad 2^{k-1}+2^{k-n}(2v+1)-1$$

and

$$2^{k-n}(4v+3) \quad \text{on} \quad 2^{k-1}+2^{k-n}(2v+1) \qquad (k=2,3,4,\ldots) \ .$$

(7) $\rho(k)$ maps 0 on 0, 2^k-1 on 1, and for $n=1,\ldots,k-1$ and $m \in Z_{2^{n-1}}$ (if $k \geqslant 2$), $\rho(k)$ maps :

$$2^{k-n}(2m+1)-1 \quad \text{on} \quad 2^n+2m$$

and

$$2^{k-n}(2m+1) \quad \text{on} \quad 2^n+2m+1 \qquad (k=1,2,3,\ldots)$$

(8) $\gamma(k) = \rho(k-1) \cup (\alpha(k-1).\rho(k-1))$ $(k=2,3,4,\ldots)$

It can be checked that $\gamma(k)$ maps 0 on 0, $2^{k-1}-1$ on 1, 2^{k-1} on $2^{k-1}+1$, 2^k-1 on 2^{k-1} and for $u=2,\ldots,k-1$ and $m \in Z_{2^{u-2}}$ (if $k \geqslant 3$), it maps :

$$2^{k-u}(2m+1)-1 \quad \text{on} \quad 2^{u-1}+2m \ ,$$

$$2^{k-u}(2m+1) \quad \text{on} \quad 2^{u-1}+2m+1 \ ,$$

$$2^{k-1}+2^{k-u}(2m+1)-1 \quad \text{on} \quad 2^{k-1}+2^{u-1}+2m$$

and

$$2^{k-1}+2^{k-u}(2m+1) \quad \text{on} \quad 2^{k-1}+2^{u-1}+2m+1 \qquad (k=2,3,4,\ldots)$$

<u>Step 2.</u> If $k \geqslant 2$ and if $x \in Z_{2^k} \setminus \{0, 2^{k-1}-1, 2^{k-1}, 2^k-1\}$, then $x\delta(k)\,\pi(k) = x\pi(k)\,\tau(k)$.

Indeed $\{x, x\delta(k)\}$ is a pair of the form $\{2m-1, 2m\}$. Now it is easily checked that $\pi(k)$ maps such a pair on a pair $\{n, n+2^{k-1}\}=\{n, n\tau(k)\}$, where $n \in Z_{2^{k-1}}$. As $\delta(k)=\delta(k)^{-1}$ and $\tau(k)=\tau(k)^{-1}$, the result follows.

Step 3. If $k \geqslant 2$, then $\tau(k)\ \sigma(2,2^{k-1})=\sigma(2,2^{k-1})\varepsilon(k)$.
This is due to the fact that if $m \in Z_{2^{k-1}}$, then $\sigma(2,2^{k-1})$ maps m on 2m and $m+2^{k-1}$ on 2m+1, and that $\tau(k)$ permutes the pairs $\{m, m+2^{k-1}\}$, while $\varepsilon(k)$ permutes the pairs $\{2m, 2m+1\}$

Step 4. If $k \geqslant 2$, then $\alpha(k)\delta(k)\pi(k) = \pi(k)\tau(k)\beta(k)$.
Proof. Clearly, both $\pi(k)$ and $\tau(k)$ preserve the set $\{0, 2^{k-1}, 2^{k-1}, 2^k-1\}$. It follows that if $x \in Z_{2^k} \setminus \{0, 2^{k-1}-1, 2^{k-1}, 2^k-1\}$, then $x\pi(k)\tau(k) \neq 0, 2^{k-1}, 2^{k-1}-1, 2^k-1$. Thus $x\pi(k)\tau(k)\beta(k)=x\pi(k)\tau(k)$

$= x\delta(k)\pi(k)$ (by Step 2)

$= x\alpha(k)\delta(k)\pi(k)$ since $\alpha(k)$ fixes x.

Now we check that :

$0\alpha(k)\delta(k)\pi(k) = (2^k-1)\delta(k)\pi(k) = (2^k-1)\pi(k) = 2^k-1$

$= 2^{k-1}\ \beta(k) = 0\tau(k)\beta(k) = 0\pi(k)\tau(k)\beta(k)$.

$(2^{k-1}-1)\alpha(k)\delta(k)\pi(k) = (2^{k-1}-1)\delta(k)\pi(k)=2^{k-1}\pi(k) = 2^{k-1}$

$= (2^k-1)\beta(k) = (2^{k-1}-1)\tau(k)\beta(k) = (2^{k-1}-1)\pi(k)\tau(k)\beta(k)$.

$2^{k-1}\alpha(k)\delta(k)\pi(k) = 2^{k-1}\delta(k)\pi(k) = (2^{k-1}-1)\pi(k) = 2^{k-1}-1$

$= 0\beta(k) = 2^{k-1}\tau(k)\beta(k) = 2^{k-1}\pi(k)\tau(k)\beta(k)$.

$(2^k-1)\alpha(k)\delta(k)\pi(k) = 0\delta(k)\pi(k) = 0\pi(k) = 0$

$= (2^{k-1}-1)\beta(k) = (2^k-1)\tau(k)\beta(k) = (2^k-1)\pi(k)\tau(k)\beta(k)$.

Step 5. If $k \geqslant 2$, then $\pi(k)\gamma(k)=\rho(k)\sigma(2^{k-1},2)$.

Proof. If $m \in Z_{2^{k-1}}$, then $\sigma(2^{k-1},2)$ maps $2m$ on m and $2m+1$ on $m+2^{k-1}$.
Now we check that :

$0\pi(k)\gamma(k) = 0\gamma(k) = 0$,

$(2^{k-1}-1)\pi(k)\gamma(k) = (2^{k-1}-1)\gamma(k) = 1$,

$2^{k-1}\pi(k)\gamma(k) = 2^{k-1}\gamma(k) = 2^{k-1}+1$,

$(2^{k}-1)\pi(k)\gamma(k) = (2^{k}-1)\gamma(k) = 2^{k-1}$,

and for $n=2,\dots,k-1$ and $v \in Z_{2^{n-2}}$ if ($k \geqslant 3$), we have :

$(2^{k-n}(4v+1)-1)\pi(k)\gamma(k) = (2^{k-n}(2v+1)-1)\gamma(k)= 2^{n-1}+2v$,

$(2^{k-n}(4v+3)-1)\pi(k)\gamma(k) = (2^{k-n}(2v+1))\gamma(k) = 2^{n-1}+2v+1$,

$(2^{k-n}(4v+1))\pi(k)\gamma(k) = (2^{k-1}+2^{k-n}(2v+1)-1)\gamma(k) = 2^{k-1}+2^{n-1}+2v$,

$(2^{k-n}(4v+3))\pi(k)\gamma(k) = (2^{k-1}+2^{k-n}(2v+1))\gamma(k) = 2^{k-1}+2^{n-1}+2v+1$.

Thus $\pi(k)\gamma(k)$ is known. Then we check that :

$0\rho(k)\sigma(2^{k-1},2) = 0\sigma(2^{k-1},2) = 0$,

$(2^{k-1}-1)\rho(k)\sigma(2^{k-1},2) = 2\sigma(2^{k-1},2) = 1$,

$2^{k-1}\rho(k)\sigma(2^{k-1},2) = 3\sigma(2^{k-1},2) = 2^{k-1}+1$,

$(2^{k}-1)\rho(k)\sigma(2^{k-1},2) = 1\sigma(2^{k-1},2) = 2^{k-1}$,

and for $n=2,\dots,k-1$ and $v \in Z_{2^{u-2}}$ (if $k \geqslant 3$), we have :

$(2^{k-n}(4v+1)-1)\rho(k)\sigma(2^{k-1},2) = (2^{n}+4v)\sigma(2^{k-1},2) = 2^{n-1}+2v$,

$(2^{k-n}(4v+3)-1)\rho(k)\sigma(2^{k-1},2) = (2^{n}+4v+2)\sigma(2^{k-1},2) = 2^{n-1}+2v+1$,

$(2^{k-n}(4v+1))\rho(k)\sigma(2^{k-1},2) = (2^{n}+4v+1)\sigma(2^{k-1},2) = 2^{k-1}+2^{n-1}+2v$,

$(2^{k-n}(4v+3))\rho(k)\sigma(2^{k-1},2)=(2^{n}+4v+3)\sigma(2^{k-1},2)= 2^{k-1}+2^{n-1}+2v+1$.

We see then that $\pi(k)\gamma(k) =\rho(k)\sigma(2^{k-1},2)$.

Step 6. For $k \geqslant 2$, define $R_2(k)=2^{k-1}\,P(2)$ and $R_1(k)=S\cup((2^{k-1}-1)P(2))\cup S$. Then we have the following :

$$T'(2^k) = R_1(k).\pi(k).(2T'(2^{k-1})).\sigma(2,2^{k-1}).R_2(k).$$

Proof. Delete the last stage of copies of $P(2)$ in $Y(2^k)$. Then there remains two copies of $Y(2^{k-1})$. For $n=1,\ldots,k-1$, delete the first stage of copies of $P(2)$ in $Z(2^n)$. Then there remains two copies of $Z(2^{n-1})$ if $n \geqslant 2$ and 2 copies of S if $n=1$. Now, by definition of $T'(2^k)$, it is easily seen that the copies of $Z(2^{n-1})$ $(n=2,\ldots,k-1)$ and $Y(2^{k-1})$ form together two copies of $T'(2^{k-1})$. Clearly, the first stage of copies of $P(2)$ which has been deleted is equal to $R_1(k)$, while the last one is equal to $R_2(k)$. Thus :

$$T'(2^k) = R_1(k)\,.\pi.(2T'(2^{k-1})).\sigma.R_2(k)\ ,$$

where σ is the interconnection permutation in the last stage of $Z(2^k)$ and π is the interconnection permutation linking the first stage of copies of $P(2)$ to the two copies of $T'(2^{k-1})$. Now $Z(2^k)=Z(2^{k-1})\times P(2)=Z(2^{k-1}).\sigma(2,2^{k-1})$ $.R_2(k)$ by definition of the product $\times$. Thus $\sigma=\sigma(2,2^{k-1})$.

Let us now look at π. Clearly π fixes 0 and 2^{k-1}. Now $2^{k-1}-1$ and 2^{k-1} are also fixed by π, since $I_{2^{k-1}-1}$ and $I_{2^{k-1}}$ are connected to $Z(2)$. If $x \in Z_{2^k}\setminus\{0,2^k-1,\ 2^{k-1}-1,\ 2^{k-1}\}$, then I_x is connected to some $I_u(Z(2^n))$, and $I_{x\pi}$ is connected to $I_{u\sigma(2^{n-1},2)}(Z(2^n))$, because $\sigma(2^{n-1},2)$ is the interconnection permutation between the first stage of copies of $P(2)$ and the two copies of $Z(2^{n-1})$ in $Z(2^n)$. Thus we get the following for $n=2,\ldots,k-1$ and $v \in Z_{2^{n-2}}$:

$$
\begin{array}{llll}
x & \to u \to & u\sigma(2^{n-1},2) & \to x\pi \\
2^{k-n}(4v+1)-1 & \to 4v \to & 2v & \to 2^{k-n}(2v+1)-1 \\
2^{k-n}(4v+3)-1 & \to 4v+2 \to & 2v+1 & \to 2^{k-n}(2v+1) \\
2^{k-n}(4v+1) & \to 4v+1 \to & 2v+2^{n-1} & \to 2^{k-n}(2v+2^{n-1}+1)-1 \\
2^{k-n}(4v+3) & \to 4v+3 \to & 2v+1+2^{n-1} & \to 2^{k-n}(2v+2^{n-1}+1) .
\end{array}
$$

Thus $\pi=\pi(k)$ and the result follows.

Note : Step 6 is illustrated in Fig. 11.

