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CONVERGENCE OF DISCRETE ASYNCHRONOUS ITERATIONS

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A convergence theorem of asynchronous iterations of discrete systems partitioned into blocks is given. The mathematical model developed generalizes several classical block discrete models: parallel (Jacobi), series (Gauss-Seidel) and chaotic. Numerical applications with boolean networks show convergences predicted by this theorem.

Keywords: Discrete asynchronous iterations; discrete dynamic systems; contraction function; numerical simulations

C. R. Category: G.1.3

1. INTRODUCTION

Asynchronous evolution of general discrete systems with n components is studied in this paper. Boolean networks represent particular cases of such systems.

Each component i takes a finite number of values x_i, $i \in \{1, \dots, n\}$. This system is partitioned into α blocks. Each block i has n_i components, $\sum_{i=1}^{\alpha} n_i = n$. The value of a block i is denoted by X_i and the value of the block system, by $X = (X_1, \dots, X_\alpha)$. The dynamic of the system is described

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according to a function f

$$f(x) = (f_1(x_1, \dots, x_n), \dots, f_n(x_1, \dots, x_n)).$$

This function f is partitioned into a compatible way, *i.e.*,

$$f(x) = F(X) = (F_1(X_1, \dots, X_\alpha), \dots, F_\alpha(X_1, \dots, X_\alpha)).$$

The iterations considered are asynchronous, *i.e.*, chaotic iterations taking into account delays which may be generated by different communications and computation sizes of the elements of the system. The state of a system (resp. a block system) at the time t is represented by x^t (resp. X^t), or more precisely by

$$x^t = X^t = (x_1^t, \dots, x_n^t) = (X_1^t, \dots, X_\alpha^t).$$

In the case of an asynchronous system, the block i at the time t is either iterated by using some blocks j with states $X_j^{s_j^i(t)}$ available at the previous time $s_j^i(t) = t - r_j^i(t) \leq t$ where $r_j^i(t)$ is the delay of the block j compared to the block i at the time t, *i.e.*, $X_i^{t+1} = F_i\,(X_1^{s_1^i(t)}, \dots, X_\alpha^{s_\alpha^i(t)})$, or not iterated, *i.e.*, $X_i^{t+1} = X_i^t$. These two alternatives are described by the iteration strategy $\{J(t)\}_{t\in\mathbb{N}}$, $J(t) \subset \{1, \dots, \alpha\}$, $\forall t \in \mathbb{N}$. The dynamic of the system is then described by the following algorithm

$$\begin{cases} \text{Given } X^0 = (X_1^0, \dots, X_\alpha^0) \\ t = 0, 1, \dots \\ \quad i = 1, \dots, \alpha \\ \quad\quad X_i^{t+1} = \begin{cases} F_i(X_1^{s_1^i(t)}, \dots, X_\alpha^{s_\alpha^i(t)}) \text{ if } i \in J(t) \\ X_i^t \text{ if } i \notin J(t). \end{cases} \end{cases}$$

There is neither synchronization condition nor management of the critical section which can be found, for example, in synchronous algorithms.

The convergence results in the asynchronous continuous framework are well known and are based on a contraction hypothesis with respect to a maximum norm, see *e.g.* [3, 5, 4, 7, 1, 2]. However, this approach cannot be applied to the discrete framework. Indeed, the hypothesis in the continuous framework leads to constant functions in the discrete framework (detailed in Remark 3.1). A new study is necessary for the discrete case. We suppose that

F is a contraction with respect to a vectorial distance. The contraction with respect to a vectorial distance was first introduced by [6] in order to study a particular case of asynchronous iterations, namely the chaotic ones.

For analyzing the convergence of discrete systems, preliminary results concerning the boolean matrices [6] and the decomposition of a discrete system are introduced in Section 2. The asynchronous discrete model is formulated in Section 3. The convergence theorem is given in Section 4. Finally in Section 5, classical particular cases of the asynchronous discrete model and numerical applications with a boolean network are presented.

