# Formalization of Foundations of Geometry

## An overview of the GeoCoq library

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1. Interactive Theorem Proving for the Education

2. Overview of GeoCoq
   - Foundations
   - Arithmetization of Geometry
     - Addition
     - Multiplication
   - Automation
   - Continuity
   - Logic
   - 34 parallel postulates
   - Two formalizations of the Elements
   - Some high-school examples

# Using a computer to teach maths

## Software are used in the classroom

1. for numerical computations
2. for symbolic computations (Maple. . . )
3. for producing conjectures (dynamic geometry software)
4. for construction exercises (Euclidea, . . . ,)
5. for checking conjectures using probabilistic or algebraic methods (Cabri, GeoGebra, . . . )

## But

the use of software for checking **proofs** is not widespread !

# Teaching the concept of proof

- Maths teachers often do not know about logic.
- Only some of the reasoning rules are given: proof by contradiction, contrapositive, reasoning by cases.
- Semantics checks are used rather than syntactic checks.
- Learning by imitation (a proof is what makes the teacher happy/ good marks).

"If you can't explain mathematics to a machine, it is an illusion to think you can explain it to a student." De Bruijn "Invited lecture at the Mathematics Knowledge Management Symposium", 25-29 November 2003, Heriot-Watt University, Edinburgh, Scotland

# Different potential goals

- To teach what is a proof
- To teach logic
- To teach software foundations
- To automate proof checking
- To teach maths in general
- To automate feedback in general

# Different potential goals

- To teach what is a proof
- To teach logic
- To teach software foundations
- To automate proof checking
- ~~To teach maths in general~~
- ~~To automate feedback in general~~

I do not need a tutor, just a proof checker.

# ITP, why ?

- Clarify the rules of the game: the deduction rules are explicit.
- Clarify the language: axiom, theorem, lemma, hypotheses, definition, conjecture, counter-example. . .
- Objective criterion for the validity of a proof.
- Interactivity: feedback during homework.
- Motivation: theorem proving as a game.

## Challenges

- Find a good language/user interface.
- Build the needed libraries.
- Automate what should be automatized and not more (depending on the context).

# About the language, proof rules, user interface

I would like deduction rules which are:

- sound
- explicit
- clear
- complete
- not necessarily minimal
- not too far from the mathematical practice

# Coherent logic

$$\forall x, H_1(x) \wedge \ldots \wedge H_n(x) \;\rightarrow\; \begin{array}{ll} & \exists y, P_1(x,y) \wedge \ldots P_k(x,y) \\ \vee & \ldots \end{array}$$

Several authors have identified independently this fragment of FOL.
Allows proofs to be **somewhat** readable [1].

[1] Sana Stojanović et al. (2014). "A Vernacular for Coherent Logic". English. In: Intelligent Computer Mathematics. Vol. 8543. Lecture Notes in Computer Science

# Existing tools

Two communities:

1. Didactics of mathematics
2. Interactive theorem proving

# Didactics of mathematics Community

Geometry Tutor [2], MENTONIEZH [3], DEFI [4], CHYPRE [5], Geometrix [6], Cabri Euclide [7], Baghera [8], AgentGeom, geogebraTUTOR and Turing [9]

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[2] John R. Anderson, C. F. Boyle, and Gregg Yost (1985). "The geometry Tutor". In: IJCAI Proceedings

[3] Dominique Py (1990). "Reconnaissance de plan pour l'aide à la démonstration dans un tuteur intelligent de la géométrie". PhD thesis. Université de Rennes

[4] Ag-Almouloud (1992). "L'ordinateur, outil d'aide à l'apprentissage de la démonstration et de traitement de données didactiques". PhD thesis. Université de Rennes

[5] Philippe Bernat (1993). CHYPRE: Un logiciel d'aide au raisonnement. Tech. rep. 10. IREM

[6] Jacques Gressier (1988). Geometrix.

[7] Vanda Luengo (1997). "Cabri-Euclide: Un micromonde de Preuve intégrant la réfutation". PhD thesis. Université Joseph Fourier

[8] Nicolas Balacheff et al. (1999). Baghera.

[9] Philippe R. Richard et al. (2011). "Didactic and theoretical-based perspectives in the experimental development of an intelligent tutorial system for the learning of geometry". en. In: ZDM 43.3

# ITP Community

## Computer Science

- logic
- proof of programs, semantics, software foundations

U-Penn, Portland, Princeton, Harvard, Warsaw, CNAM, Lyon, Nice, Paris, Strasbourg, ...

## Maths

- Bachelor - Logic: Bordeaux, Warsaw, Pohang, Strasbourg, ...
- Bachelor - Maths: Nijmegen (ProofWeb), Nice (CoqWeb), ...
- ...

Two kinds of systems:

1. Syntactic sugar added over a state of the art proof assistant
   - PCoq [10]
   - Coq Web [11]
   - ProofWeb [12]
   - Edukera [13]
2. Natural language + Automatic Theorem Proving

[10] Ahmed Amerkad et al. (2001). "Mathematics and Proof Presentation in Pcoq". In: Workshop Proof Transformation and Presentation and Proof Complexities in connection w

[11] Jérémy Blanc et al. (2007). "Proofs for freshmen with Coqweb". In: PATE'07

[12] CS Kaliszyk et al. (2008). "Deduction using the ProofWeb system". In:

[13] Benoit Rognier and Guillaume Duhamel (2016). "Présentation de la plateforme edukera". In: Vingt-septièmes Journées Francophones des Langages Applicatifs (JFLA 2016)

Two kinds of systems:

1. Syntactic sugar added over a state of the art proof assistant
2. Natural language + Automatic Theorem Proving
   - SAD [10]
   - Naproche [11]
   - Lurch [12]
   - ELFE [13]
   - CalcCheck [14]
   - Mendes' system [15]

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[10] Alexander Lyaletski, Andrey Paskevich, and Konstantin Verchinine (2006). "SAD as a mathematical assistant—how should we go from here to there?" In: Journal of Applied Logic. Towards Computer Aided Mathematics 4.4