Step 7. For any $k \geqslant 1$, $T'(2^k) \approx \alpha(k)\ T'(2^k)$

Proof. We use induction on k. The result is obviously true for k=1. Suppose that $k > 1$ and that the result is true for k-1. By Step 6, we have :

$$
\begin{aligned}
T'(2^k) &= R_1(k).\pi(k).(2T'(2^{k-1})).\sigma(2,2^{k-1}).R_2(k) \\
&\cong R_1(k).\pi(k).\tau(k).(2T'(2^{k-1})).\tau(k).\sigma(2,2^{k-1}).R_2(k)
\end{aligned}
$$

by Lemma 1, since $\tau(k)=\tau(1).2^{k-1}$. By Step 3, we get :

$$
\begin{aligned}
T'(2^k) &\cong R_1(k).\pi(k).\tau(k).(2T'(2^{k-1})).\sigma(2,2^{k-1}).(\varepsilon(k).R_2(k)) \\
&\approx R_1(k).\pi(k)\tau(k).(2T'(2^{k-1})).\sigma(2,2^{k-1}).R_2(k) \\
&\approx \alpha(k).R_1(k).\alpha(k)^{-1}.\pi(k).\tau(k).(2T'(2^{k-1})).\sigma(2,2^{k-1}).R_2(k)
\end{aligned}
$$

since $R_1(k)$ does not act on 0 and 2^k-1. Using induction hypothesis, we have $T'(2^{k-1}) \approx \alpha(k-1).T'(2^{k-1})$ and so $2T'(2^{k-1}) \approx 2(\alpha(k-1).T'(2^{k-1})) \approx (2\alpha(k-1)).(2T'(2^{k-1})) \approx \beta(k).(2T'(2^{k-1}))$ by Proposition 4(viii). Thus :

$$
\begin{aligned}
T'(2^k) &\approx \alpha(k).R_1(k).\alpha(k)^{-1}.\pi(k).\tau(k).\beta(k).(2T'(2^{k-1})).\sigma(2,2^{k-1}).R_2(k) \\
&\approx \alpha(k).R_1(k).\alpha(k)^{-1}.\alpha(k).\delta(k).\pi(k).(2T'(2^{k-1})).\sigma(2,2^{k-1}).R_2(k) \\
&\approx \alpha(k).(R_1(k).\delta(k)).\pi(k).(2T'(2^{k-1})).\sigma(2,2^{k-1}).R_2(k)
\end{aligned}
$$

by Step 4. Hence :

$$T'(2^k) \approx \alpha(k).R_1(k).\pi(k).(2T'(2^{k-1})).\sigma(2,2^{k-1}).R_2(k)$$

$$\approx \alpha(k)\ T'(2^k)$$

and the result follows.

Step 8. For any $k \geqslant 1$, $T'(2^k) \approx \rho(k).W(2^k)^*$.

Proof. We use induction on k. Theresult is true for k=1. Suppose that $k > 1$ and that the result if true for k-1. Then we have :

$T'(2^{k-1}) \approx \rho(k-1).W(2^{k-1})^*$ and

$T'(2^{k-1}) \approx \alpha(k-1).T'(2^{k-1}) \approx \alpha(k-1).\rho(k-1).W(2^{k-1})^*$

by Step 7. It follows that :

$$2T'(2^{k-1}) = T'(2^{k-1}) \cup T'(2^{k-1})$$

$$\approx (\rho(k-1).W(2^{k-1})^*) \cup (\alpha(k-1).\rho(k-1).W(2^{k-1})^*).$$

$$\approx (\rho(k-1) \cup (\alpha(k-1)\rho(k-1))).(2W(2^{k-1})^*).$$

$$\approx \gamma(k).(2W(2^{k-1})^*)$$

Using Step 6, we get :

$$T'(2^k) \approx R_1(k).\pi(k).\gamma(k).(2W(2^{k-1})^*).\sigma(2,2^{k-1}).R_2(k).$$

$$\approx R_1(k).\rho(k).\sigma(2^{k-1},2).(2W(2^{k-1})^*).\sigma(2,2^{k-1}).R_2(k)$$

by Step 5.

Now let $R_0(k)=2S \cup ((2^{k-1}-1)(P(2))$. Then $R_1(k) \cong \rho(k).R_0(k).\rho(k)^{-1}$ and so we get :

$$T'(2^k) \approx \rho(k).R_0(k).\rho(k)^{-1}.\rho(k).\sigma(2^{k-1},2).(2W(2^{k-1})^*).\sigma(2,2^{k-1}).R_2(k)$$

$$\approx \rho(k).R_0(k).\sigma(2^{k-1},2).(2W(2^{k-1})^*).\sigma(2,2^{k-1}).R_2(k)$$

$$\approx \rho(k).W(2^k)^*.$$

It follows then that $T(2^k) \sim W(2^k)^*$.

Remark. From Step 6, it follows that the "looping" algorithm can be used for the control of $T'(2^k)$ (See the remark at the end of Section A).

It could perhaps be possible to design nested trees with copies of $P(n)$ $(n > 2)$ instead of $P(2)$, and the result would probably be quasiequivalent with $P(n) \wedge (P(n) \wedge (\ldots \wedge(P(n) \wedge P(n))\ldots))$.

C. Joel's serial construction and the triangular network [10,11].

This construction allows us to design a permutation network on n+1 bits using a permutation network on n bits and n copies of P(2).

Two possible designs are shown in Fig. 12 in the case where n=4.

The control is easy. Let $\pi \in \mathrm{Sym}(n+1)$. If $n\pi=m$, then in the design (a) we put the cells whose label is a number $\geq m$ in the 1-state. Together they form a permutation ρ. Now $\pi\rho^{-1} \in \mathrm{Sym}(n)$ and we have only to realize $\pi\rho^{-1}$ in P(n). If $m\pi=n$, then in the design (b) we put all cells whose labels is a number $\geq m$ in the 1-state. Together they form a permutation τ. Now $\tau^{-1}\pi \in \mathrm{Sym}(n)$ and we have only to realize $\tau^{-1}\pi$ in P(n).

We can use these two designs in iteration to form a permutation network on an arbitrary number n of bits. Both lead to the same network, called the serial network [10] or the triangular array [11], whose design is shown in Fig.13 for n=5.

This network has cost n(n-1)/2 and delay 2n-3.

Sequential realizations of this network can be found in [9].

As the triangular array is a sorting network (see [3] for a definition), it has the following control algorithm :

- The signal on I_i is given the weight $i\pi$, where π is the permutation to be realized by the network.
- When two signals reach a cell C, then C is put in the 0-state if the weight of the signal on $I_1(C)$ is superior to the weight of the signal on $I_0(C)$, and C is put in the 1-state otherwise.

D. <u>The diamond array</u> [11].

The diamond array D(n) is a permutation network on n bits whose design is shown in Fig. 14 for n=4 and n=5. It has cost n(n-1)/2 and delay n.

As D(n) is a sorting network, it has the same control algorithm as the triangular network. This algorithm is called the "<u>decentralized control</u>"

E. <u>The Bose-Nelson array</u> [3,11] .

Bose and Nelson [3] designed a sorting network, which can also be considered as a permutation network with decentralized control.

We first define a partial permutation network P(I,J). Let n be an integer larger than 1, let I and J be two parts of Z_n such that $||I|-|J|| \leqslant 1$ and $i < j$ for every $i \in I$ and $j \in J$. Define the following four sets :

- $I_1 = \{\lfloor \frac{1}{2} |I| \rfloor$ smallest elements of $I\}$.
- $I_2 = I \setminus I_1$.
- $J_1 = \{\lfloor \frac{1}{2} |I| \rfloor$ smallest elements of $J\}$ if $|I|=|J| \equiv 1 \pmod 2$.

  $= \{\lceil \frac{1}{2} |I| \rceil$ smallest elements of $J\}$ otherwise.
- $J_2 = J \setminus J_1$.

Now $P(I,J)$ is defined recursively as follows for any suitable I and J:

- If $I=\emptyset$ or $J=\emptyset$, then $P(I,J)=nS$
- If $|I|=|J|= 1$, then $P(I,J)$ consists of a 2-cell between I and J and n-2 simple interconnections
- If $\max \{|I|,|J|\} \geq 2$, then we have :

  $P(I,J) = P(I_1,J_1).P(I_2,J_2).P(I_2,J_1)$.

Several examples of $P(I,J)$ are drawn on Fig. 15. If $I=\{i_1,\dots,i_m\}$ and $J = \{j_1,\dots,j_n\}$, then we will write $P(i_1 i_2 \dots . i_m, j_1 j_2 \dots . j_n)$ for $P(I,J)$.

Now let K be a subset of Z_N. Let $K_1= \{\lfloor \frac{1}{2} |K| \rfloor$ smallest elements of $K\}$ and $K_2=K \setminus K_1$. Then we define $P^*(K)$ recursively as follows :

- $P^*(K) = nS$ if $|K| < 2$.
- If $|K| \geq 2$, then $P^*(K)=P^*(K_1).P^*(K_2).P(K_1,K_2)$.

If $K=Z_n$, then write $P^*(n)$ for $P^*(K)$.

Bose and Nelson [3] proved that $P^*(n)$ is a permutation network on n bits. Moreover, if $n= \sum_{i=1}^{t} 2^{r_i}$, where $r_1 < r_2 < \dots < r_t$, then $P^*(n)$ has cost :

$$\sum_{i=1}^{t} (3^{r_i} - 2^{r_i}) + \sum_{i=1}^{t-1} (\tfrac{3}{2})^{r_i} \left(\sum_{j=i+1}^{t} 2^{r_j - j + i + 1} \right)$$

In particular, $P^*(2^r)$ has cost $3^r - 2^r$ and $P^*(2^r+1)$ has cost 3^r.

Several examples of $P^*(n)$ are drawn in Fig. 16.

F. Some other networks.

We give here a list of arrays to be found in [11] , together with their cost and delay :

Type	n	Cost	Delay
rectangular	even	$n^2/2$	n
	odd	$(n^2-1)/2$	n+1
pruned rectangular	even	$n(3n-2)/8$	n-1
	odd	$3(n^2-1)/8$	n
rhombiodal	even	$n^2/2$	n
	odd	$(n^2-1)/2$	n+1?
almost square	$\equiv 2 \pmod 4$	$(3n-2)^2/16$	
	$\equiv 0 \pmod 4$	$3n(3n-4)/16$	at most $\frac{3n}{2}$?
	$\equiv 1 \pmod 4$	$(3n+1)(3n-3)/16$	
	$\equiv 3 \pmod 4$	$(3n+3)(3n-5)/16$	

Waksman [21] designed a permutation network on n bits with cost $\frac{3}{2} n \log_2(n) - \frac{5}{2} n + 3$ when n is a power of 2.

Tsao-Wu [19] defined a sorting network which has cost $n(n-1)/2-(n\ \log_2(n)/2-n+1)$.

§ VI. Optimization of cost and delay.

A. Permutation networks designed with 2-cells.

Let P be a permutation network designed with 2-cells. Then the cost $\gamma(P)$ of P is the number of 2-cells used in the design of P. The delay of P is the maximum number $\delta(P)$ of cells that a signal may traverse between an input and an output. The network P can be divided in $\delta(P)$ stages, where the ith stage $(i=1,\dots,\delta(P))$ is the set of all cells such that $i-1$ is the maximum number of cells traversed by an input signal before reaching an input of this cell.

Let γ_n and δ_n be the minimum cost and delay of a permutation network on n bits. We have the following lower bounds :

PROPOSITION 10. (i) $\gamma_n \geqslant \lceil \log_2(n!) \rceil$ [6] .

$$\text{(ii)}\quad \delta_n \geqslant \gamma_n / \lfloor \tfrac{n}{2} \rfloor \geqslant \lceil \log_2(n!) \rceil / \lfloor \tfrac{n}{2} \rfloor .$$

Proof. Let P be a permutation network such that $\gamma(P)=\gamma_n$. Then the set of 2-cells of P has 2^{γ_n} states and must be able to realize n! permutations. Thus $2^{\gamma_n} \geqslant n!$ and so (i) follows.

Let Q be a permutation network on n bits such that $\delta(Q)=\delta_n$. Now every stage of Q has no more than $\lfloor\frac{n}{2}\rfloor$ 2-cells. Hence $\lfloor\frac{n}{2}\rfloor.\delta_n \geqslant \gamma(Q) \geqslant \gamma_n$ and so (ii) follows.

By Stirling's formula, $\log_2(n!)$ is asymptotically $n \log_2(n)$. Hence the asymptotic values for the lower bounds on γ_n and δ_n are $n \log_2(n)$ and $2\log_2(n)$. As these two numbers are the asymptotic values of $\gamma(G(n))$ and $\delta(G(n))$, it follows that G(n) is asymptotically optimal.

It is easily seen that $\gamma_2=\psi(2)=1$, $\gamma_3=\psi(3)=3$ and $\gamma_4=\psi(4)=5$. Green [7] has shown that $\gamma_5=\psi(5)=8$. It is not known wether $\gamma_n=\psi(n)$ for $n > 5$. We will show that G(n) has minimal cost amongst all networks built from 2-cells with the operation (10) of §IV.

Let us prove first the following preliminary result :

PROPOSITION 11. Let a,b and k be integers such that $b \geqslant 2$, $a \geqslant 3$ and $a-1 \geqslant k \geqslant 0$. Then $\psi(ab-k) < 2(b-1)\psi(a)+\psi(a-k)+k\psi(b-1)+(a-k)\psi(b)$.

Proof. Let $\phi(a,b,k)$ be the right-hand side of this inequality. The proof consists in six steps :

Step 1. The result holds for a=3 and k=1.