2. PRELIMINARY RESULTS

2.1. Specific Results of Boolean Matrices (Detailed in [6])

DEFINITION 2.1 Consider the Cartesian product $E = \prod_{i=1}^{n} E_i$ where E_i represents the finite set of possible values x_i of the component i of the discrete system. The vectorial distance $d: E \times E \to \{0,1\}^n$ is defined by for all $(x, y) \in E \times E$

$$(x, y) \to d(x, y) = (\delta(x_1, y_1), \ldots, \delta(x_n, y_n))$$

where

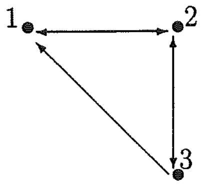
$$\delta(x_i, y_i) = \begin{cases} 1 & \text{if } x_i \neq y_i \\ 0 & \text{if } x_i = y_i. \end{cases}$$

DEFINITION 2.2 Consider a discrete system whose dynamic is described according to a function $f: E \to E$. The boolean matrix $B(f)$ associated with f is defined by its general term b_{ij}, $i, j \in \{1, \ldots, n\}$, so that

$$b_{ij} = \begin{cases} 1 & \text{if the } i\text{th component of } f \text{ depends on } x_j \\ 0 & \text{otherwise.} \end{cases}$$

As $B(f)$ is a boolean matrix, its only possible eigenvalues are 0 or 1.

Example 2.1 The following simple example shows that to have the boolean matrix $B(f)$ is equivalent to have the connexion graph of the discrete system. Consider a discrete system with 3 components 1, 2 and 3. Assume that the notation $1 \to 2$ means that component 1 informs component 2. If these components are connected as shown in the figure below



then, the contraction matrix of any mapping $f = (f_1, f_2, f_3)$ describing the dynamic of the system according to the above graph is

$$B(f) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

PROPOSITION 2.1 *$d(f(x), f(y)) \leq B(f)d(x, y)$, for all $(x, y) \in E \times E$ where the componentwise order relation $\leq$ is defined in $\{0, 1\}^n$ by $0 \leq 0 \leq 1 \leq 1$.*

DEFINITION 2.3 The function f is a contraction if the associated matrix $B(f)$ has all eigenvalues equal to 0, *i.e.*, with a spectral radius $\rho(B) = 0$. $B(f)$ is called the contraction matrix of f.

PROPOSITION 2.2 *$B(f)$ is a contraction matrix if and only if there exists a permutation matrix P so that $P^T BP$ is a strictly lower triangular matrix where P^T is the transpose of P.*

PROPOSITION 2.3 *If the function f is a contraction on $E = \prod_{i=1}^{n} E_i$ then there exists a unique $x^* \in E$ so that $x^* = f(x^*)$. x^* is called the fixed point of f.*

2.2. Decomposition of the Discrete System

Consider α integers n_i so that $\Sigma_{i=1}^{\alpha} n_i = n$. Define the integers q_i, $i \in \{1, \ldots, \alpha\}$, by

$$\begin{cases} q_1 = 0 \\ q_i = n_1 + \cdots + n_{i-1}. \end{cases}$$

Then $x = (x_1, \ldots, x_n) \in \prod_{i=1}^{n} E_i$ is partitioned as follows:

$$\begin{cases} x = X = (X_1, \ldots, X_\alpha) \\ X_i = (x_{q_i+1}, \ldots, x_{q_i+n_i}) \in \prod_{j=1}^{n_i} E_{q_i+j}. \end{cases}$$

The function f is partitioned into a compatible way

$$\begin{cases} f = F = (F_1, \ldots, F_\alpha) \\ F_i = (f_{q_i+1}, \ldots, f_{q_i+n_i}). \end{cases}$$

$\{0,1\}^n$ is considered as $\prod_{i=1}^{\alpha}\{0,1\}^{n_i}$.

DEFINITION 2.4 The block vectorial distance is defined as follows

$$d(X, Y) = (\delta(X_1, Y_1), \ldots, \delta(X_\alpha, Y_\alpha)),$$

such that

$$\delta(X_i, Y_i) = \begin{cases} 1 & \text{if } X_i \neq Y_i \\ 0 & \text{if } X_i = Y_i. \end{cases}$$

3. ASYNCHRONOUS DISCRETE MODEL

DEFINITION 3.1 Let the strategy $\{J(t)\}_{t\in\mathbb{N}}$ be a sequence of non-empty subsets of $\{1, \ldots, \alpha\}$ at the time t. Let $\{s_j^i(t)\}_{t\in\mathbb{N}}$, $i, j \in \{1, \ldots, \alpha\}$, be a sequence of integers at the time t satisfying the 3 following conditions:

(i) $s_j^i(t) = t - r_j^i(t)$ with $0 \leq r_j^i(t) \leq t$, $r_j^i(t)$ being the delay of the block j compared to the block i.
(ii) $\forall i, j \in \{1, \ldots, \alpha\}$, $\lim_{t\to\infty} s_j^i(t) = \infty$, *i.e.*, the delays associated with the block i are unbounded but follow the iterations of the system.
(iii) $\forall i \in \{1, \ldots, \alpha\}$, Card $(\{t, i \in J(t)\}) = \infty$, *i.e.*, no block is definitively lost.

