

Formalizing projective geometry in Coq

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Outline

- 1 Objectives
- 2 Formal proofs in geometry
- 3 Projective geometry
- 4 A case study : Desargues' theorem
- 5 Formalization in Coq
- 6 Future work and conclusion

Geometric constraint solving and automated deduction in geometry

- 1 Proofs of theorems in geometry are needed for solving geometric constraint systems.
- 2 Most methods in automated deduction in geometry (Wu's method, the area method) can only deal with geometric statements expressed as a ruler and compass *construction*.

We need a system to prove both mathematical theorems and programs!

Geometry

- Hilbert's Grundlagen [DDS00, MF03]
- Tarski's geometry [Nar07]
- High-school geometry [Gui05]
- Hessenberg's theorem [BH08]
- Convex-hulls algorithm [MF05] [PB01]
- Image segmentation algorithm [Duf07]

Degenerated cases

- *Using Three-Valued Logic to Specify and Verify*, Brandt and Schneider [BS05]

Using matroids

- Incidence constraints, a combinatorial approach, Schreck-Michelucci[MS06]

Why projective geometry ?

- Non degeneracy conditions
- A simple framework

Axiom system (2D)

Line-Existence

$$\forall A B : \text{Point}, (\exists l : \text{Line}, A \in l \wedge B \in l)$$

Point-Existence

$$\forall l m : \text{Line}, (\exists A : \text{Point}, A \in l \wedge A \in m)$$

Axiom Uniqueness

$$\forall A B : \text{Point}, \forall l m : \text{Line}, \\ A \in l \Rightarrow B \in l \Rightarrow A \in m \Rightarrow B \in m \Rightarrow A = B \vee l = m$$

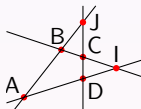
Axiom Four Points

$$\exists A : \text{Point}, \exists B : \text{Point}, \exists C : \text{Point}, \exists D : \text{Point}, \\ \text{distinct4 } A B C D \wedge \\ (\forall l : \text{Line}, (A \in l \wedge B \in l \Rightarrow C \notin l \wedge D \notin l) \wedge \\ (A \in l \wedge C \in l \Rightarrow B \notin l \wedge D \notin l) \wedge \\ (A \in l \wedge D \in l \Rightarrow B \notin l \wedge C \notin l) \wedge \\ (B \in l \wedge C \in l \Rightarrow A \notin l \wedge D \notin l) \wedge \\ (B \in l \wedge D \in l \Rightarrow A \notin l \wedge C \notin l) \wedge \\ (C \in l \wedge D \in l \Rightarrow A \in l \wedge B \in l))$$

Axiom system ($\geq 3D$)

Line-Existence $\forall A B : \text{Point}, \exists l : \text{Line}, A \in l \wedge B \in l$

Pasch

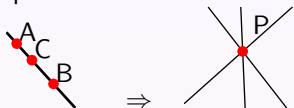
$$\forall A B C D : \text{Point}, \forall l_{AB} l_{CD} l_{AC} l_{BD} : \text{Line},$$
$$A \neq B \wedge A \neq C \wedge A \neq D \wedge B \neq C \wedge B \neq D \wedge C \neq D \wedge$$
$$A \in l_{AB} \wedge B \in l_{AB} \wedge C \in l_{CD} \wedge D \in l_{CD} \wedge$$
$$A \in l_{AC} \wedge C \in l_{AC} \wedge B \in l_{BD} \wedge D \in l_{BD} \wedge$$
$$(\exists l : \text{Line}, l \in l_{AB} \wedge l \in l_{CD}) \Rightarrow$$
$$(\exists J : \text{Point}, J \in l_{AC} \wedge J \in l_{BD})$$


Uniqueness $\forall A B : \text{Point}, \forall l m : \text{Line},$
 $A \in l \wedge B \in l \wedge A \in m \wedge B \in m \Rightarrow A = B \vee l = m$

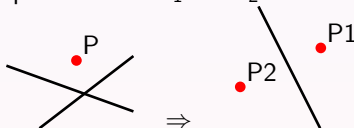
Three-Points $\forall l : \text{Line}, \exists A B C : \text{Point},$
 $A \neq B \wedge B \neq C \wedge A \neq C \wedge A \in l \wedge B \in l \wedge C \in l$

Lower-Dimension $\exists l m : \text{Line}, \forall p : \text{Point}, p \notin l \vee p \notin m$

- Proving some theorems by duality:
 - 3 points on a line \Rightarrow 3 lines through a point



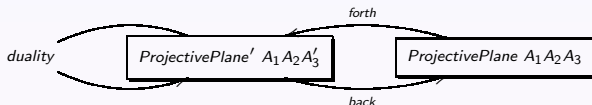
- 1 point outside l_1 and $l_2 \Rightarrow$ 1 line not through P_1 and P_2



- The solution: a functor from Projective Plane Geometry to itself

Implementation in Coq !