Proof : We have $\phi(3,b,1)=6(b-1)+1+\psi(b-1)+2\psi(b) = 6b-5+\psi(b-1)+2\psi(b)$.

$$\text{Now } \psi(3b-1) = \sum_{y=1}^{3b-1} \lceil \log_2(y) \rceil$$

$$= \psi(3) + \sum_{m=2}^{b-1} (\lceil \log_2(3m) \rceil + \lceil \log_2(3m-1) \rceil + \lceil \log_2(3m-2) \rceil)$$

$$+ \lceil \log_2(3b-2) \rceil + \lceil \log_2(3b-1) \rceil .$$

Now, if we take m=2, we get :

$$\lceil \log_2(3m-2) \rceil = \lceil \log_2(4) \rceil < \lceil \log_2(6) \rceil = \lceil \log_2(3m) \rceil .$$

Hence we have the following :

$$\psi(3b-1) < \psi(3) + 3 \sum_{m=2}^{b-1} \lceil \log_2(3m) \rceil + 2 \lceil \log_2(3b) \rceil$$

$$< 3 + 3 \sum_{m=2}^{b-1} (\lceil \log_2(3) \rceil + \lceil \log_2(m) \rceil) + 2 (\lceil \log_2(3) \rceil + \lceil \log_2(b) \rceil)$$

$$< 3 + 3(b-2) \lceil \log_2(3) \rceil + 3 \sum_{m=2}^{b-1} \lceil \log_2(m) \rceil + 2\lceil \log_2(3) \rceil + 2\lceil \log_2(b) \rceil$$

$$< 3 + (3b-4) \lceil \log_2(3) \rceil + 2 \sum_{m=2}^{b} \lceil \log_2(m) \rceil + \sum_{m=2}^{b-1} \lceil \log_2(m) \rceil$$

$$< 3 + 2(3b-4) + 2\psi(m) + \psi(m-1) = \phi(3,b,1) .$$

Step 2. The result is true for a=3.

Proof. We have :

(i) $\phi(3,b,0)-\phi(3,b,1)=\psi(3)-\psi(2)+\psi(b)-\psi(b-1)$

$$=\lceil \log_2(3) \rceil + \lceil \log_2(b) \rceil \geqslant \lceil \log_2(3b) \rceil = \psi(3b) - \psi(3b-1) .$$

Thus $\psi(3b) = \psi(3b-1) + (\psi(3b)-\psi(3b-1))$

$$< \phi(3,b,1)+ (\phi(3,b,0)-\phi(3,b,1)) = \phi(3,b,0) .$$

(ii) $\phi(3,b,1)-\phi(3,b,2) = \psi(2) -\psi(1) +\psi(b) -\psi(b-1)$

$$= \lceil \log_2(2) \rceil + \lceil \log_2(b) \rceil = \lceil \log_2(2b) \rceil \leqslant \lceil \log_2(3b-1) \rceil$$

$$= \psi(3b-1) - \psi(3b-2).$$

Thus $\psi(3b-2) = \psi(3b-1)- (\psi(3b-1)-\psi(3b-2))$

$$< \phi(3,b,1)-(\phi(3,b,1)-\phi(3,b,2)) = \phi(3,b,2).$$

Step 3. The result is true for a=4.

Proof. Using Lemma 6 (ii) and (iii), it is easy to check that for $k=0,1,2,3$, we have :

$$\psi(4b-k) = 8b-3-2k+(4-k)\psi(b) + k\psi(b-1)$$

Now $\phi(4,b,k)=2(b-1)\psi(4)+\psi(4-k)+ k\psi(b-1)+(4-k)\psi(b)$

$$=10b-10 +\psi(4-k)+ k\psi(b-1)+(4-k)\psi(b)$$

Thus $\phi(4,b,k)-\psi(4b-k) = 2b-7+2k+\psi(4-k)$.

Now we check easily that $2k+\psi(4-k) \geqslant 5$.

As $2b-2 > 0$, it follows that $\phi(4,b,k)-\psi(4b-k) > 0$.

Step 4. For any integer $x \geqslant 2$, $2^{\lceil \log_2(x) \rceil} \leqslant 2(x-1)$.

Indeed, suppose that $2^{k-1} < x \leqslant 2^k$, where $k \geqslant 1$. Then $k= \lceil \log_2(x) \rceil$ and so $2^{\lceil \log_2(x) \rceil} = 2^k$. But $x-1 \geqslant 2^{k-1}$. Hence $2(x-1) \geqslant 2^k = 2^{\lceil \log_2(x) \rceil}$.

Step 5. The result is true for k=0 and any a.

Proof. We use induction on a. The result is true for a=3,4. Suppose that $a \geqslant 5$ and that the result is true for a-1 (and k=0). We have :

(i) $$\psi(ab)-\psi((a-1)b) = \sum_{x=ab-b+1}^{ab} \lceil \log_2(x) \rceil \leqslant b \lceil \log_2(ab) \rceil$$

$$\leqslant b(\lceil \log_2(a) \rceil + \lceil \log_2(b) \rceil) .$$

(ii) $$\phi(a,b,0)-\phi(a-1,b,0)=(2b-1)(\psi(a)-\psi(a-1))+\psi(b)$$

$$=(2b-1) \lceil \log_2(a) \rceil + \psi(b) = (2b-1) \lceil \log_2(a) \rceil + b \lceil \log_2(b) \rceil -2^{\lceil \log_2(b) \rceil}+1$$

by Lemma 6(i). Using Step 4,we get :

$$\phi(a,b,0)-\phi(a-1,b,0) \geqslant (2b-1)\lceil \log_2(a) \rceil + b\lceil \log_2(b) \rceil - 2(b-1)+1$$

$$\geqslant (b-1) \lceil \log_2(a) \rceil - 2(b-1)+1 + b(\lceil \log_2(a) \rceil + \lceil \log_2(b) \rceil)$$

$$\geqslant (b-1) (\lceil \log_2(a) \rceil -2)+1 +\psi(ab)-\psi((a-1)b)$$

by (i). Now, as $a > 4$, $\lceil \log_2(a) \rceil > 2$ and so $\phi(a,b,0)-\phi(a-1,b,0) > \psi(ab)-\psi((a-1)b)$. By induction hypothesis, $\phi(a-1,b,0) > \psi((a-1)b)$. By adding these two inequalities, we get :

$$\phi(a,b,0) > \psi(ab) .$$

<u>Step 6.</u> The result is true for any $a \geqslant 3$ and $k \in Z_a$.

<u>Proof</u>. We use induction on a. The result is true for a=3,4. Suppose now that $a \geqslant 5$ and that the result is true for a-1. By Step 5, we can suppose that $k > 0$. We have :

$$\text{(i)} \quad \psi(ab-k)-\psi((a-1)b-(k-1)) = \sum_{x=ab-b-k+2}^{ab-k} \lceil \log_2(x) \rceil$$

$$\leqslant (b-1) \lceil \log_2(ab) \rceil \leqslant (b-1)(\lceil \log_2(a) \rceil + \lceil \log_2(b) \rceil) .$$

$$\text{(ii)} \quad \phi(a,b,k)-\phi(a-1,b,k-1) = 2(b-1)(\psi(a)-\psi(a-1))+\psi(b-1)$$

$$= 2(b-1) \lceil \log_2(a) \rceil + \psi(b-1)$$

$$= 2(b-1) \lceil \log_2(a) \rceil + (b-1) \lceil \log_2(b-1) \rceil - 2^{\lceil \log_2(b-1) \rceil} +1$$

by Lemma 6(i). Now we have two cases :

- <u>$b > 2$</u> : Using step 4, we get :

$$\phi(a,b,k)-\phi(a-1,b,k-1) \geqslant 2(b-1) \lceil \log_2(a) \rceil + (b-1) \lceil \log_2(b-1) \rceil - 2(b-2)+1$$

Inserting (i), we get :

$$(\phi(a,b,k)-\phi(a-1,b,k-1))-(\psi(ab-2)-\psi((a-1)b-(k-1))$$

$$\geqslant (b-1)(\lceil \log_2(a) \rceil + (\lceil \log_2(b-1) \rceil - \lceil \log_2(b) \rceil)) -2(b-2)+1$$

$$\geqslant (b-1)\ (3-1) - 2(b-2)+1=3 > 0, \text{ since } a > 4.$$

- b=2 : Then $\psi(ab-k) - \psi((a-1)b-(k-1)) \leqslant |\log_2(a)| + 1$, while $\phi(a,b,k)-\phi(a-1,b,k) = 2\lceil\log_2(a)\rceil > \lceil\log_2(a)\rceil + 1$, since $a > 2$.

Thus $\phi(a,b,k)-\phi(a-1,b,k-1) > \psi(ab-k)-\psi((a-1)b-(k-1))$ in both cases. Now, as $0 \leqslant k-1 < a-1$, we have by induction hypothesis :

$$\phi(a-1,b,k-1) > \psi((a-1)b-(k-1))$$

By adding both inequalities, we get the required result, namely that $\phi(a,b,k) > \psi(ab-k)$.

Let us now prove our result. If A and B are permutation networks, then we can write $A \wedge B$ as $(A,A,B,\bar{B})$, where $\bar{B}$ is a permutation network on b-1 bits, with B=B(b). This corresponds to the case where k=0.

Let Π be the family defined recursively as follows :

- S and P(2) belong to Π
- If P(n), P(n-k), P(r) and P(r-1) belong to Π, where $n,r \geqslant 2$ and $k \in Z_n$, then (P(n), P(n-k), P(r), P(r-1)) belong to Π.

Our result is the following :

THEOREM 12. If P(n) belongs to Π, then P(n)=G(n) or $\gamma(P(n)) > \psi(n)$.

Proof. We use induction on n. The result is obvious for n=1,2. Suppose now that $n \geqslant 3$ and that the result is true for $n' \in \{1,\ldots,n-1\}$.

Let $P(n) \in \Pi$. Then there are three numbers a,b,k such that $a \geqslant 2$, $b \geqslant 2$, $k \in Z_a$ and n=ab-k, and four permutation networks P(a), P(a-k), P(b), P(b-1) belonging to Π, such that P(n)=(P(a),P(a-k),P(b),P(b-1)).

It is easy to check that $a < n$ and $b < n$. We may thus apply induction : For $i=a,\ a-k,\ b,\ b-1$, $\gamma(P(i)) \geqslant \psi(i)$, and the equality holds if and only if $P(i)=G(i)$.

Now $P(n)$ is built with $2(b-1)$ copies of $P(a)$, one of $P(a-k)$, k of $P(b-1)$ and $(a-k)$ of $P(b)$. Thus

$$\begin{aligned}\gamma(P(n)) &= 2(b-1)\gamma(P(a))+\gamma(P(a-k))+ k\gamma(P(b-1))+(a-k)\gamma(P(b))\\ &\geqslant 2(b-1)\psi(a) +\psi(a-k) + k\psi(b-1)+(a-k)\psi(b)\\ &\geqslant\psi(n)\ ,\end{aligned}$$

and the equality holds if and only if $a=2$ and both $P(b)=G(b)$ and $P(b-1)=G(b-1)$ (by Proposition 11).

Thus $\gamma(P(n))=\psi(n)$ if and only if $P(n)=(G(2),G(2-k),G(b),G(b-1))=G(n)$.

We will prove a relatively similar result on the delay :

<u>THEOREM 13.</u> If $P(n)\in \Pi$, then $\delta(P(n)) \geqslant \delta(G(n))$

<u>Proof</u>. In fact, we will prove the slightly stronger following result :

If all cells of $P(n)$ are in the 0-state, then the signal on $I_{n-1}(P(n))$ reaches $O_{n-1}(P(n))$ after traversing at least $\delta(G(n))$ cells.

We use induction on n. The result is true for $n=1,2$. Suppose now that $n > 3$ and that the result is true for $n' \in \{1,\dots,n-1\}$. We can write $P(n)=(P(a),\ P(a-k),P(b),P(b-1))$, where $a\geqslant 2$, $b\geqslant 2$, $k\in Z_a$ and $P(i) \in \Pi$ for $i=a,\ a-k,\ b,\ b-1$.

Now put all the cells of $P(n)$ in the 0-state. Then the signal on $I_{n-1}(P(n))$ has the following trajectory :

$I_{n-1}(P(n)) \to I_{a-1}(P(a)) \to$ cells of $P(a) \to O_{a-1}(P(a)) \to I_{b-1}(P(b))$ $\to$ cells of $P(b) \to O_{b-1}(P(b)) \to I_{a-1}(P(a)^*) \to$ cells of $P(a)^* \to O_{a-1}(P(a)^*) \to O_{n-1}(P(n))$.

By induction hypothesis, it traverses at least $2\delta(G(a))+\delta(G(b))$ cells. Thus we have only to prove that $\delta(G(n)) \leqslant 2\delta(G(a))+\delta(G(b))$.

By Proposition 8, $\delta(G(x))=2\lceil\log_2(x)\rceil-1$ for $x=2,3,4,\ldots$

Thus we get :

$2\delta(G(a))+\delta(G(b))= 4\lceil\log_2(a)\rceil+ 2\lceil\log_2(b)\rceil-3$

$= 2(\lceil\log_2(a)\rceil+\lceil\log_2(b)\rceil) -1 + 2(\lceil\log_2(a)\rceil-1)$

$\geqslant 2\lceil\log_2(ab)\rceil- 1 + 2\lceil(\log_2(a)\rceil-1)$

$\geqslant 2\lceil\log_2(ab-k)\rceil-1 + 2(\lceil\log_2(2)\rceil-1)$

$\geqslant 2\lceil\log_2(n)\rceil-1 = \delta(G(n))$.