Then, the asynchronous iterations with delays $\{r_j^i(t)\}$ according to the strategy $\{J(t)\}$ are described by the algorithm

$$\begin{cases} \text{Given } X^0 = (X_1^0, \ldots, X_\alpha^0) \\ t = 0, 1, \ldots \\ \quad i = 1, \ldots, \alpha \\ \quad\quad X_i^{t+1} = \begin{cases} F_i(X_1^{s_1^i(t)}, \ldots, X_\alpha^{s_\alpha^i(t)}) \text{ if } i \in J(t) \\ X_i^t \text{ if } i \notin J(t). \end{cases} \end{cases} \tag{1}$$

If the block i belongs to the strategy $J(t)$ at the time t, then its state X_i^{t+1} is iterated by F_i, *i.e.*, $X_i^{t+1} = F_i(X_1^{s_1^i(t)}, \ldots, X_\alpha^{s_\alpha^i(t)})$ otherwise its state X_i^{t+1} is not iterated, *i.e.*, $X_i^{t+1} = X_i^t$.

Remark 3.1 Even if the convergence of asynchronous iterations is well studied in the continuous framework, the discrete framework remains particular and needs a specific convergence analysis. Indeed, the convergence in the continuous framework is based on a contraction hypothesis of F with respect to the maximum norm $\max_{1 \leq i \leq n} \delta(x_i, y_i)$. In the discrete framework, this hypothesis implies constant functions which have no interest in simulation. Therefore, a new study for the discrete framework is necessary with a contraction hypothesis with respect to a new distance: the vectorial distance here.

4. RESULTS

DEFINITION 4.1 Consider the strictly increasing sequence of integers $\{p_l\}_{l \in \mathbb{N}}$ as follows:

$$\begin{cases} p_0 = 0 \\ p_{l+1} \text{ is the smallest integer satisfying} \\ \qquad \bigcup_{p_l \leq s_{\min}(t) \leq t < p_{l+1}} J(t) = \{1, \ldots, \alpha\} \\ \quad \text{where } J(t) \subset \{1, \ldots, \alpha\} \text{ and } s_{\min}(t) = \min_{1 \leq i, j \leq \alpha}\{s_j^i(t)\}. \end{cases}$$

This sequence $\{p_l\}$ is well defined thanks to the conditions (ii) and (iii) of Definition 3.1.

THEOREM 4.1 *Let a discrete dynamic system of n components be partitioned into α blocks and described by an iteration function $F = (F_1, \ldots, F_\alpha)$. If F is a contraction with respect to the block vectorial distance of Definition* 2.4 *on the finite Cartesian product set $E = \prod_{i=1}^{\alpha} E_i$ and if the* 3 *conditions of Definition* 3.1 *are satisfied, then all asynchronous iterations from any initial state $X^0 = (X_1^0, \ldots, X_\alpha^0)$ converge to a unique fixed point X^* within p_α steps, i.e.,*

$$X^* = X^t, \ t = p_\alpha, p_\alpha + 1, \ldots$$

where X^t is defined in (1) *and $\{p_l\}_{l \in \mathbb{N}}$, in Definition* 4.1.

Proof Denote: For $t \in \mathbb{N}^*$ and $i, j \in \{1, \ldots, \alpha\}$

$$X^t = (X_1^t, \ldots, X_\alpha^t),$$
$$X^{s^i(t)} = \left(X_1^{s_1^i(t)}, \ldots, X_\alpha^{s_\alpha^i(t)}\right),$$
$$X^{s(t)} = \left(X^{s^1(t)}, \ldots, X^{s^\alpha(t)}\right),$$

and A_{t-1}, a $n \times n$ matrix defined by

$$(A_{t-1})_{ij} = \begin{cases} 1 & \text{if } i = j \notin J(t-1) \\ 0 & \text{if } i = j \in J(t-1) \\ 0 & \text{if } i \neq j. \end{cases} \tag{2}$$

The proof is divided into 3 parts.