[11] Marcos Cramer et al. (2010). "The Naproche Project Controlled Natural Language Proof Checking of Mathematical Texts". In: Controlled Natural Language

[12] Nathan C. Carter and Kenneth G. Monks. "Lurch: a word processor built on OpenMath that can check mathematical reasoning". In:

[13] Maximilian Doré (2018). "The ELFE Prover". In: 25th Automated Reasoning Workshop

[14] Wolfram Kahl (2018). "CalcCheck: A Proof Checker for Teaching the "Logical Approach to Discrete Math"". en. In: Interactive Theorem Proving. Lecture Notes in Computer Science

[15] Alexandra Mendes and João F. Ferreira (2018). "Towards Verified Handwritten

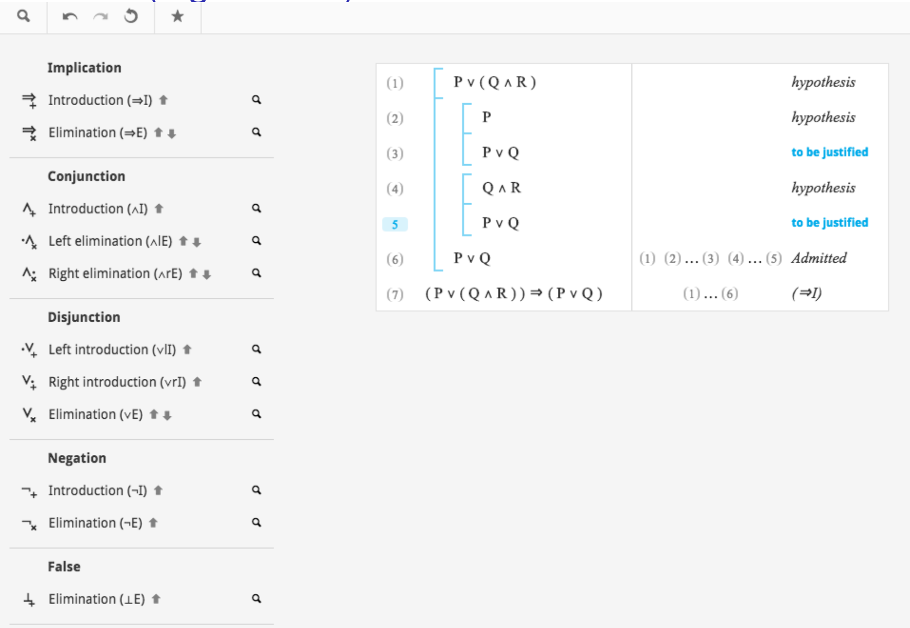
# Edukera (Rognier and Duhamel)

- Web-application
- Coq is hidden inside the web-page
- LCF style interaction + proof displayed in a pen and paper style.
- Some users in France (about 1000 students, 70k exercises)
- No textual input ”proof by pointing”, syntactically correct by construction (as using Scratch)
- Easy to learn using a tutorial
- Always correct applications of a logic rule
- Meta-variables

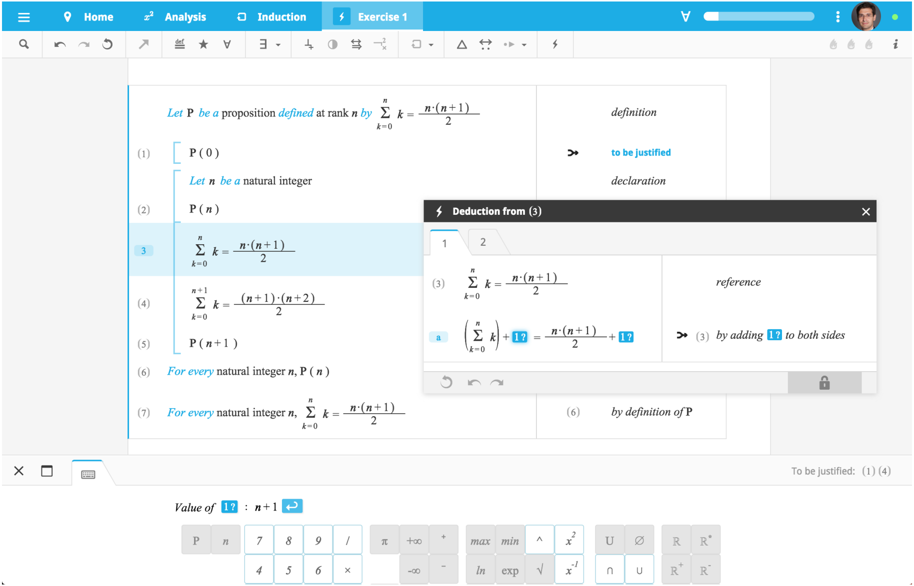
# Two modes

1. Logic
   - Use natural deduction rules.
   - Can display proof tree (Fitch's or Gentzen's style).
   - Backward reasoning
2. Maths
   - Forward/Backward reasoning.
   - Less fine grained proof steps than in logic mode.

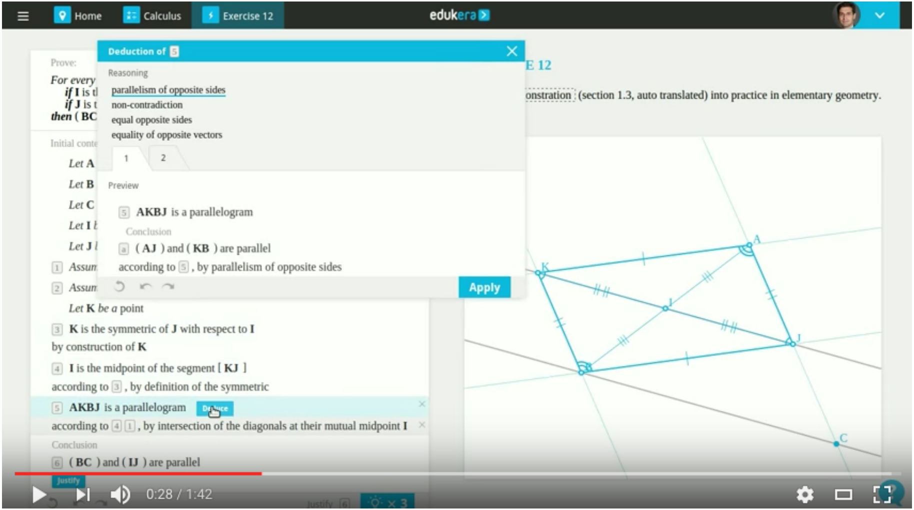
# Edukera (logic mode)