Note that if $a > 2$, then $\lceil\log_2(a)\rceil -1 > 0$ and so $2\delta(G(a))+\delta(G(b)) > \delta(G(n))$, which implies that $\delta(P(n)) > \delta(G(n))$.

But we can nevertheless have $\delta(P(n))=\delta(G(n))$ when $P(n)\neq G(n)$.

Indeed, take $n=17$, $a=2$, $b=9$ and $k=1$. Take $P(17)=(P(2), S, G(9), P(8))$, where $P(8)= G(4) \wedge G(2)$, then $P(17)\neq G(17)$, since $P(8)\neq G(8)$. But we have :

$$\delta(G(9)) = 2\lceil\log_2(9)\rceil- 1 = 7$$

$$\delta(P(8)) = 2\ \delta(G(4))+\delta(G(2)) = 7$$

and so $\delta(P(17)) = \delta(P(2)) + \max\{\delta(G(9)), \delta(P(8))\}$

$= 2 +\delta(G(9)) = \delta(G(17))$.

B. Permutation networks designed with different cells.

Suppose that we have different prefabricated networks $P(x), P(y), P(z),\ldots$, having respective costs $\gamma_x,\gamma_y,\gamma_z,\ldots$, and delays $\delta_x,\delta_y,\delta_z,\ldots$ We want to build larger permutation networks using $P_x,P_y,P_z,\ldots$.

as cells. How to minimize cost and delay?

To simplify our notation, we will write $\gamma_{f(x,y,z,\ldots)}$ for $\gamma(f(P(x), P(y), P(z),\ldots)$ and $\delta_{f(x,y,z,\ldots)}$ for $\delta(f(P(x),P(y),P(z),\ldots)$ whenever $f(*,*,*,\ldots)$ is a function of the type $N(I,0;*,*,*,\ldots.;\Gamma)$, as defined in §II.

Let Γ be the set of prefabricated cells P(x), P(y), P(z),... . We have first the following result :

PROPOSITION 14. For any P(x), P(y), P(z) built from cells in Γ, we have :

(i) $\gamma_{x\wedge y} = (2y-1)\gamma_x + x\,\gamma_y$.

(ii) $\delta_{x\wedge y} = 2\delta_x + \delta_y$.

(iii) $\gamma_{(x\wedge y)\wedge z} > \gamma_{x\wedge(y\wedge z)}$.

(iv) $\delta_{(x\wedge y)\wedge z} > \delta_{x\wedge(y\wedge z)}$.

(v) $\gamma_{x\wedge y} < \gamma_{y\wedge x}$ if and only if $\frac{\gamma_x}{x-1} < \frac{\gamma_y}{y-1}$.

(vi) $\gamma_{x\wedge(y\wedge z)} < \gamma_{y\wedge(x\wedge z)}$ if and only if $\frac{\gamma_x}{x-1} < \frac{\gamma_y}{y-1}$.

(vii) $\delta_{x\wedge y} < \delta_{y\wedge x}$ if and only $\delta_x < \delta_y$.

(viii) $\delta_{x\wedge(y\wedge z)} = \delta_{y\wedge(x\wedge z)}$

(ix) $\gamma_{x\wedge(\ldots\wedge(x\wedge x)\ldots)} = (2kx^{k-1} - \frac{x^k-1}{x-1})\gamma_x$.
k factors

(x) $\delta_{x\wedge(\ldots\wedge(x\wedge x)\ldots)} = (2k-1)\delta_x$.
k factors

Proof. (i) and (ii) are obvious. A repeated use of (i) implies that :

$\gamma_{(x\wedge y)\wedge z} = (2z-1)(2y-1)\gamma_x + (2z-1)x\gamma_y + xy\,\gamma_z$.

and $\gamma_{x\wedge(y\wedge z)} = (2yz-1)\gamma_x + (2z-1)x\gamma_y + xy\,\gamma_z$. (*)

Now $(2z-1)(2y-1)-(2yz-1)=2(y-1)(z-1) > 0$ and so (iii) follows.

A repeated use of (ii) implies that :

$\delta_{(x\wedge y)\wedge z} = 4\delta_x + 2\delta_y + \delta_z$

and $\delta_{x\wedge(y\wedge z)} = 2\delta_x + 2\delta_y + \delta_z$. (**)

Thus (iv) follows.

Now (v) follows from (i), (vi) from (*), (vii) from (ii) and (viii) from (**).

Finally, (ix) and (x) are proved by induction, using (i) and (ii) respectively.

We wish to study functions of elements of Γ used with the operation $\wedge$. These functions belong to the sets $F_n (n=1,2,3,\ldots)$ defined recursively as follows :

- $F_1 = \Gamma$
- For $n > 1$, $F_n = \{\phi = \phi(P(x_1),\ldots,P(x_n)) \mid$ there is some $k \in \{1,\ldots,n-1\}$, $\zeta \in F_k$ and $\xi \in F_{n-k}$ such that for any permutation networks $P_1,\ldots,P_n$, $\phi(P_1,P_2,\ldots,P_n) = \zeta(P_1,\ldots,P_k) \wedge \xi(P_{k+1},\ldots,P_n)\}$.

From Proposition 14, we deduce the following :

COROLLARY 15. Let $\phi \in F_n$. Let $P(x_i) \in \Gamma (i=1,\ldots,n)$. Then :

(i) $\phi(P(x_1),\ldots,P(x_n))$ has minimum cost and delay if and only if $\phi(P(x_1,\ldots,P(x_n)) = P(x_1) \wedge (P(x_2) \wedge (\ldots \wedge(P(x_{n-1}) \wedge P(x_n))\ldots))$.

(ii) If π is a permutation of $1,\ldots,n$, then $P(x_{1\pi}) \wedge (P(x_{2\pi}) \wedge (\ldots \wedge (P(x_{(n-1)\pi}) \wedge P(x_{n\pi}))\ldots))$ has minimum cost if $\gamma_{x_{i\pi}}/(x_{i\pi}-1) \leqslant \gamma_{x_{j\pi}}/(x_{j\pi}-1)$ for any i,j such that $1 \leqslant i \leqslant j \leqslant n$, and minimum delay if $\delta_{x_{n\pi}} = \max\{\delta_{x_i} \mid i=1,\ldots,n\}$.

This is a direct consequence of the results (v), (vi), (vii) and (viii) of Proposition 14.

It can generally be assumed that $\delta_x < \delta_y$ when $x < y$, because $P(x)$ can be considered as a part of $P(y)$.

It is also reasonable to suppose that $\gamma_x/(x-1) < \gamma_y/(y-1)$ whenever $x < y$. This is indeed the case for the following two choices of Γ :

- take $\Gamma=\{G(n) \mid n \geqslant 2\}$. Then $\gamma_n=\psi(n)$ and we check that for any $n \geqslant 2$,

$$\gamma_{n+1}/n-\gamma_n/(n-1) = \frac{(n-1)\psi(n+1)-n\psi(n)}{n(n-1)}$$

$$= \frac{1}{n(n-1)}\left((n-1)\left\lceil \log_2(n+1)\right\rceil-\psi(n)\right)$$

$$= \frac{1}{n(n-1)}\left(\sum_{i=2}^{n}\left(\left\lceil \log_2(n+1)\right\rceil-\left\lceil \log_2(i)\right\rceil\right)\right) > 0$$

- take Γ to be the set of all $n\times n$ crossbar switches. Then $\gamma_n=n^2$ and so $\gamma_n/(n-1)=n+1+\frac{1}{n-1}$, which is a strictly increasing function of n for $n \geqslant 2$.

While Corollary 15 indicated us in which way to build a permutation network on $x_1\ldots x_n$ bits using x_i-cells $(i=1,\ldots,n)$, we need to know when we can further decompose x_i into $x_{i0}\cdot x_{i1}$ and use copies of $P(x_{i0})$ and $P(x_{i1})$ instead of $P(x_i)$.

Let $P=f(P(y),\ldots,P(y_k)) \in F_k$ for some $k \geqslant 1$. Write γ_f and δ_f for the cost and delay of P. Suppose that f is on z bits, where $z \geqslant 2$.

PROPOSITION 16.

(i) If $\gamma_{xy} \leqslant y\gamma_x + x\gamma_y$, then $\gamma_{xy\wedge f} < \gamma_{x\wedge(y\wedge f)}$ for every value of z.

(ii) If $y\gamma_x + x\gamma_y < \gamma_{xy} < \frac{4y-1}{3}\gamma_x + x\gamma_y$, then $\gamma_{xy\wedge f} \geqslant x_{x\wedge(y\wedge f)}$ if $z \geqslant z_0$, where z_0 is a fixed integer bigger than 2.

(iii) If $\gamma_{xy} \geqslant \frac{4y-1}{2}\gamma_x + x\gamma_y$, then $\gamma_{xy\wedge f} \geqslant \gamma_{x\wedge(y\wedge f)}$ for every value of z bigger than 1.

(iv) If $\gamma_{xy} \geqslant (2y-1)\gamma_x + x\gamma_y$, then $\gamma_{xy} \geqslant \gamma_{x\wedge y}$.

Proof. We easily compute that :

$$\gamma_{x\wedge(y\wedge f)} = (2yz-1)\gamma_x + x((2z-1)\gamma_y + y\gamma_f)$$

and $\gamma_{xy\wedge f} = (2z-1)\gamma_{xy} + xy\gamma_f$.

Thus $\gamma_{xy\wedge f} \geqslant \gamma_{x\wedge(y\wedge f)}$ if and only if $\gamma_{xy} \geqslant \frac{2yz-1}{2z-1}\gamma_x + x\gamma_y$.

Let $f(z) = \frac{2yz-1}{2z-1}$. Then f(z) is a strictly monotonous decreasing function of z, with $f(2) = \frac{4y-1}{3}$ and $\lim_{z\to\infty} f(z) = y$.

If $\gamma_{xy} \leqslant y\gamma_x + x\gamma_y = (\lim_{z\to\infty} f(z))\gamma_x + x\gamma_y$, then $\gamma_{xy} < f(z)\gamma_x + x\gamma_y$ for any $z \geqslant 2$. Thus $\gamma_{xy\wedge f} < \gamma_{x\wedge(y\wedge f)}$ and (i) holds.

If $\gamma_{xy} \geqslant \frac{4y-1}{3}\gamma_x + x\gamma_y = f(2)\gamma_x + x\gamma_y$, then $\gamma_{xy} \geqslant f(z)\gamma_x + x\gamma_y$ for any $z \geqslant 2$. Thus $\gamma_{xy\wedge f} \geqslant \gamma_{x\wedge(y\wedge f)}$ and (iii) holds.

If $y\gamma_x + x\gamma_y < \gamma_{xy} < \frac{4y-1}{3}\gamma_x + x\gamma_y$, then $\gamma_{xy} = u\gamma_x + x\gamma_y$, where $\lim_{z\to\infty} f(z) < u < f(2)$. Thus there is some $z_0 > 2$ such that for $z \geqslant z_0$, $u \geqslant f(z)$. Thus $\gamma_{xy} \geqslant f(z)\gamma_x + x\gamma_y$ and so $\gamma_{xy\wedge f} \geqslant \gamma_{x\wedge(y\wedge f)}$ if $z \geqslant z_0$ and (ii) holds.

Now (iv) follows from Proposition 14(i).

Using the results of this section, one can easily derive the optimization of such constructions using square $n \times n$ crossbar switches as cells (see [14]).

It seems difficult to get a result similar to Theorems 12 and 13 when we take different cells and give their respective cost and delay.

<u>Note</u>. Additional informations on permutation networks can be found in [18].

Appendix.

The Goldstein-Leibholz construction.

Goldstein and Leibholz [6] proved that if A and B are permutation networks, then $A \wedge B$ is a permutation network. This generalizes the theorem of Slepian and Duguid [5,17], which asserts that $A \times\times B$ is a permutation network.

We will prove here the theorem of Goldstein and Leibholz. Our proof is similar to Benes'proof of the theorem of Slepian and Duguid [2, Theorem 3.1] .

We will use the following theorem due to P. Hall [8] :

A finite family $\{A_0,\ldots,A_{n-1}\}$ of subsets of a set A has a set of distinct representitives if and only if

$$\left|\bigcup_{i \in I} A_i\right| \geqslant |I| \text{ for any } I \subseteq \{0,\ldots,n-1\}$$

Let us make a few definitions :

Let X and Y be two sets. A partial bijection $\pi\colon X \to Y$ is a bijection from a part X' of X onto a part Y' of Y. We will say that X' is the domain of π and Y' is the image of π, and we will write $X'=\mathrm{Dom}(\pi)$ and $Y'=\mathrm{Im}(\pi)$.

Consider two sets X and Y of size nr, where n and r are integers larger than 1. Write then $X=\{x_{i,j} \mid i \in Z_r \text{ and } j \in Z_n\}$ and $Y=\{y_{i,j} \mid i \in Z_r \text{ and } j \in Z_n\}$. Let us define for $i \in Z_r$ the sets $X_i=\{x_{i,j} \mid j \in Z_n\}$ and $Y_i=\{y_{i,j} \mid j \in Z_n\}$. Let $\Sigma=\{X_i \mid i \in Z_r\}$ and $\Omega=\{Y_i \mid i \in Z_r\}$.