(i) $d(X^t, X^*) \leq A_{t-1} d(X^{t-1}, X^*) + B(F)\, d(X^{s(t-1)}, X^*)$.

Define the vectors X_{I}^t and X_{II}^t as follows

$$\text{for } i \notin J(t-1),\ (X_{I}^t)_i = X_i^t \text{ and for } i \in J(t-1),\ (X_{II}^t)_i = X_i^t,$$

so according to (1), we have

$$(X_{I}^t)_i = X_i^{t-1} \text{ and } (X_{II}^t)_i = F_i\big(X^{s^i(t-1)}\big).$$

By Proposition 2.3, F has a unique fixed point denoted by X^* so that $X^* = F(X^*)$. By applying Definition 2.4 of the vectorial distance d and after reordering the blocks

$$d(X^t, X^*) = \begin{pmatrix} d(X_{I}^t, X^*) \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ d(X_{II}^t, X^*) \end{pmatrix}.$$

By using the componentwise order relation $\leq$ defined in $\{0,1\}^n$ by $0 \leq 0 \leq 1 \leq 1$, we obtain

$$d(X^t, X^*) \leq A_{t-1} d(X^{t-1}, X^*) + d\big(F\big(X^{s(t-1)}\big), F(X^*)\big),$$

where for all integers $t \geq 1$, A_{t-1} is defined by (2).

By applying Proposition 2.1, we deduce

$$d(X^t, X^*) \leq A_{t-1} d(X^{t-1}, X^*) + B(F) d\big(X^{s(t-1)}, X^*\big). \tag{3}$$

(ii) $d(X^t, X^*) \leq \sum_{j=1}^{q} (B(F))^{k_j} d(X^0, X^*)$ with $k_j \geq \alpha$ for $t \geq p_\alpha$.

(iia) $d(X^t, X^*) \leq \sum_{j=1}^{q} (B(F))^{k_j} d(X^0, X^*)$.

By remarking that for all t, $A_t B(F) \leq B(F)$ and $B(F) A_t \leq B(F)$, the induction formulae (3) leads to

$$d(X^t, X^*) \leq \prod_{j=0}^{t-1} A_j d(X^0, X^*) + \sum_{j=1}^{q} (B(F))^{k_j} d(X^0, X^*),$$

where q is a finite integer.

Due to Definition 4.1 of $\{p_l\}_{l\in\mathbb{N}}$,

$$\forall i\in\{1,\ldots,\alpha\},\forall l\in\mathbb{N},\exists j\in\{p_l,\ldots,p_{l+1}-1\}\text{ so that }(A_j)_{ii}=0. \quad (4)$$

Therefore,

$$\prod_{j=0}^{t-1}A_j=0\text{ if }t\geq p_1.$$

Then for $t\geq p_\alpha$,

$$d(X^t,X^*)\leq\sum_{j=1}^{q}(B(F))^{k_j}d(X^0,X^*).$$

(iib) $k_j\geq\alpha$ for $t\geq p_\alpha$.

Due to the construction of $\{p_l\}_{l\in\mathbb{N}}$ in Definition 4.1, if $t\geq p_\alpha$ then $k_j\geq\alpha$. Indeed, for $t=p_1, d(X^t,X^*)\leq\sum_{j=1}^{q}(B(F))^{k_j}d(X^0,X^*)$ thanks to (4) and $k_j\geq 1$, $\forall j\in\{1,\ldots,q\}$ as all the blocks are updated at least one time. Suppose that for $\{p_1,\ldots,p_\alpha\}$ we have for all $t\geq p_\alpha$,

$$\begin{cases} d(X^t,X^*)\leq\sum_{j=1}^{q}(B(F))^{k_j}d(X^0,X^*)\\ k_j\geq\alpha,\forall j\in\{1,\ldots,q\},\end{cases}$$

if $t\geq p_\alpha$ then,

$$\begin{cases} d(X_i^t,X_i^*)\leq\sum_{j=1}^{q}\sum_{l=1}^{\alpha}\left((B(F))^{k_j}\right)_{il}d(X_l^0,X_l^*)\\ k_j\geq\alpha,\forall j\in\{1,\ldots,q\}.\end{cases}$$