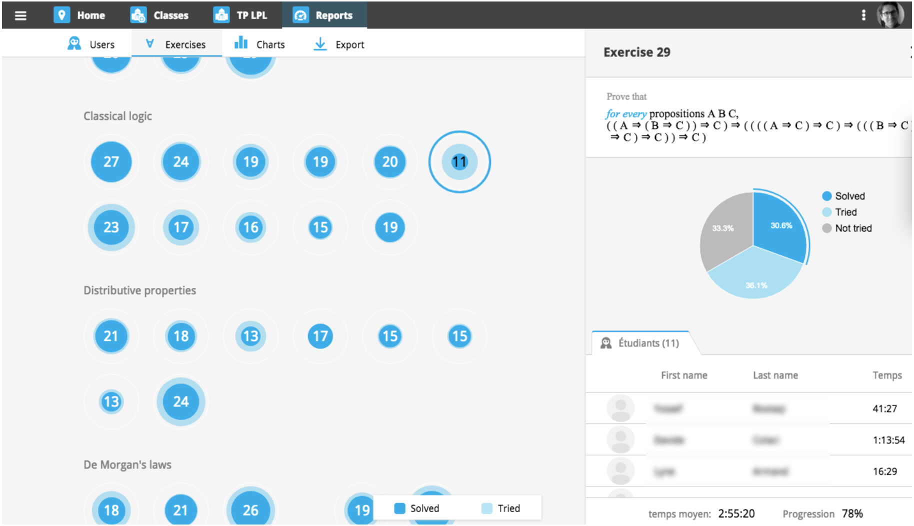


# Edukera (math mode)



# Edukera (prototype for geometry)




Home
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Distributive properties
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De Morgan's laws
18
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26
19
Solved
Tried
Exercise 29
Prove that
for every propositions A B C,
((A ⇒ (B ⇒ C)) ⇒ C) ⇒ ((((A ⇒ C) ⇒ C) ⇒ (((B ⇒ C
⇒ C) ⇒ C)) ⇒ C)
Solved
Tried
Not tried
33.3%
30.6%
36.1%
Étudiants (11)
First name
Last name
Temps
41:27
1:13:54
16:29
temps moyen: 2:55:20
Progression 78%

# Some experiments

Previous years:

- Undergraduate computer-science logic course: natural deduction (Edukera/Logic Mode)
- Graduate computer-science formal theorem proving course (Edukera Logic Mode+Coq)
- Graduate computer-science software-foundations course (Coq, Frama-c, why3)

Starting September:

- First-year undergraduate maths/computer science: The concept of proof and very basics results about relations/functions/sets (Edukera Maths Mode).

# Results in a logic course

- 36 students, $>$ 2000 exercises in natural deduction
- positive student feedback
- need a scientific evaluation

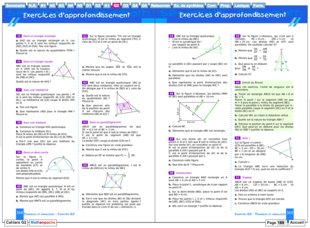
# Results in a course about formal theorem proving

- Edukera as a tool to learn natural deduction.
- Coq tactics are then learned quicker.

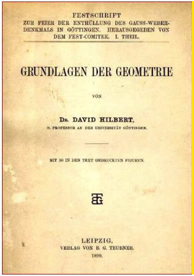
# Outline

# GeoCoq

- An Open Source library about foundations of geometry
- Michael Beeson, Gabriel Braun, Pierre Boutry, Charly Gries, Julien Narboux, Pascal Schreck
- Size: $>$ 3900 Lemmas, $>$ 130000 lines
- License: LGPL3




Exercises


FESTSCHRIFT
ZUR FEIER DER ENTHÜLLUNG DES GAUSS-WEBER-DENKMALS IN GÖTTINGEN. HERAUSGEGEBEN VON DEM FEST-COMITEE. I. THEIL.

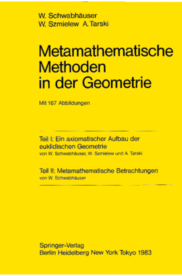
GRUNDLAGEN DER GEOMETRIE

VON

DR. DAVID HILBERT,
O. PROFESSOR AN DER UNIVERSITÄT GÖTTINGEN.

MIT 50 IN DEN TEXT GEDRUCKTEN FIGUREN.

LEIPZIG,
VERLAG VON B. G. TEUBNER.
1899.


W. Schwabhäuser
W. Szmielew A. Tarski

Metamathematische Methoden in der Geometrie

Mit 167 Abbildungen

Teil I: Ein axiomatischer Aufbau der euklidischen Geometrie
von W. Schwabhäuser, W. Szmielew und A. Tarski

Teil II: Metamathematische Betrachtungen
von W. Schwabhäuser

Springer-Verlag
Berlin Heidelberg New York Tokyo 1983

Euclide Hilbert Tarski

## What we have:

**Axiom systems** Tarski's, Hilbert's, Euclid's and variants.

**Foundations** In arbitrary dimension, in neutral geometry. Betweenness, Two-sides, One-side, Collinearity, Midpoint, Symmetric point, Perpendicularity, Parallelism, Angles, Co-planarity, . . .

**Classic theorems** Pappus, Pythagoras, Thales' intercept theorem, Thales' circle theorem, nine point circle, Euler line, orthocenter, circumcenter, incenter, centroid, quadrilaterals, Sum of angles, Varignon's theorem, . . .