LEMMA A. Let π be a bijection $X \to Y$. Then there is a partial bijection π' included in π such that for $i \in Z_r$, $|X_i \cap \mathrm{Dom}(\pi')| = |Y_i \cap \mathrm{Im}(\pi')| = 1$.

Proof. For any $X_i \in \sum$ and $Y_j \in \Omega$, write $X_i \sim Y_j$ is there is some $x_{i,k} \in X_i$ such that $x_{i,k}\pi \in Y_j$. For any $i \in Z_r$, write $A_i = \{Y_j \in \Omega \mid X_i \sim Y_j\}$. Let $L \subseteq \sum$ and let $M = \bigcup_{X_i \in L} A_i$. Syppose that $|L| = m$ and $|M| = m'$. Clearly, $|\bigcup_{X_i \in L} X_i| = nm$ and $|\bigcup_{Y_j \in M} |Y_j = nm'$. If $x_{i,k} \in X_i$, then $x_{i,k}\pi \in \bigcup_{Y_j \in A_i} Y_j$. Thus $X_i\pi \subseteq \bigcup_{Y_j \in A_i} Y_j$. It follows that $(\bigcup_{X_i \in L} X_i)\pi \in \bigcup_{Y_j \in M} Y_j$.

Hence : $nm = |\bigcup_{X_i \in L} X_i| = |(\bigcup_{X_i \in L} X_i)\pi| \leqslant |\bigcup_{Y_j \in M} Y_j| = nm'$

Therefore $m' \geqslant m$ and Hall's theorem implies that the sets A_i have distinct representitives, in other words, for $i=0,1,\ldots,r-1$, there is some $Y_{j_i} \in A_i$ such that for $i' \neq i$, $Y_{j_{i'}} \neq Y_{j_i}$. By definition of A_i, there is for each $i \in Z_r$ an integer $f(i)$ such that $x_{i,f(i)}\pi \in Y_{j_i}$. Take $\pi' = \{(x_{i,f(i)}, x_{i,f(i)}\pi) \mid i \in Z_r\}$. As $\{j_i \mid i \in Z_r\} = Z_r$, π' has the required properties and the result follows.

PROPOSITION B. Let π be a bijection $X \to Y$. Then π is the disjoint union of n partial bijections $\pi_0, \ldots, \pi_{n-1}$ such that for $j \in Z_n$ and $i \in Z_r$, $|X_i \cap \mathrm{Dom}(\pi_j)| = |Y_i \cap \mathrm{Dom}(\pi_j)| = 1$.

Proof. We use induction on n. The result is obviously true for n=1. Suppose that $n > 1$ and that the result is true for n-1. By Lemma 1, there is partial bijection $\pi_{n-1} \subseteq \pi$ such that for $i \in Z_r$, $|X_i \cap \mathrm{Dom}(\pi_{n-1})| = |Y_i \cap \mathrm{Im}(\pi_{n-1})| = 1$. Write $\pi' = \pi \setminus \pi_{n-1}$, $X' = X \setminus \mathrm{Dom}(\pi_{n-1})$, $Y' = Y \setminus \mathrm{Im}(\pi_{n-1})$, $X_i' = X_i \cap X'$ and $Y_i' = Y_i \cap Y'$ $(i \in Z_r)$. Then π', X', Y' and the sets X_i' and Y_i' $(i \in Z_r)$ satisfy the hypothesis with n-1 instead of n. Applying induction hypothesis, we find that π' is the disjoint union of partial bijections $\pi_0,\dots,\pi_{n-2}$ such that for $i \in Z_r$ and $j \in Z_{n-1}$, $|X'_i \cap \mathrm{Dom}(\pi_j)| = |Y'_i \cap \mathrm{Im}(\pi_j)| = 1$. Now $\pi_0,\dots,\pi_{n-2}$ are partial bijections $X \to Y$ and we have $|X_i \cap \mathrm{Dom}(\pi_j)| = |Y_i \cap \mathrm{Im}(\pi_j)| = 1$ for $i \in Z_r$ and $j \in Z_{n-1}$. Therefore the result holds for n.

Remarks. (1) We can choose the labelling of the bijections π_j in such a way that for $j \in Z_n$, $y_{0,j} \in Y_0 \cap \mathrm{Im}(\pi_j)$. With this additional hypothesis, the labelling becomes unique.

(2) Every π_j $(j \in Z_n)$ induces a bijection $\sum \to \Omega$. We write it $\bar{\pi}_j$.

(3) For any $i \in Z_r$, there is a permutation ρ_i of X_i which maps the unique element of $X_i \cap \mathrm{Dom}(\pi_k)$ on $x_{i,k}$ for $k \in Z_n$.

(4) Similarly, for any $i \in Z_r$, there is a permutation τ_i of Y_i which maps $y_{i,k}$ on the unique element of $Y_i \cap \mathrm{Im}(\pi_k)$ for $k \in Z_n$. It follows from our choice of labelling in Remark 1 that $\tau_0 = 1_{Y_0}$.

Now define the n partial bijections

$$\pi'_k : \{x_{i,k} \mid i \in Z_r\} \to \{y_{i,k} \mid i \in Z_r\}$$

$$x_{i,k} \to y_{j,k}, \text{ where } Y_j = X_i\bar{\pi}_k \quad (k \in Z_n) .$$

Note that $y_{j,k} = x_{i,k}\rho_i^{-1}.\pi.\tau_j^{-1}$.

If $\rho=\rho_0 \dots \rho_{r-1}$, $\tau=\tau_0\dots\tau_{r-1}$ and $\pi'=\pi'_0\dots\pi'_{n-1}$, then $\pi=\rho.\pi'.\tau$.

Now let us show how π can be realized on $A \wedge B$, where $A=A(n)$ and $B=B(r)$.

- In the first stage, we realize ρ_i on the ith copy of A $(i \in Z_r)$.

- In the second stage, we realize $\bar{\pi}_k$ (or π'_k) on the kth copy of B ($k \in Z_n$).

- In the third state, we realize τ_i on the ith copy of A^* ($i \in Z_r$).

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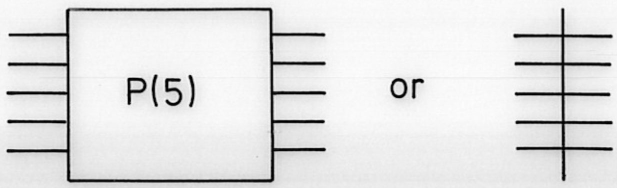
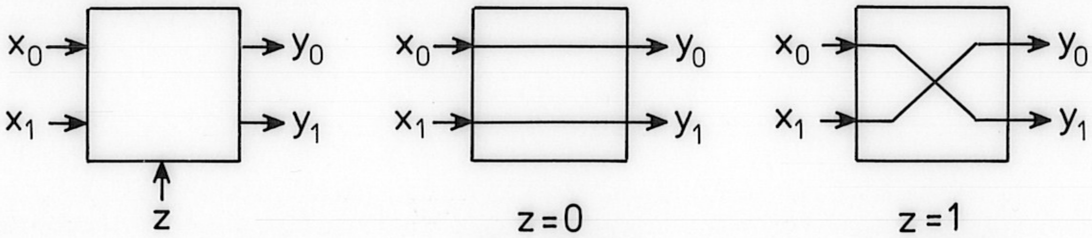
(a)



A5 - cell

(b)



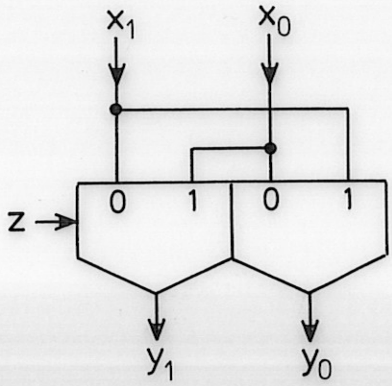
z=0
0-state

z=1
1-state

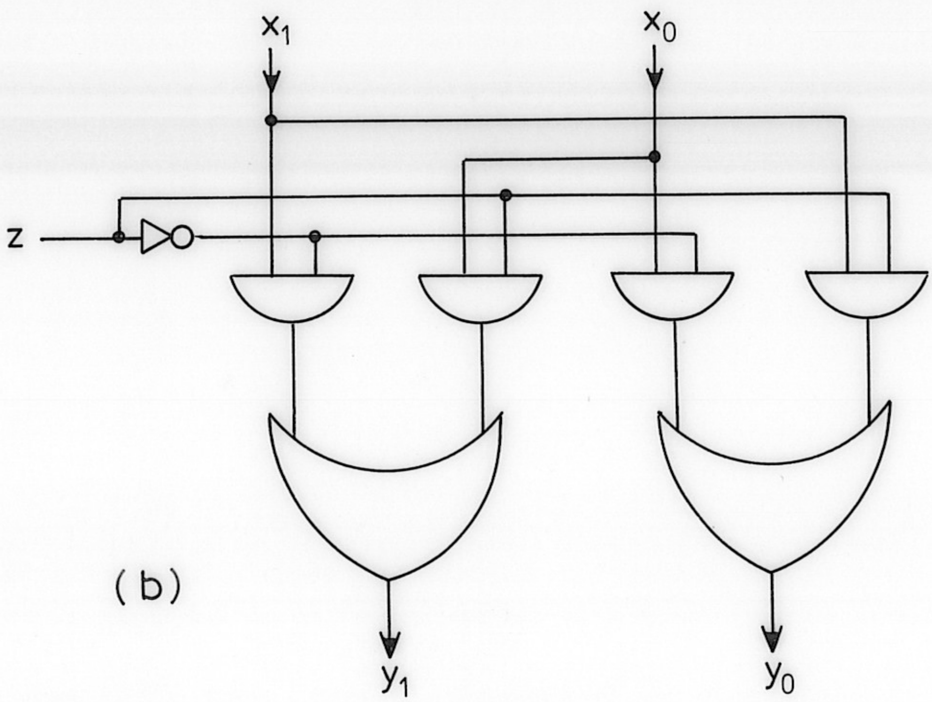
A2-cell and its 2 states

FIG. 1

(a)



Design of a 2-cell using multiplexers

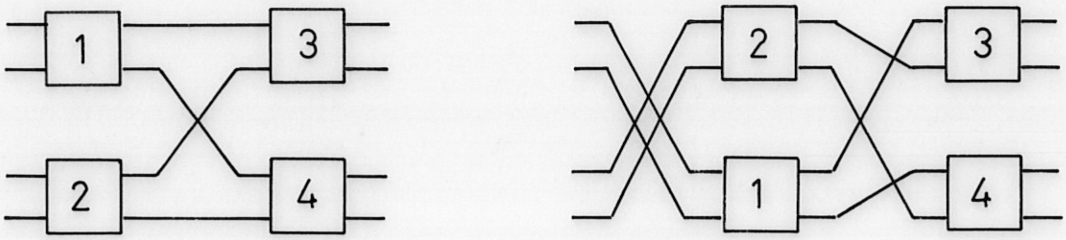


(b)

Design of a 2-cell using logical gates.