- Module types (boxes) and Functors (arrows)



- Duality as a functor

```
Module swap (M' : ProjectivePlane') <: ProjectivePlane'.
Definition Point := M'.Line.
Definition Line := M'.Point.
Definition Incid := fun (x:Point) (y:Line) => M'.Incid y x.
[...]
Definition a1_exist := M'.a2_exist.
Definition a2_exist := M'.a1_exist.
[...]
```

- Application

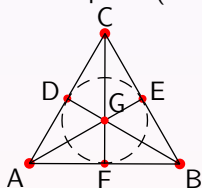
```
Lemma outsider : forall l1 l2: Line, { P:Point | ~Incid P l1/\~Incid P l2}.
Proof. [...] Qed.
```

```
Lemma dual_example :
forall P1 P2 : Point,{l : Line | ~ Incid P1 l /\ ~ Incid P2 l}.
Proof. apply ProjectivePlaneFacts_m.outsider. Qed.
```

- Finite Models

$PG(dim, b)$ where given a point on a line, b is the number of other lines through the point.

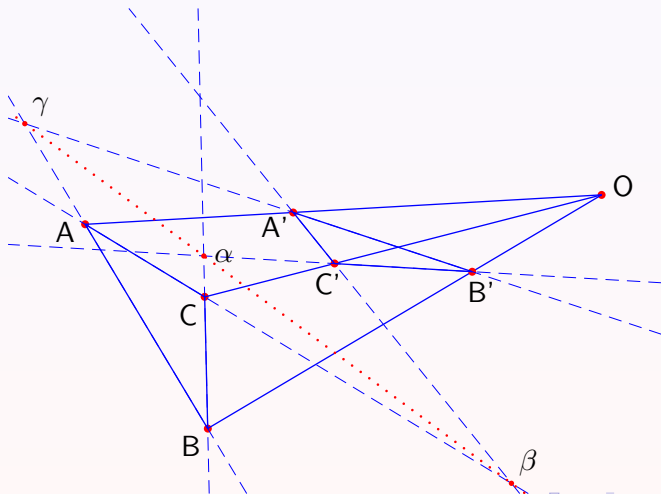
- Fano's plane (= $PG(2, 2)$ the smallest projective plane)



- $PG(2,5)$ (31 points and as many lines, thus 923 521 cases)
- Infinite Model: Homogeneous Coordinates

Desargues' theorem

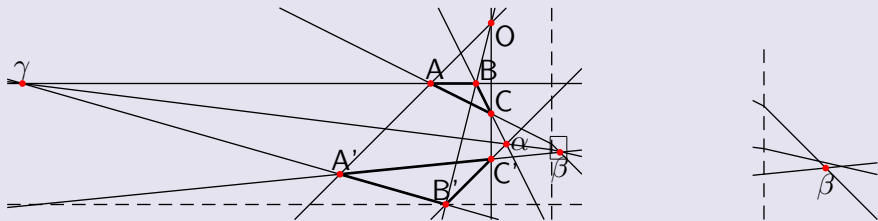
let E be a 3D or higher projective space, if the 3 lines joining the corresponding vertices of triangles ABC and $A'B'C'$ all meet in a point O , then the 3 intersections of pairs of corresponding sides α , β and γ lie on a line.



Why 3D ?

Desargue's theorem is false in some 2D models.

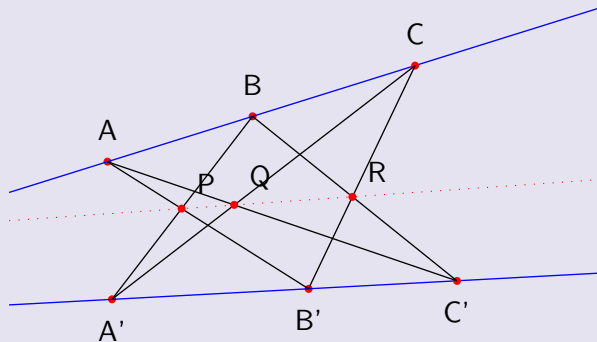
Moulton's plane



Hessenberg's theorem

If Pappus holds then Desargues holds as well.