**Arithmetization** Coordinates

**High-school** Some exercises

## What is missing:

- Consequence of continuity: trigonometry, areas
- link with Complex numbers

# Foundations of geometry

1. Synthetic geometry
2. Analytic geometry
3. Metric geometry
4. Transformations based approaches

# Synthetic approach

Assume some undefined geometric objects + geometric predicates + axioms . . .
The name of the assumed types are not important.

- Hilbert's axioms:

  types: points, lines and planes
  predicates: incidence, between, congruence of segments, congruence of angles

- Tarski's axioms:

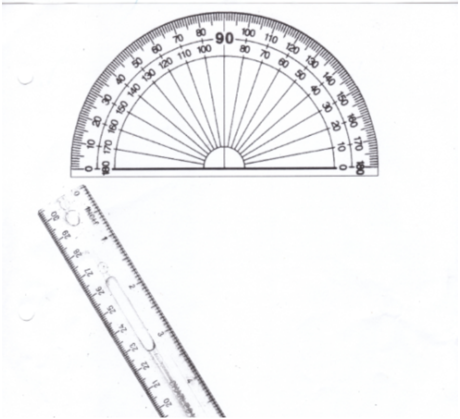
  types: points
  prédicats: between, congruence

- . . . many variants

# Analytic approach

We assume we have numbers (a field $\mathbb{F}$).
We define geometric objects by their coordinates.
Points := $\mathbb{F}^n$

# Metric approach



Compromise between synthetic and metric approach.
We assume both:

- numbers (a field)
- geometric objects
- axioms

- Birkhoff's axioms: points, lines, reals, ruler and protractor
- Chou-Gao-Zhang's axioms: points, numbers, three geometric quantities

# Transformation groups

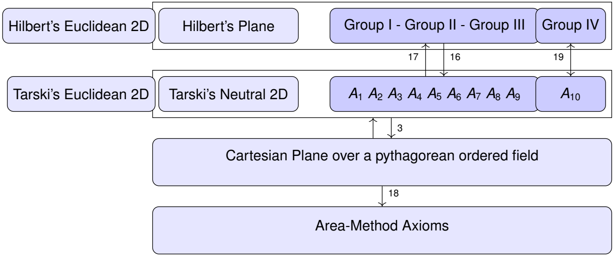
Erlangen program. Foundations of geometry based on group actions and invariants.

Felix Klein

# Comparison

| | Synthetic | Analytic |
|---|---|---|
| Logical Reasoning | ☺ | ☹ |
| Proof reuse between geometries | ☺ | ☹ |
| Computations | ☹ | ☺ |
| Automatic proofs | ☹ | ☺ |

# Overview of the axiom systems




[16] Gabriel Braun, Pierre Boutry, and Julien Narboux (2016). "From Hilbert to Tarski". In: Eleventh International Workshop on Automated Deduction in Geometry. Proceedings of ADG 2016

[17] Gabriel Braun and Julien Narboux (2012). "From Tarski to Hilbert". English. In: Post-proceedings of Automated Deduction in Geometry 2012. Vol. 7993. LNCS

[18] Pierre Boutry, Gabriel Braun, and Julien Narboux (2017). "Formalization of the Arithmetization of Euclidean Plane Geometry and Applications". In: Journal of Symbolic Computation

[19] **boutry˙parallel˙2015**

# An "axiom free" development

Axiom = global variable

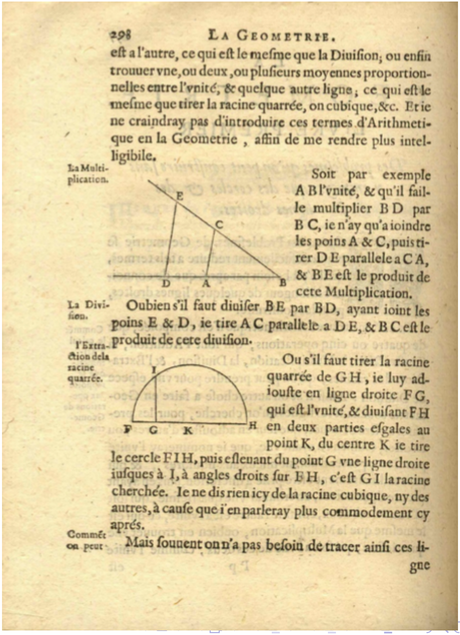
```
Class Tarski_neutral_dimensionless :=
{
 Tpoint : Type;
 Bet : Tpoint -> Tpoint -> Tpoint -> Prop;
 Cong : Tpoint -> Tpoint -> Tpoint -> Tpoint -> Prop;
 cong_pseudo_reflexivity : forall A B, Cong A B B A;
 cong_inner_transitivity : forall A B C D E F,
   Cong A B C D -> Cong A B E F -> Cong C D E F;
 cong_identity : forall A B C, Cong A B C C -> A = B;
 segment_construction : forall A B C D,
   exists E, Bet A B E /\ Cong B E C D;
   ...
```

# Then, we can also formalize some meta-theoretical results:

```
Instance Hilbert_euclidean_follows_from_Tarski_euclidean :
  Hilbert_euclidean
  Hilbert_neutral_follows_from_Tarski_neutral.
```

# Arithmetization of Geometry

René Descartes (1925).
La géométrie.





eſt a l'autre, ce qui eſt le meſme que la Diuiſion; ou enfin trouuer vne, ou deux, ou pluſieurs moyennes proportionnelles entre l'vnité, & quelque autre ligne; ce qui eſt le meſme que tirer la racine quarrée, on cubique, &c. Et ie ne craindray pas d'introduire ces termes d'Arithmetique en la Geometrie, affin de me rendre plus intelligibile.

La Multiplication.

Soit par exemple A B l'vnité, & qu'il faille multiplier B D par B C, ie n'ay qu'a ioindre les poins A & C, puis tirer D E parallele a C A, & B E eſt le produit de cete Multiplication.



La Diuiſion.

Oubien s'il faut diuiſer B E par B D, ayant ioint les poins E & D, ie tire A C parallele a D E, & B C eſt le produit de cete diuiſion.

l'Extraction de la racine quarrée.