FIG. 2

(a)

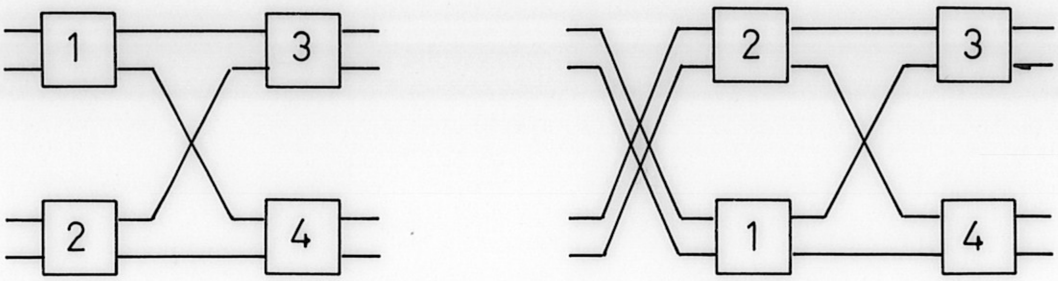


Two isomorphic networks

(b)

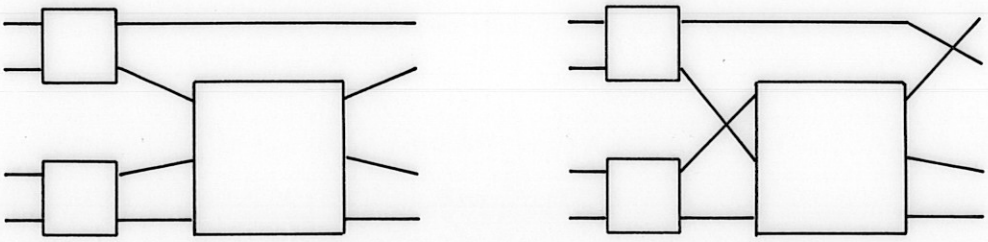
Two equivalent networks

(c)



Two quasiisomorphic networks

(d)



Two quasiequivalent networks

FIG. 3

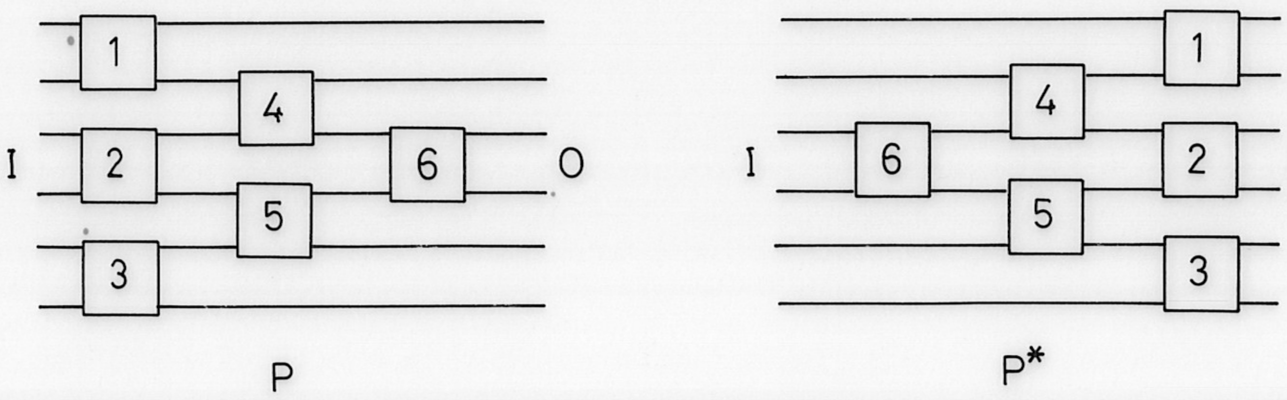


FIG. 4 A network and its dual

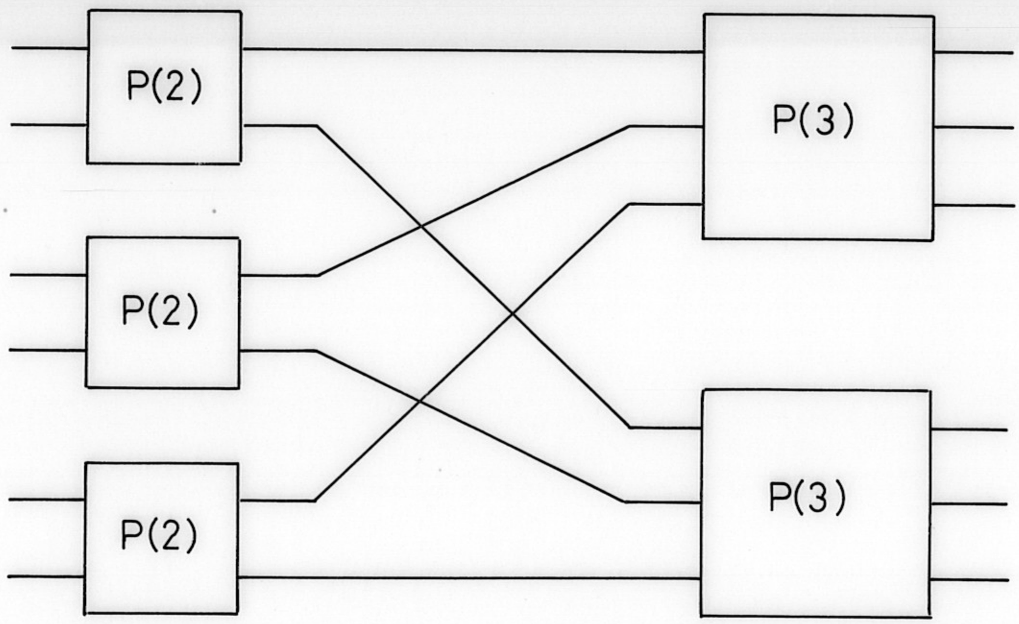


FIG. 5 P(2) x P(3)

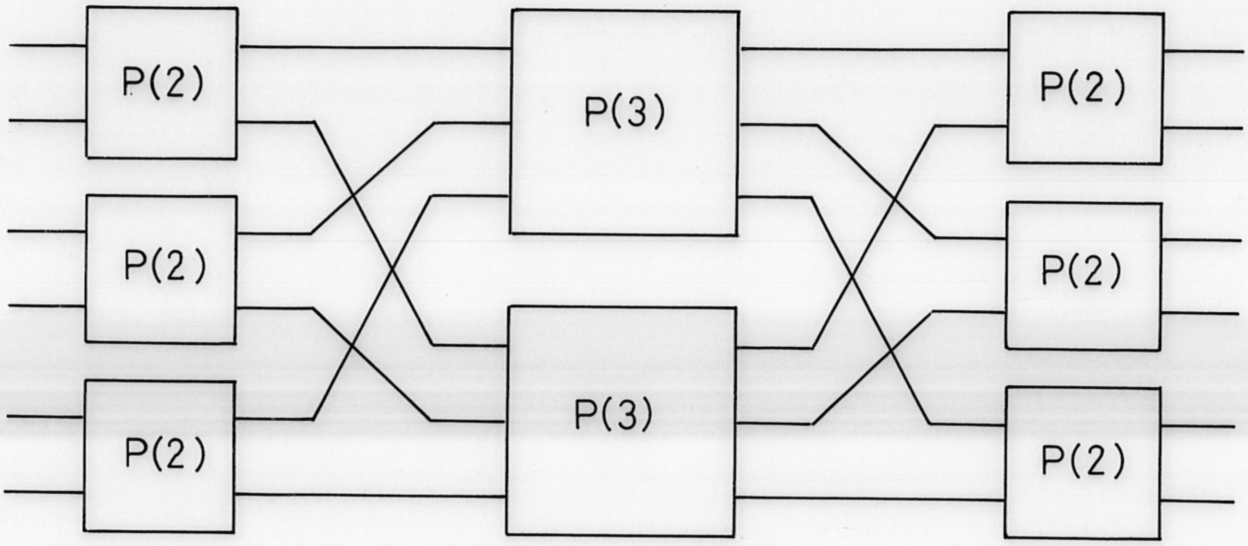


FIG. 6 P(2) x x P(3)

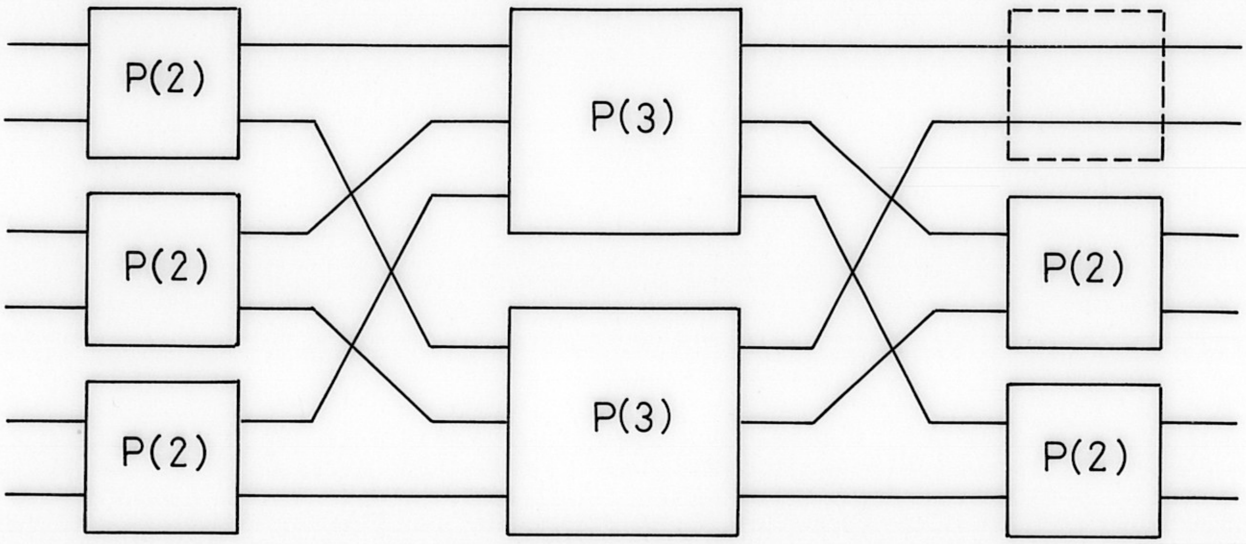


FIG. 7 P(2) ∧ P(3)

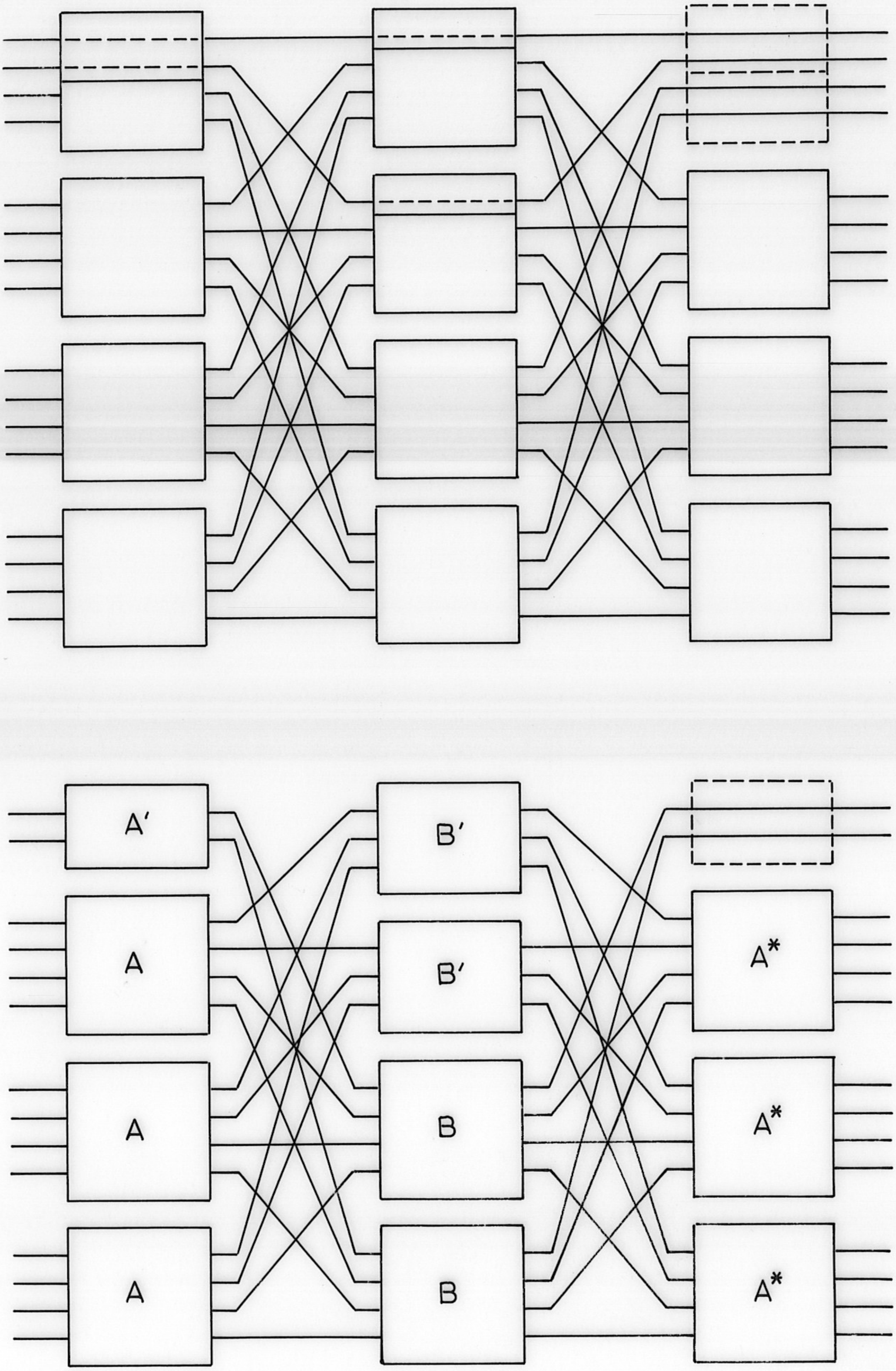
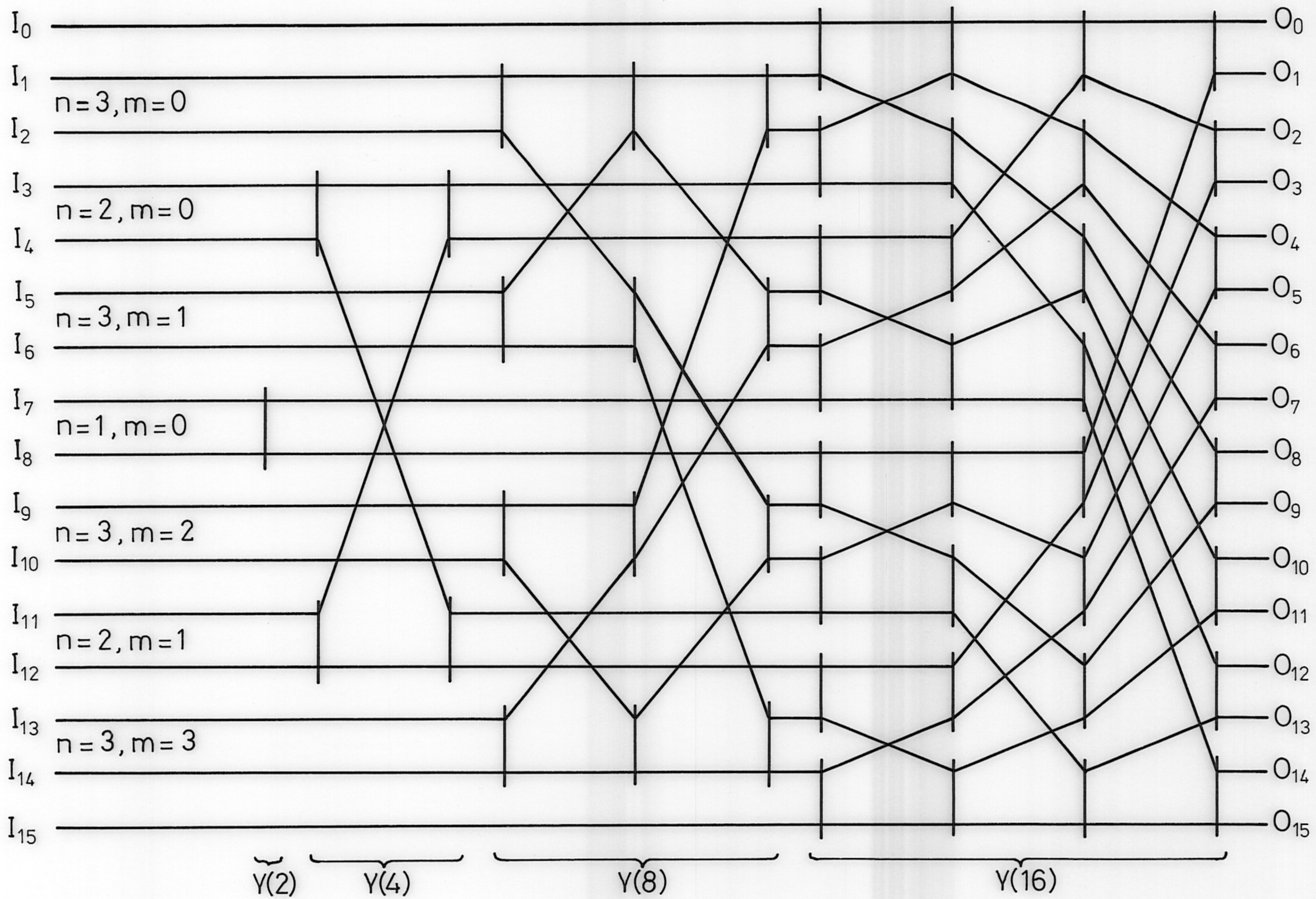