Let $t\geq p_{\alpha+1}$, then due to the construction of $\{p_l\}_{l\in\mathbb{N}}$, $\forall i\in\{1,\ldots,\alpha\}$, $\exists q$ so that $p_\alpha\leq s_{\min}(q)\leq q<p_{\alpha+1}$ and $X_i^t=F_i(X^{s^i(q)})$ so,

$$\begin{aligned} d(X_i^t,X_i^*)&\leq\left(B(F)d\left(X^{s^i(q)},X^*\right)\right)_i\\ &\leq\sum_{m=1}^{\alpha}B(F)_{im}d\left(X_m^{s^i(q)},X_m^*\right),\end{aligned}$$

as $p_\alpha\leq s_{\min}(q)$, we have by the induction hypothesis

$$\begin{cases} d(X_i^t,X_i^*)\leq\sum_{m=1}^{\alpha}B(F)_{im}\sum_{j=1}^{q}\sum_{l=1}^{\alpha}\left((B(F))^{k_j}\right)_{ml}d(X_l^0,X_l^*)\\ k_j\geq\alpha,\forall j\in\{1,\ldots,q\},\end{cases}$$

so

$$d(X_i^t, X_i^*) \leq \sum_{j=1}^{q} \sum_{l=1}^{\alpha} \sum_{m=1}^{\alpha} B(F)_{im} \left((B(F))^{k_j} \right)_{ml} d(X_l^0, X_l^*),$$

and

$$d(X_i^t, X_i^*) \leq \sum_{j=1}^{q} \left((B(F))^{k_j+1} d(X^0, X^*) \right)_i,$$

so

$$\begin{cases} d(X^t, X^*) \leq \sum_{j=1}^{q} (B(F))^{h_j} d(X^0, X^*) \\ h_j \geq \alpha + 1, \forall j \in \{1, \ldots, q\}. \end{cases}$$

(iib) $\forall t \geq p_\alpha$, $d(X^t, X^*) = 0$.

As F is a contraction, Proposition 2.2 leads to

$$\sum_{j=1}^{q} (B(F))^{k_j} = \sum_{j=1}^{q} (PLP^T)^{k_j},$$

where L is a strictly lower triangular block matrix. As $PP^T = I$ (identity matrix), then,

$$\sum_{j=1}^{q} (B(F))^{k_j} = P \sum_{j=1}^{q} L^{k_j} P^T.$$

As $k_j \geq \alpha$, L is a strictly lower triangular matrix and as the product of α strictly lower triangular block matrices of dimension α leads to a null matrix, then for all $j \in \{1, \ldots, q\}$,

$$L^{k_j} = 0.$$

So

$$d(X^t, X^*) = 0.$$

In conclusion, the asynchronous discrete model converges to the unique fixed point X^* at most after p_α steps.

Remark 4.1 By taking $E = \{0, 1\}$, the discrete system is boolean.

5. PARTICULAR CASES OF THE ASYNCHRONOUS DISCRETE MODEL

- Componentwise iterations: $n_i = 1, \forall i \in \{1, \dots, \alpha = n\}$.
- Block parallel iterations (block Jacobi iterations): $s_j^i(t) = t, \forall i,j \in \{1, \dots, \alpha\}$ (no delays) and $J(t) = \{1, \dots, \alpha\}$.
- Block series iterations (block Gauss-Seidel): $s_j^i(t) = t, \forall i,j \in \{1, \dots, \alpha\}$ and $J(t) = 1 + t \bmod \alpha$.
- Block chaotic iterations: $s_j^i(t) = t$, $\forall i,j \in \{1, \dots, \alpha\}$, $J(t) \neq \emptyset$ and $\forall i \in \{1, \dots, \alpha\}$, Card $(\{t, i \in J(t)\}) = \infty$ (case studied by [6, 7]).

6. NUMERICAL APPLICATIONS

A research software DSE (Discrete System Evolution) has been developed in order to simulate different convergence strategies of discrete models with asynchronous iterations. The discrete models studied are boolean networks, each component takes only 2 states $\{0, 1\}$. DSE is based on 3 functionalities: the dynamic of the network, the choice of different network parameters and the graphical representation of results. DSE verifies the contraction hypothesis. The results are presented as follows. As the state x_i^t of each boolean component i is defined on the binary system $E = \{0, 1\}$ of $\text{Card}(E) = 2$, the state x^t of the network with n components can be associated with the boolean number $x_E^t = x_1^t \dots x_n^t$. This number x_E^t can be represented on the decimal system $\mathfrak{D} = \{0, \dots, 9\}$ as follows $x_{\mathfrak{D}}^t = x_1^t 2^{n-1} + x_2^t 2^{n-2} + \cdots + x_n^t$. The curves obtained are represented as follows: the abscissa shows the time t and the ordinate gives the state of the network on the decimal system $x_{\mathfrak{D}}$.