Ou s'il faut tirer la racine quarrée de G H, ie luy adiouſte en ligne droite F G, qui eſt l'vnité, & diuiſant F H en deux parties eſgales au point K, du centre K ie tire le cercle F I H, puis eſleuant du point G vne ligne droite iuſques à I, à angles droits ſur F H, c'eſt G I la racine cherchée. Ie ne dis rien icy de la racine cubique, ny des autres, à cauſe que i'en parleray plus commodement cy aprés.

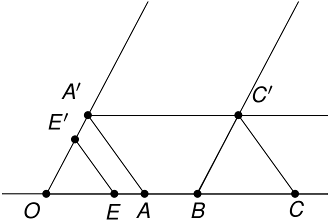


Commét on peut

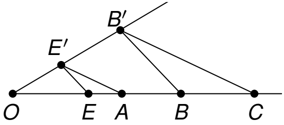
Mais ſouuent on n'a pas beſoin de tracer ainſi ces li-

gne

# Addition



# Multiplication



# Characterization of geometric predicates

| Geometric predicate | Characterization |
| --- | --- |
| $AB \equiv CD$ | $(x_A - x_B)^2 + (y_A - y_B)^2 - (x_C - x_D)^2 + (y_C - y_D)^2 = 0$ |
| Bet $A\,B\,C$ | $\exists t, 0 \leq t \leq 1 \wedge \begin{array}{l} t(x_C - x_A) = x_B - x_A \wedge \\ t(y_C - y_A) = y_B - y_A \end{array}$ |
| $Col\ A\ B\ C$ | $(x_A - x_B)(y_B - y_C) - (y_A - y_B)(x_B - x_C) = 0$ |
| $I$ midpoint of $AB$ | $2x_I - (x_A + x_B) = 0 \wedge$<br>$2y_I - (y_A + y_B) = 0$ |
| $PerABC$ | $(x_A - x_B)(x_B - x_C) + (y_A - y_B)(y_B - y_C) = 0$ |
| $AB \parallel CD$ | $(x_A - x_B)(x_C - x_D) + (y_A - y_B)(y_C - y_C) = 0 \wedge$<br>$(x_A - x_B)(x_A - x_B) + (y_A - y_B)(y_A - y_B) \neq 0 \wedge$<br>$(x_C - x_D)(x_C - x_D) + (y_C - y_D)(y_C - y_D) \neq 0$ |
| $AB \perp CD$ | $(x_A - x_B)(y_C - y_D) - (y_A - y_B)(x_C - x_D) = 0 \wedge$<br>$(x_A - x_B)(x_A - x_B) + (y_A - y_B)(y_A - y_B) \neq 0 \wedge$<br>$(x_C - x_D)(x_C - x_D) + (y_C - y_D)(y_C - y_D) \neq 0$ |

# Formalization technique: bootstrapping

**Manually** bet, cong, equality, col

**Automatically** midpoint, right triangles, parallelism and perpendicularity

# Using automation

Using Gröbner's bases, but this is not a theorem about polynomials:

```
Lemma centroid_theorem : forall A B C A1 B1 C1 G,
  Midpoint A1 B C ->
  Midpoint B1 A C ->
  Midpoint C1 A B ->
  Col A A1 G ->
  Col B B1 G ->
  Col C C1 G \/ Col A B C.
Proof.
intros A B C A1 B1 C1 G; convert_to_algebra; decompose_coordinates.
intros; spliter. express_disj_as_a_single_poly; nsatz.
Qed.
```

# Continuity properties

- Dedekind

# Continuity properties

- Dedekind
- Archimedes

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- Dedekind
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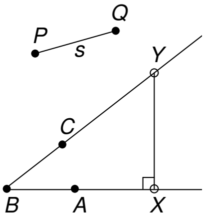
- Aristotle

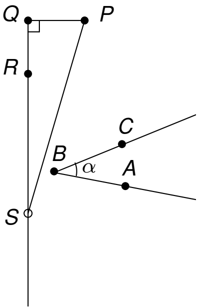
# Continuity properties

- Dedekind
- Archimedes
- Aristotle



# Continuity properties

- Dedekind
- Archimedes
- Aristotle
- Greenberg



# Continuity properties

- Dedekind
- Archimedes
- Aristotle
- Greenberg

# Continuity properties

- Dedekind

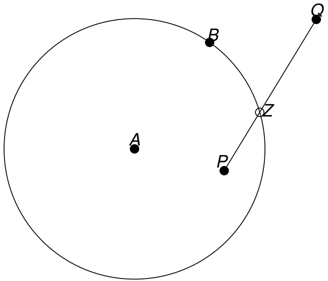
  $\Downarrow$
- Archimedes

  $\Downarrow$
- Aristotle

  $\Downarrow$
- Greenberg

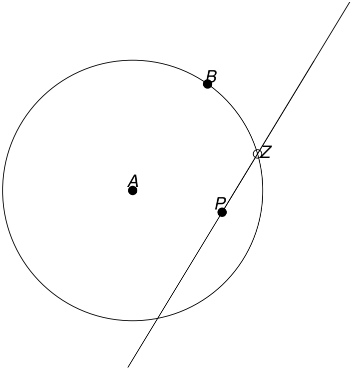
# Segment-Circle / Line-Circle continuity

- Circle-Segment



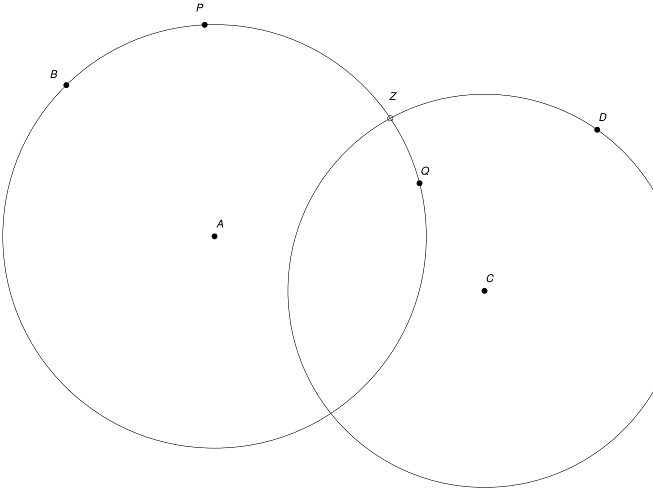
# Segment-Circle / Line-Circle continuity