FIG. 8 (A, A', B, B') from A∧B



I0
I1
n=3, m=0
I2
I3
n=2, m=0
I4
I5
n=3, m=1
I6
I7
n=1, m=0
I8
I9
n=3, m=2
I10
I11
n=2, m=1
I12
I13
n=3, m=3
I14
I15
O0
O1
O2
O3
O4
O5
O6
O7
O8
O9
O10
O11
O12
O13
O14
O15
Y(2)
Y(4)
Y(8)
Y(16)

FIG. 9 T(16)

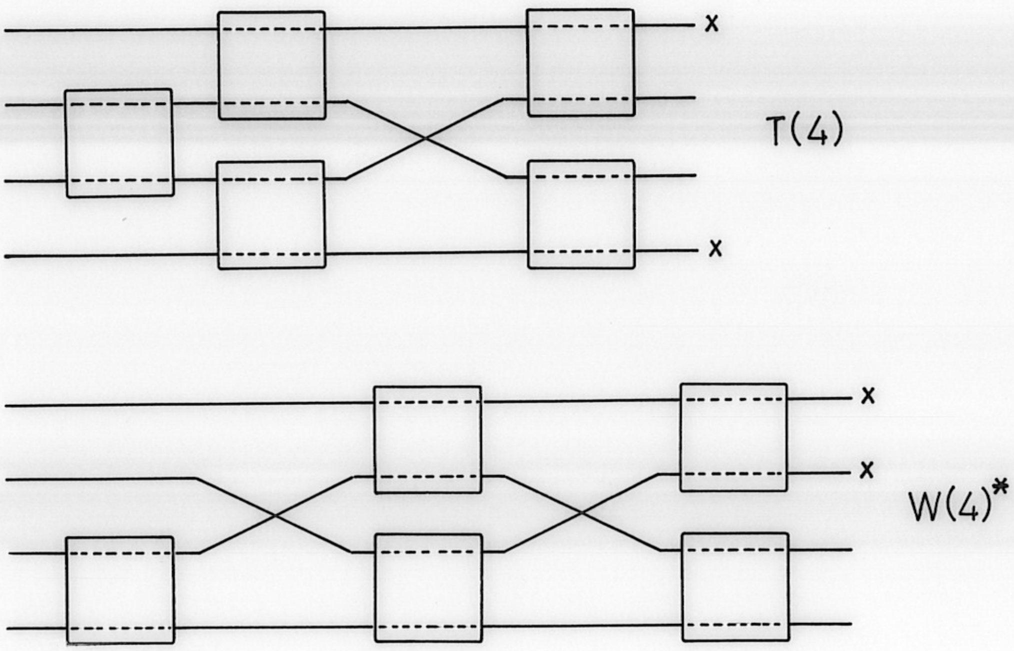


FIG. 10

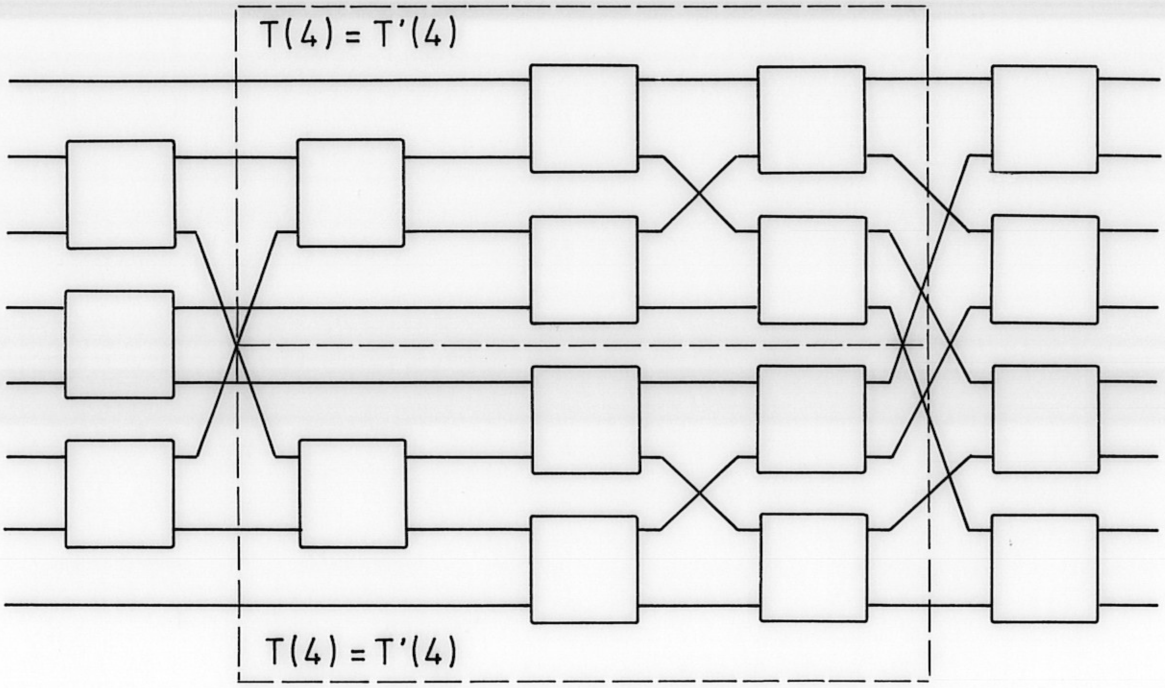


FIG. 11 T(8) = T'(8)