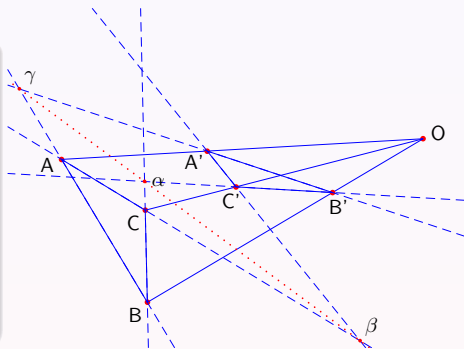
Pappus



The proof

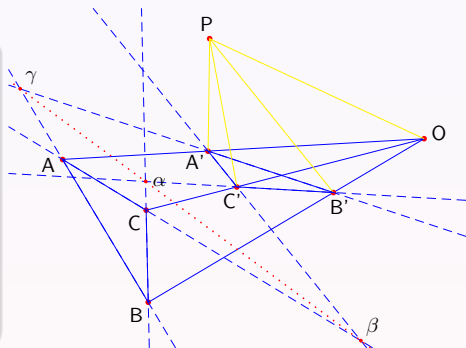
Outline of the proof:

- We prove the 3D version of Desargues' theorem.
- We lift the 2D statement to a 3D version of Desargues' theorem which projects into the 2D statement.