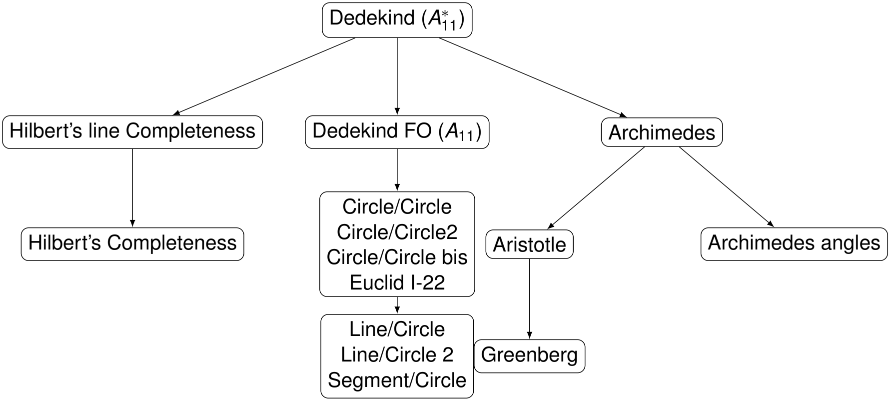
- Circle-Segment
- Circle-Line



# Segment-Circle / Line-Circle continuity

- Circle-Segment
- Circle-Line
- Circle-Circle




Dedekind ($A_{11}^*$)
Hilbert's line Completeness
Dedekind FO ($A_{11}$)
Archimedes
Hilbert's Completeness
Circle/Circle
Circle/Circle2
Circle/Circle bis
Euclid I-22
Aristotle
Archimedes angles
Line/Circle
Line/Circle 2
Segment/Circle
Greenberg

# Algebra/Geometry

| Continuity | Axiom |
| --- | --- |
| | ordered Pythagorean field[20] |
| circle/line continuity | ordered Euclidean field [21] |
| FO Dedekind cuts | real closed field [22] |
| Dedekind | reals |

[20] the sum of squares is a square
[21] positive are square
[22] F is euclidean and every polynomial of odd degree has at least one root in F.

# Logic

Intuitionist logic [23]

- Assuming : $\forall A, B$ : Points, $A = B \vee A \neq B$
- We prove : excluded middle for all other predicates,

[23] Pierre Boutry et al. (2014). "A short note about case distinctions in Tarski's geometry". In: Proceedings of the 10th Int. Workshop on Automated Deduction in Geometry. Vol. TR 2014/01. Proceedings of ADG 2014

# Logic

Intuitionist logic [23]

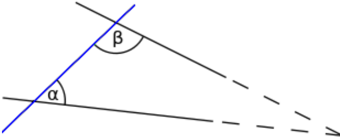
- Assuming : $\forall A, B$ : Points, $A = B \lor A \neq B$
- We prove : excluded middle for all other predicates, except line intersection

[23] Pierre Boutry et al. (2014). "A short note about case distinctions in Tarski's geometry". In: Proceedings of the 10th Int. Workshop on Automated Deduction in Geometry. Vol. TR 2014/01. Proceedings of ADG 2014

# Outline

# Euclid 5th postulate

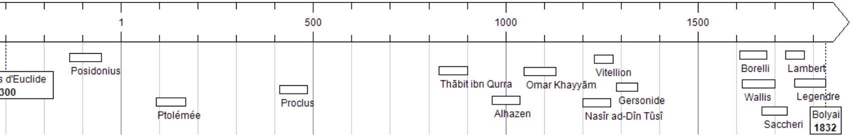
*"If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough."*



# History

- A less obvious postulate

# History




- A less obvious postulate
- Incorrect proofs during centuries

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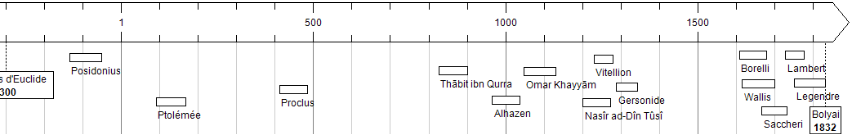
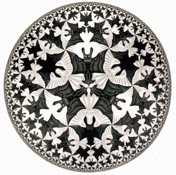
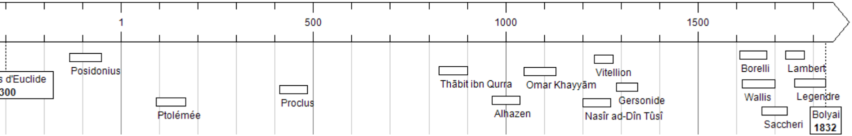



- A less obvious postulate
- Incorrect proofs during centuries
- Independence

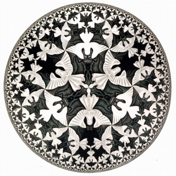


Escher, Circle Limit IV, 1960

# History




- A less obvious postulate
- Incorrect proofs during centuries
- Independence
- Some equivalent statements



Escher, Circle Limit IV, 1960

# A long history of incorrect proofs . . .

In 1763, Klügel [24] provides a list of 30 failed attempts at proving the parallel postulate.

**Examples:**

- Ptolémée uses implicitly Playfair's postulate (uniqueness of the parallel).
- Proclus uses implicitly "Given two parallel lines, if a line intersect one of them it intersects the other".
- Legendre published several incorrect proofs in its *best-seller* "Éléments de géométrie".