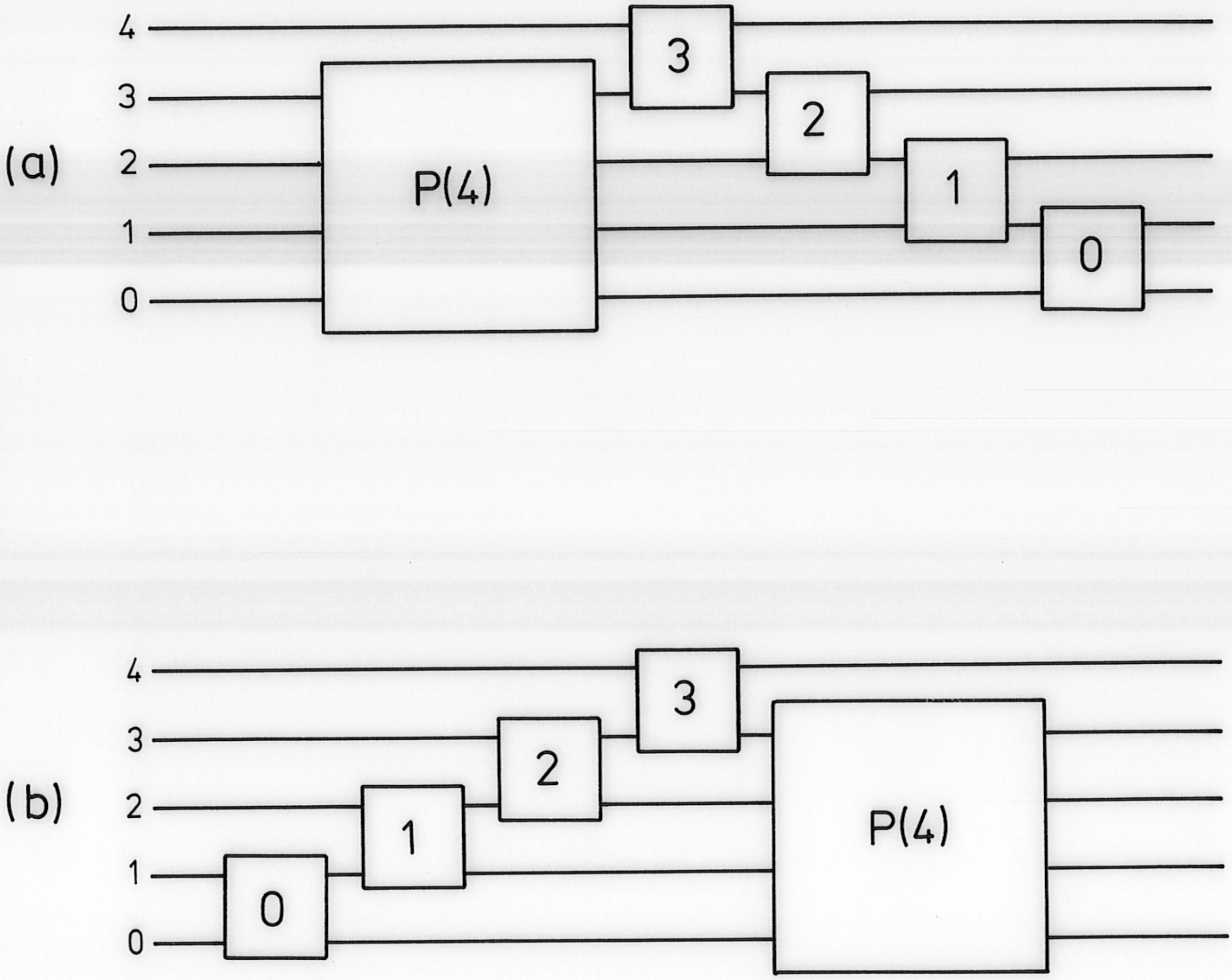


FIG. 12 Joel's serial construction

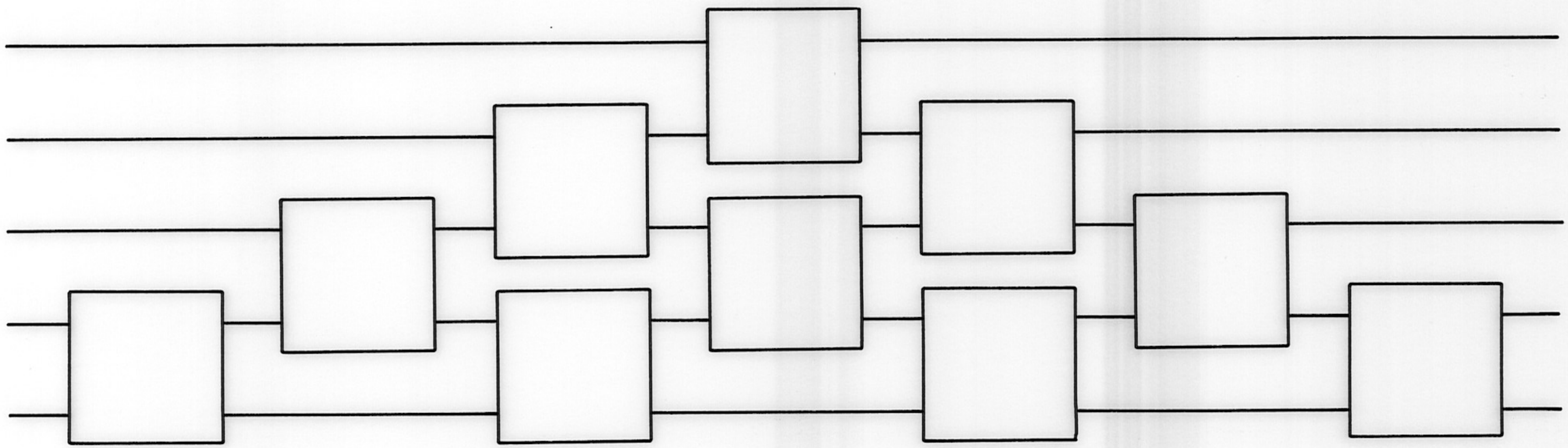
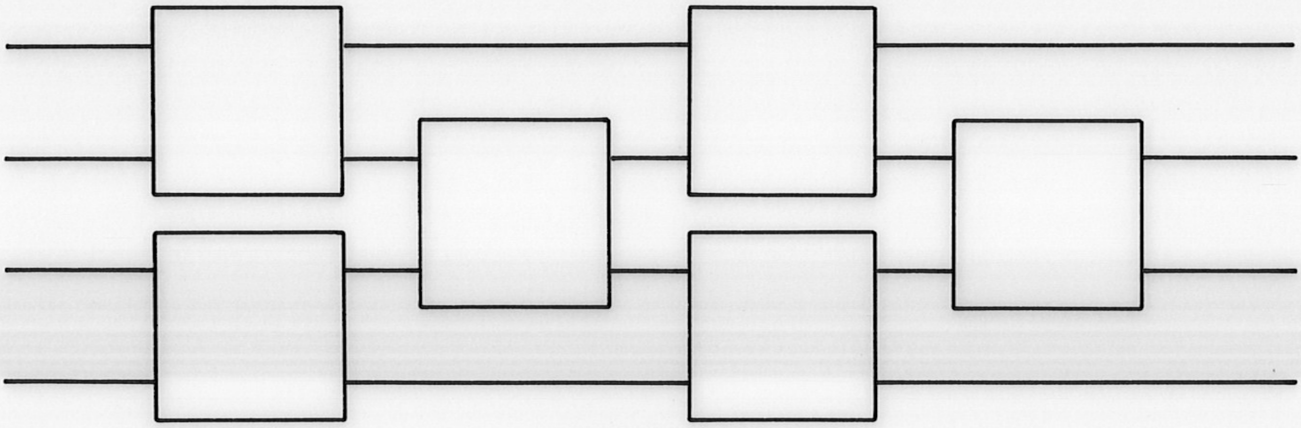
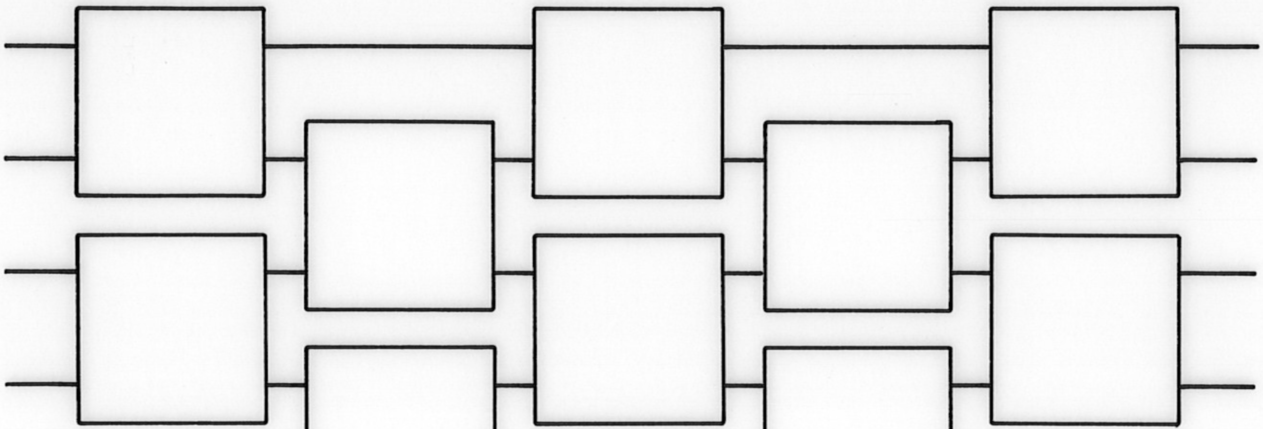


FIG. 13 The triangular array (n=5)



D(4)



D(5)

FIG. 14 The diamond array.

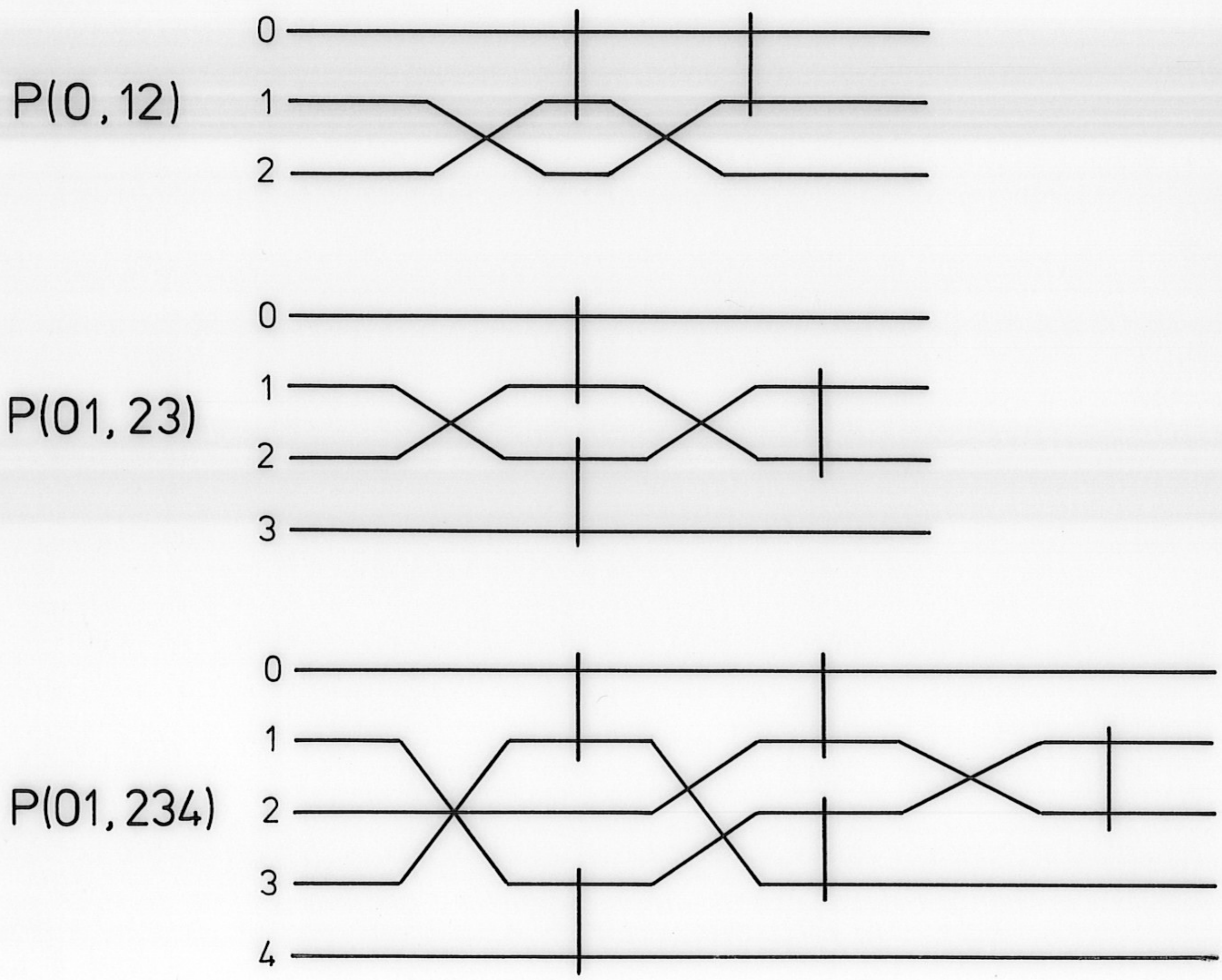


FIG. 15

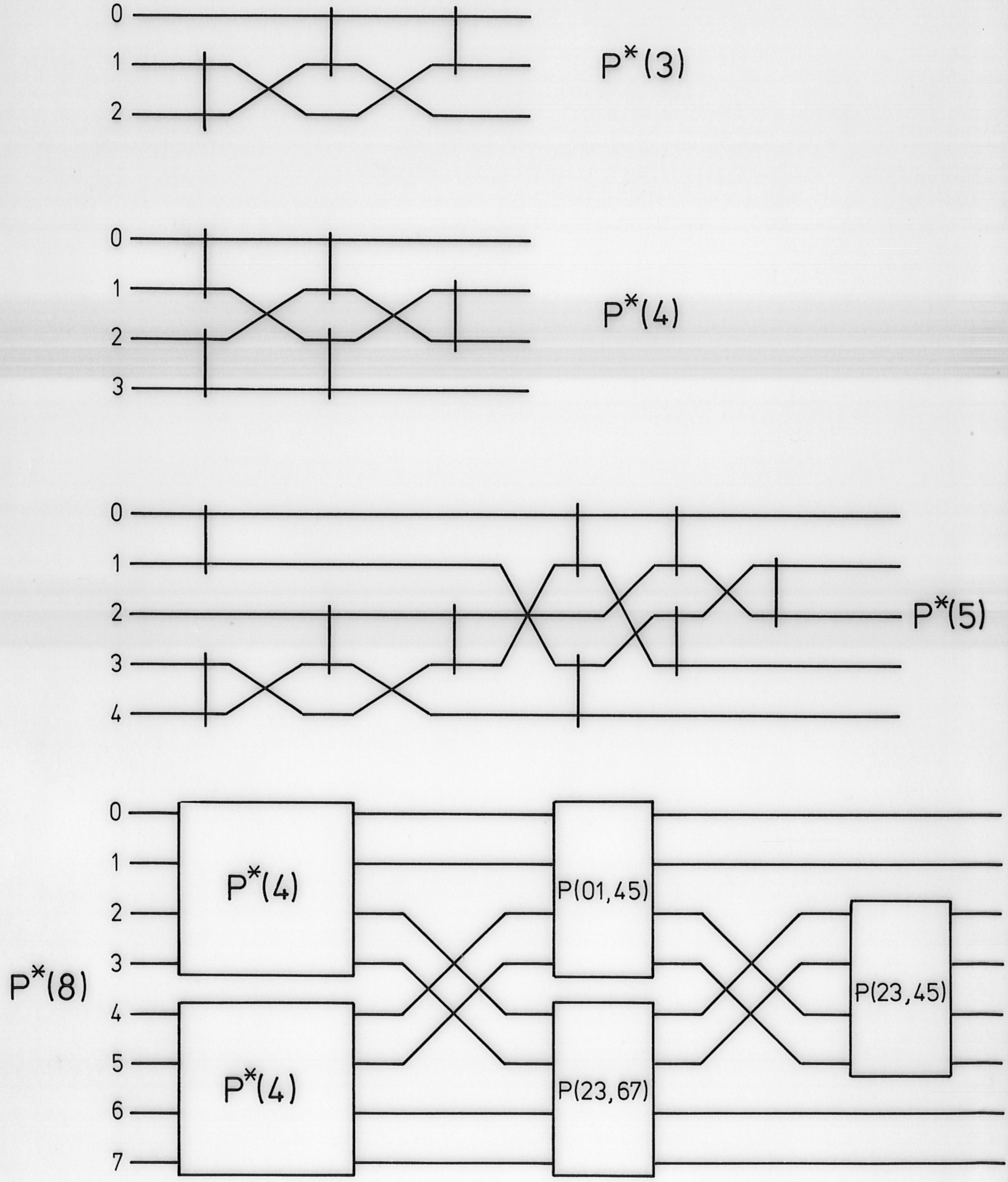


FIG. 16