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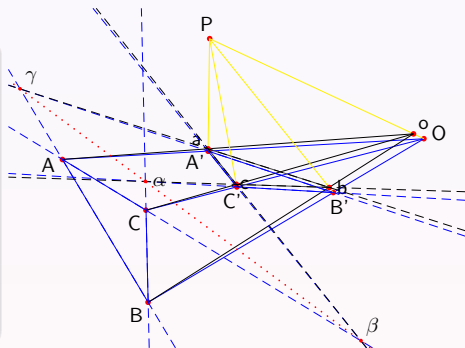
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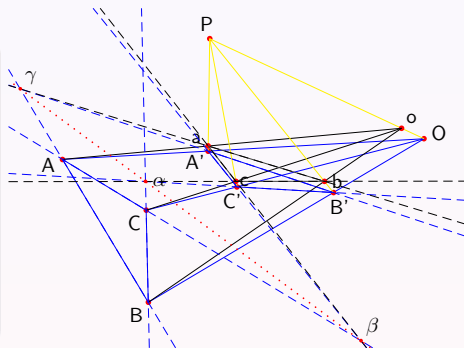
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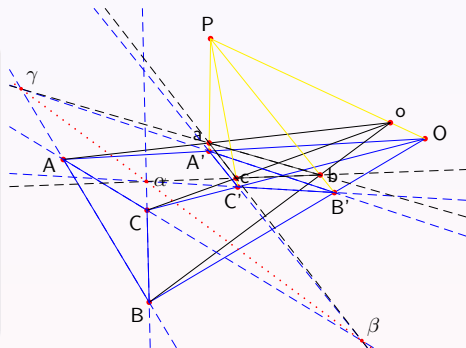
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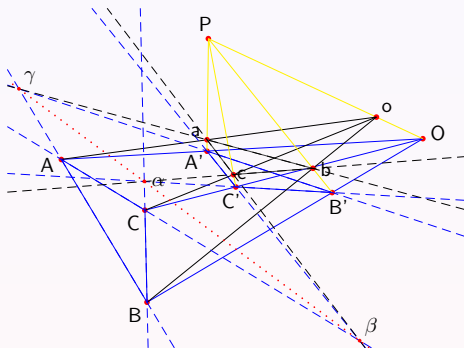
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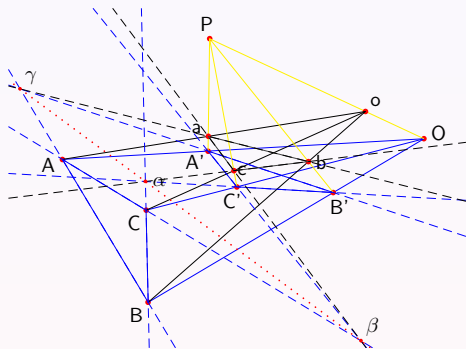
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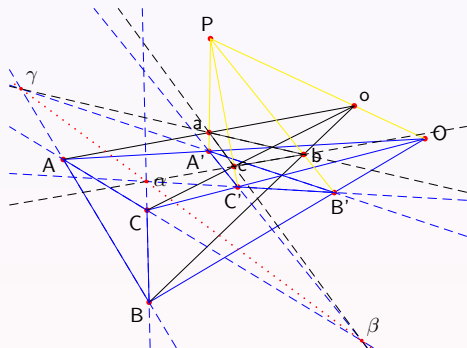
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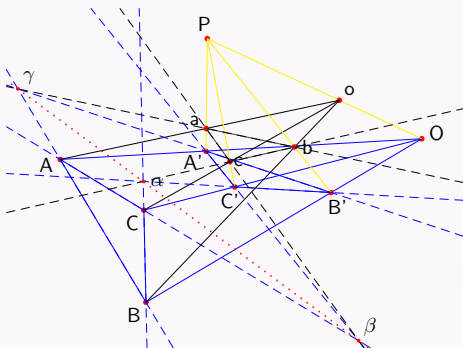
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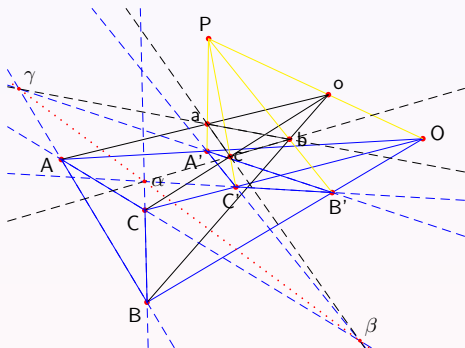
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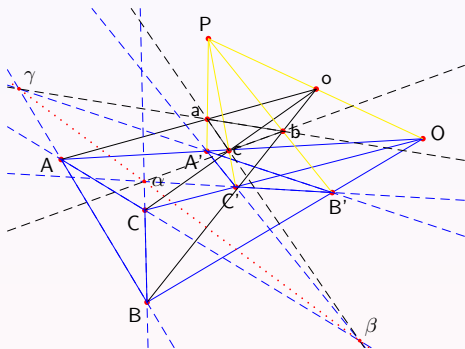
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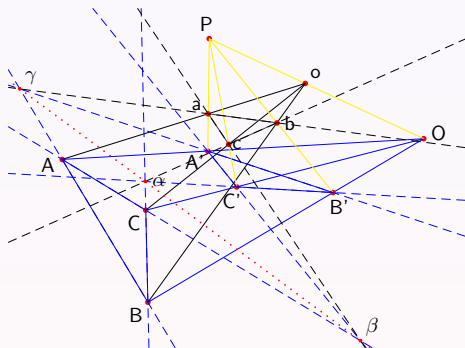
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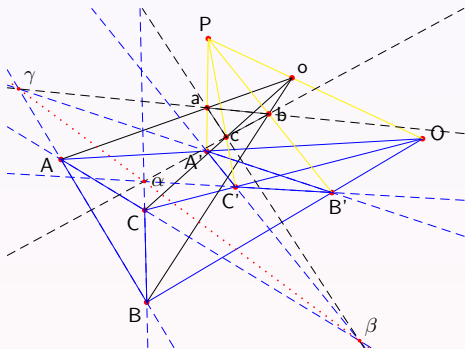
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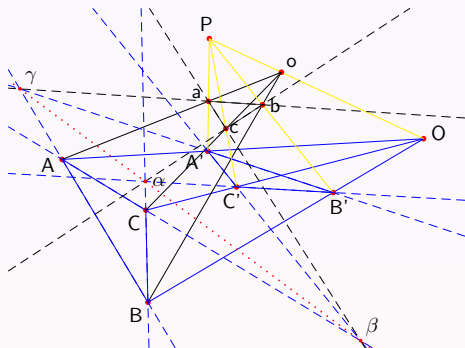
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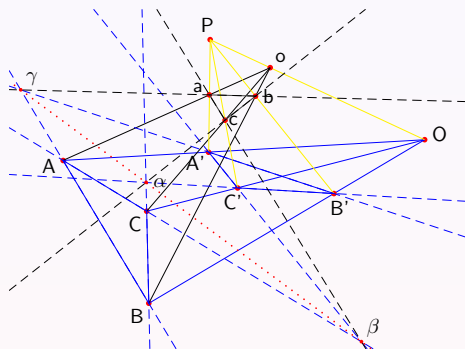
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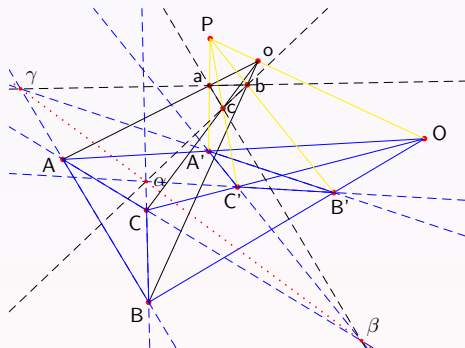
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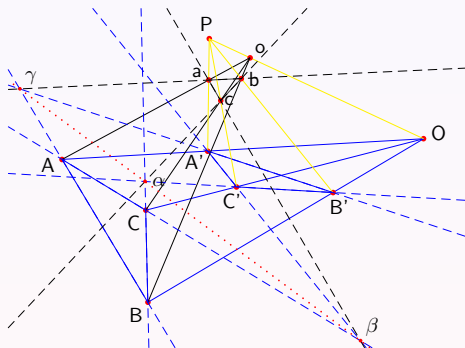
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We use the concept of rank:

R1 $\forall X \subseteq E, 0 \leq rk(X) \leq |X|$ (nonnegative and subcardinal)

R2 $\forall XY \subseteq E, X \subseteq Y \Rightarrow rk(X) \leq rk(Y)$ (nondecreasing)

R3 $\forall XY \subseteq E, rk(X \cup Y) + rk(X \cap Y) \leq rk(X) + rk(Y)$
(submodular)

Example

$rk\{A, B\} = 1$	$A = B$
$rk\{A, B\} = 2$	$A \neq B$
$rk\{A, B, C\} = 2$	A, B, C are collinear with at least two of them distinct
$rk\{A, B, C\} \leq 2$	A, B, C are collinear
$rk\{A, B, C\} = 3$	A, B, C are not collinear
$rk\{A, B, C, D\} = 3$	A, B, C, D are co-planar, not all collinear
$rk\{A, B, C, D\} = 4$	A, B, C, D are not co-planar
$rk\{A, B, C, D, E\} \leq 2$	A, B, C, D, E are all collinear

Axiom system to capture projective at least 3-dimensional space

Rk-Singleton $\forall P : \text{Point}, rk\{P\} \geq 1$

Rk-Couple $\forall P Q : \text{Point}, P \neq Q \Rightarrow rk\{P, Q\} \geq 2$

Rk-Pasch $\forall A B C D, rk\{A, B, C, D\} \leq 3 \Rightarrow$
 $\exists J, rk\{A, B, J\} = rk\{C, D, J\} = 2$

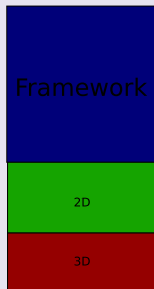
Rk-Three-Points

$\forall A B, \exists C, rk\{A, B, C\} = rk\{B, C\} = rk\{A, C\} = 2$

Rk-Lower-Dimension $\exists A B C D, rk\{A, B, C, D\} \geq 4$

Outline of the proof

- 1 general lemmas (framework)
- 2 proof of the 3D statement
- 3 proof of the 2D statement by projection



Using axiom R3

$$\forall XY, rk(X \cup Y) + rk(X \cap Y) \leq rk(X) + rk(Y)$$

↓

$$rk\{A, B, C, D, I\} + rk\{I\} \leq rk\{A, B, I\} + rk\{C, D, I\}$$

Using axiom R3

$$\forall XY, rk(X \cup Y) + rk(X \cap Y) \leq rk(X) + rk(Y)$$



$$rk\{A, B, C, D, I\} + rk\{A, I\} \leq rk\{A, B, I\} + rk\{C, D, I\}$$

This is false. If for instance $A = C$, $\{A, B, I\} \cap \{C, D, I\} = \{A, I\}$

Using axiom R3

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Lemma (R3-ALT)

$$\forall XYI, I \subseteq X \cap Y \Rightarrow rk(X \cup Y) + rk(I) \leq rk(X) + rk(Y)$$

Conclusion

- A formalization of duality
- Models
- An axiom system to capture projective geometry based on ranks.
- A way to express incidence relations thanks to ranks.
- A case study: Desargues.
- $\geq 15k$ lines of Coq

Future work

- Automatic proofs using hexamys.



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