The dynamic of a boolean network with $n = 10$ components is described according to the iteration function $f = (f_1, \dots, f_{10})$ defined as follows

$$f(x) = \begin{cases} f_1(x) = x_2\bar{x}_4x_5\bar{x}_6\bar{x}_8x_9 + x_2x_9 + x_2x_6 \\ f_2(x) = \bar{x}_8 \\ f_3(x) = \bar{x}_1 + x_2 + x_4x_5 + x_7x_8\bar{x}_9 \\ f_4(x) = \bar{x}_2 + \bar{x}_8 + \bar{x}_9 \\ f_5(x) = x_2 + x_4 + \bar{x}_8 + x_9 \\ f_6(x) = x_2\bar{x}_4x_5x_8x_9 \\ f_7(x) = x_2x_4 + x_5 + \bar{x}_6 + x_8 + \bar{x}_9 \\ f_8(x) = 0 \\ f_9(x) = \bar{x}_2 \\ f_{10}(x) = x_1\bar{x}_2\bar{x}_3 + x_5 + \bar{x}_6 + \bar{x}_9 \end{cases}$$

The matrix $B(f)$ associated with f is then deduced

$$B(f) = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

It can be verified without difficulty that $B(f)$ is a contraction matrix as $\rho\,(B) = 0$.

The solution of the function f is the fixed point $x^* = (0, 1, 1, 1, 1, 0, 1, 0, 0, 1)$ associated with $x^*_E = 0111101001$ and represented on the decimal system by $x^*_{\mathfrak{D}} = 489$. Three simulations of asynchronous componentwise iterations are given by varying the type of delay and the type of strategy. The initial state x^0 chosen for the 3 simulations is the state with greatest distance to x^*, *i.e.*, $x^0 = (1, 0, 0, 0, 0, 1, 0, 1, 1, 0)$ associated with $x^0_E = 1000010110$ and $x^0_{\mathfrak{D}} = 534$.

The results of the simulations are represented as follows: the *abscissa* shows the time t by varying t between 0 and 200, and the ordinate gives the state of the network on the decimal system $x_{\mathfrak{D}}$ with values between 0 and 1023 (2^{10} possible values).

Figure 1 shows the convergence to x^* of the boolean network with random $s^i_j(t) \in [t/2, t]$ and a random strategy $J(t)$.

The next two simulations are examples of non-convergence of the network as one of the two conditions (ii) and (iii) of (1) is not verified.

Figure 2 shows the non-convergence of the boolean network with random $s^i_j(t) \in [0, t]$ and a random strategy $J(t)$. Indeed, the condition (ii) $\lim_{t\to\infty} s^i_j(t) = \infty$ of (1) is not satisfied.

Figure 3 shows the non-convergence to x^* of the boolean network with random $s^i_j(t) \in [t/2, t]$ and a random strategy $J(t)$ when $t < 5$ but when $t \geq 5$, the component 6 is definitively lost. The curve is stabilized to $x_{\mathcal{D}} = 1017$ associated with $x_E = 1111111001$ which is the solution of a new system $\tilde{f}_i = f_i$ for $i \neq 6$ and $\tilde{f}_6$ fixed by 0 or 1. Indeed, the disregard of the component 6 does not satisfy the condition (iii) $\forall i \in \{1, \ldots, \alpha\}$, Card $(\{t, i \in J(t)\}) = \infty$ of (1).

These different simulations lead to results predicted by the theory.