[24] G. S. Klugel (1763). "Conatuum praecipuorum theoriam parallelarum demonstrandi recensio". PhD thesis. Schultz, Göttingen

# Mistakes

- Circular arguments

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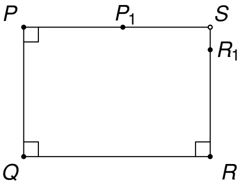
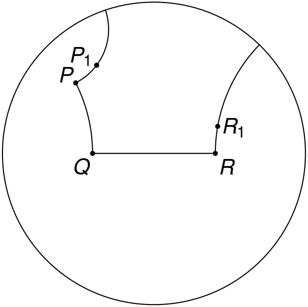
**Warning !**

(`parallelogram2` $ABCD \Leftrightarrow$ `parallelogram2` $BCDA$) $\Leftrightarrow$ `Euclid5`

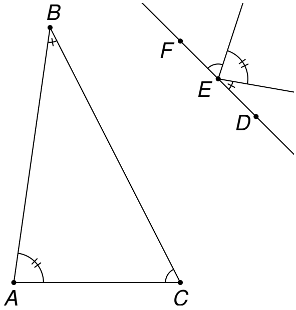
# Bachmann's Lotschnittaxiom

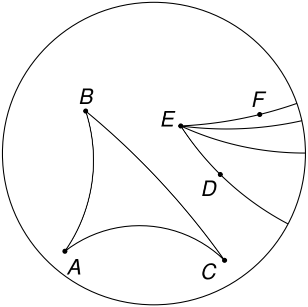
If $p \perp q$, $q \perp r$ and $r \perp s$ then $p$ and $s$ meet.



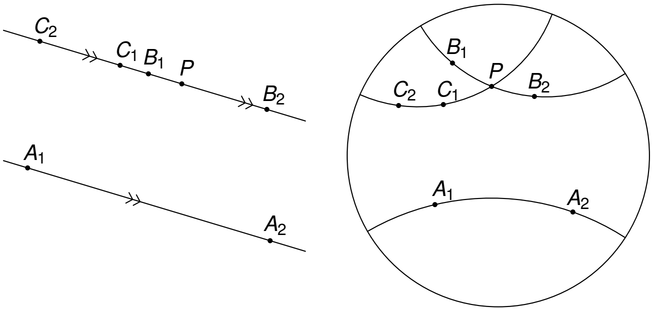



# Triangle postulate

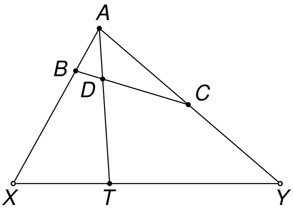


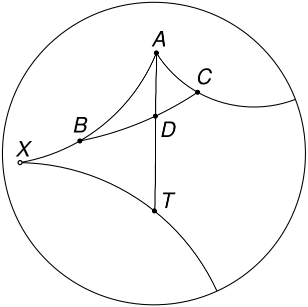



# Playfair's postulate

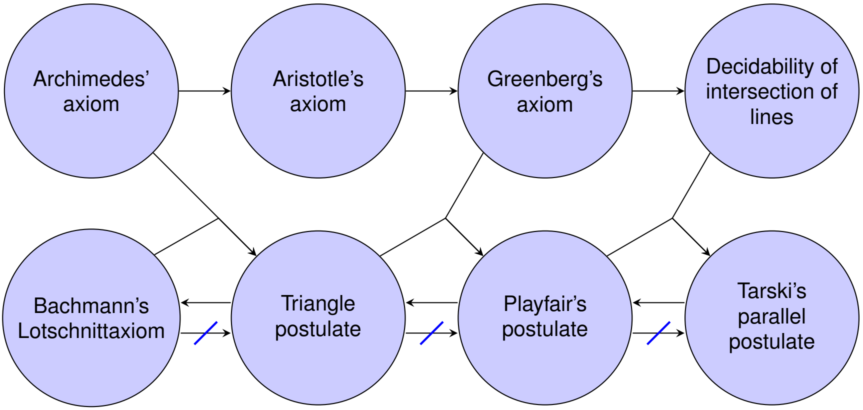


# Tarski's postulate



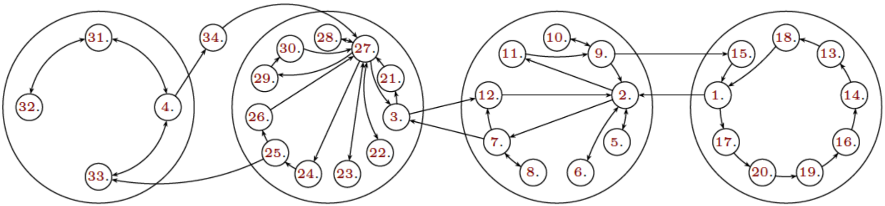



# Four groups



Archimedes' axiom
Aristotle's axiom
Greenberg's axiom
Decidability of intersection of lines
Bachmann's Lotschnittaxiom
Triangle postulate
Playfair's postulate
Tarski's parallel postulate

# Sorting 34 postulates



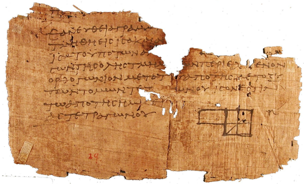
# Outline

1. Interactive Theorem Proving for the Education

2. Overview of GeoCoq
   - Foundations
   - Arithmetization of Geometry
   - Automation
   - Continuity
   - Logic
   - 34 parallel postulates
   - Two formalizations of the Elements
   - Some high-school examples

# The Elements

- A very influential mathematical book (more than 1000 editions).
- First known example of an axiomatic approach.



Book 2, Prop V, Papyrus d'Oxyrhynchus (year 100)

Euclid

# First project

- Joint work with Charly Gries and Gabriel Braun
- Mechanizing proofs of Euclid's **statements**
- Not Euclid's proofs!
- Trying to minimize the assumptions:
  - Parallel postulate
  - Elementary continuity
  - Archimedes' axiom

# Second project

- Joint work with Michael Beeson and Freek Wiedijk [25]
- Formalizing Euclid's **proofs**
- A not minimal axiom system
- Filling the gaps in Euclid

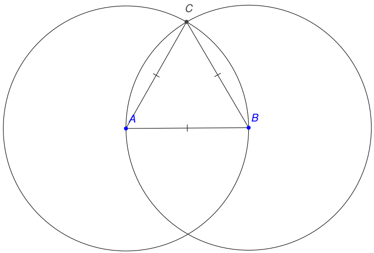
[25] Michael Beeson, Julien Narboux, and Freek Wiedijk (2017). "Proof-checking Euclid".