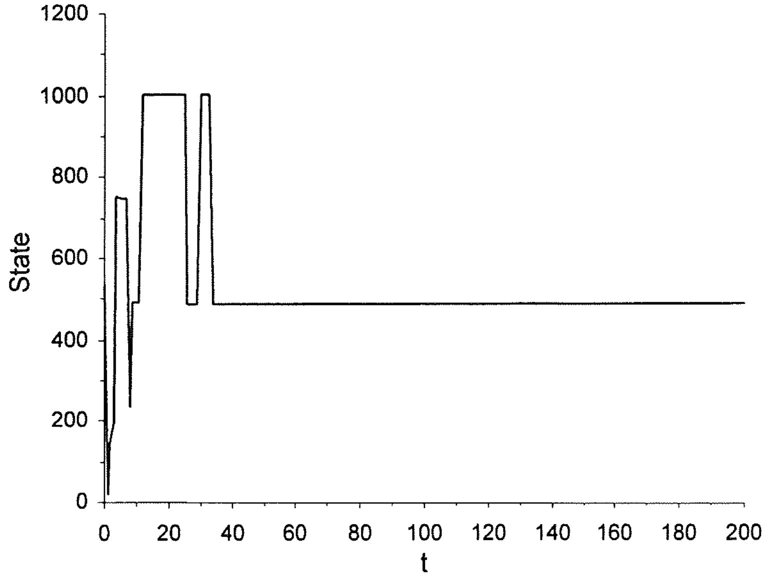


FIGURE 1 Convergence of the boolean network with asynchronous componentwise iterations and a random strategy.

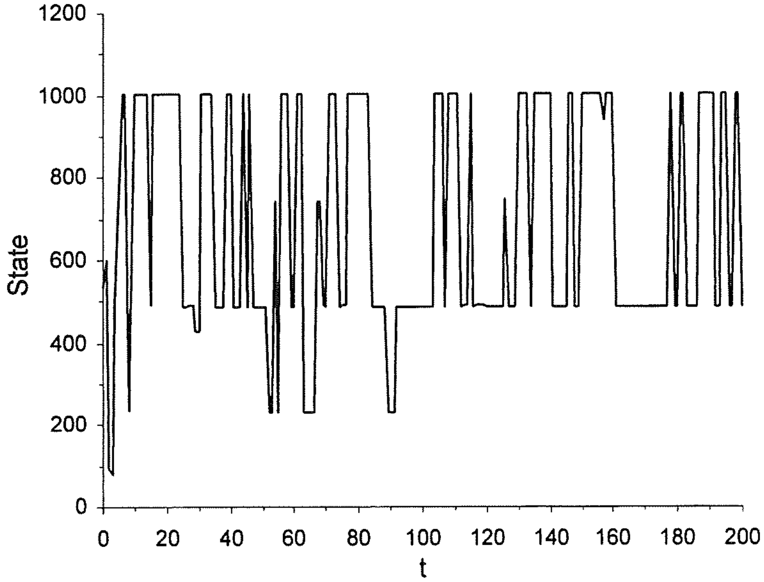


FIGURE 2 Non-convergence of the boolean network with asynchronous componentwise iterations as the condition (ii) is not verified.

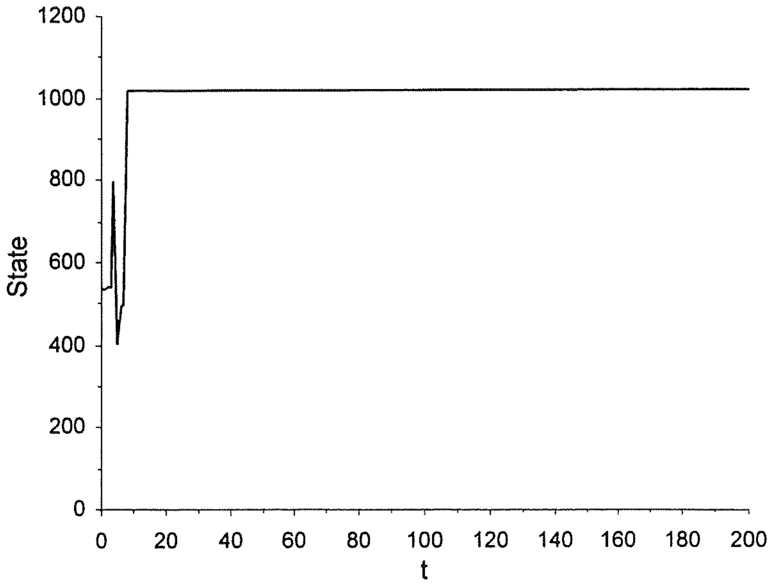


FIGURE 3 Non-convergence of the boolean network with asynchronous componentwise iterations as the condition (iii) is not verified.

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