# Example

## Proposition (Book I, Prop 1)

*Let A and B be two points, build an equilateral triangle on the base AB.*

Proof: Let $\mathcal{C}_1$ and $\mathcal{C}_2$ the circles of center $A$ and $B$ and radius $AB$. Take $C$ at the intersection of $\mathcal{C}_1$ and $\mathcal{C}_2$. The distance $AB$ is congruent to $AC$, and $AB$ is congruent to $BC$. Hence, $ABC$ is an equilateral triangle.



# Book I, Prop 1

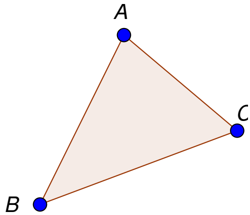
We prove two statements:

1. Assuming no continuity, but the parallel postulate.
2. Assuming circle/circle continuity, but not the parallel postulate.

Pambuccian has shown that these assumptions are minimal.

# The sum of angles of a triangle (Euclid Book I, Prop 32)

Let $l$ be a parallel to $AC$ through $B$.



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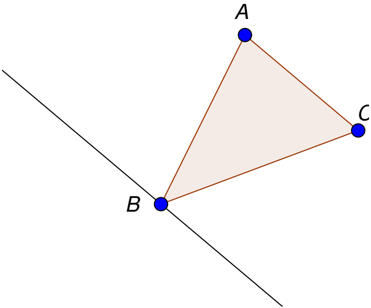

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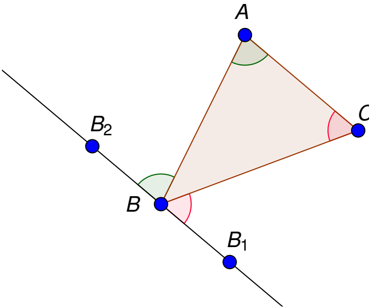
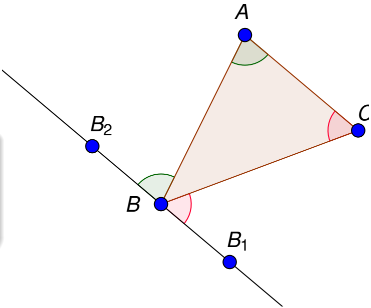


# The sum of angles of a triangle (Euclid Book I, Prop 32)

Let $l$ be a parallel to $AC$ through $B$.

**But !**

We have to prove that the angles are alternate angles.

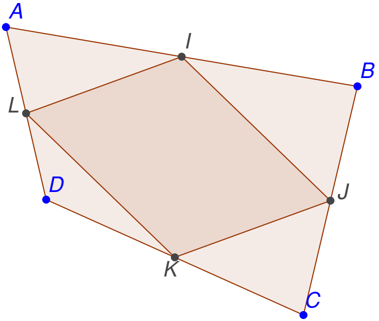


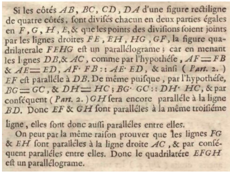
# Varignon’s theorem

**Theorem**

*Let ABCD be a quadrilateral. Let I, J, K and L the midpoints of AB, BC, CD, and AD, then IJKL is a parallelogram.*

Using the triangle midpoints theorem, in the triangle $ABC$ we have $AC \parallel IJ$. We also have $AC \parallel LK$. Hence $LK \parallel IJ$. Similarly, $IL \parallel JK$.



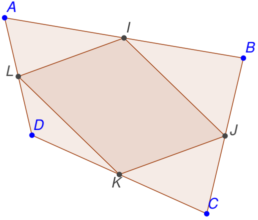



Si les côtés AB, BC, CD, DA d'une figure rectiligne de quatre côtés, ſont diviſés chacun en deux parties égales en F, G, H, E, & que les points des diviſions ſoient joints par les lignes droites FE, EH, HG, GF, la figure quadrilaterale FEHG eſt un parallélograme; car en menant les lignes DB,& AC, comme par l'hypothéſe, AF = FB & AE = ED, AF· FB :: AE· ED, & ainſi (Part. 2.) EF eſt paralléle à DB. De même puiſque, par l'hypothéſe, BG = GC, & DH = HC; BG· GC :: DH· HC, & par conſéquent (Part. 2.) GH ſera encore paralléle à la ligne BD. Donc EF & GH ſont paralléles à la même troiſiéme ligne, elles ſont donc auſſi paralléles entre elles.

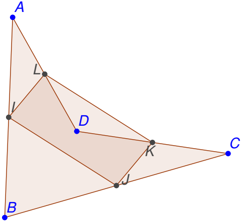
On peut par la même raiſon prouver que les lignes FG & EH ſont paralléles à la ligne droite AC, & par conſéquent paralléles entre elles. Donc le quadrilatére EFGH eſt un parallélograme.

Original proof

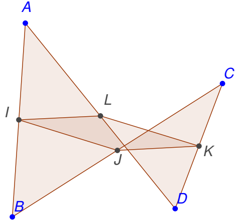
# Varignon's theorem



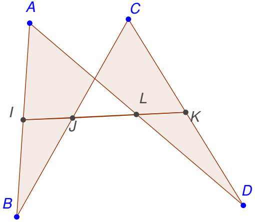

(a) Convex case




(b) Concave case




(c) Self-intersection




(d) Special case

# Challenges

- Ndgs can be easily overlooked.
- As in the Elements, text-books tend to prove properties **assuming** points in general position, but **do not check** that the points are in general position when using the properties.
- As in the Elements, text-books tend to read co-exact properties on the figure.

# Conclusion

- GeoCoq: a library for the foundations of geometry.
- Most results for high-school geometry are formalized.
- Needs integration into a GUI.
- Some challenges for automation.

Thank you