The parallel postulate: a syntactic proof of independence

Julien Narboux
joint work with
Michael Beeson and Pierre Boutry

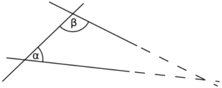
Université de Strasbourg - ICube - CNRS

Nov 2015, Strasbourg

Today's presentation

A presentation for non-specialists of:
Herbrand's theorem and non-Euclidean geometry
Michael Beeson, Pierre Boutry, Julien Narboux
Bulletin of Symbolic Logic, Association for Symbolic Logic, 2015, 21 (2), pp.12.
`https://hal.inria.fr/hal-01071431v3`

If a line segment intersects two straight lines forming two interior angles on the same side that sum to less than two right angles, then the two lines, if extended indefinitely, meet on that side on which the angles sum to less than two right angles.



A long history

From antiquity, mathematicians felt that Euclid 5th was less "obviously true" than the other axioms, and they attempted to derive it from the other axioms. Many false "proofs" were discovered and published.

Examples:

- Ptolemy assumes implicitly Playfair axioms (uniqueness of parallel).
- Proclus assumes implicitly "If a line intersects one of two parallel lines, both of which are coplanar with the original line, then it must intersect the other also."
- Legendre published several incorrect proofs of Euclid 5 in his best-seller "Éléments de géométrie".

Outline

About independence

We want to show that the parallel postulate is independent of the other axioms:

Theorem

The parallel postulate is not a theorem.

About independence

We want to show that the parallel postulate is independent of the other axioms:

Meta-Theorem

The parallel postulate is not a theorem.

A toy example

Example

The language :

One predicate : R (arity 2)
One constant : $\blacksquare$
One function symbol : μ (arity 1)

One axiom : $R(\blacksquare, \blacksquare)$

One rule : $\forall x, R(x, x) \Rightarrow R(\mu(x), \mu(x))$

Question

Is $R(\mu(\mu(\blacksquare)), \mu(\blacksquare))$ a theorem ?

Answer 1 (syntactic proof)

No, because :

1. It is not an axiom.
2. We cannot apply the rule.

Answer 2 (semantic proof)

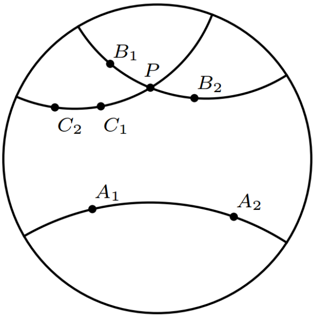
No, because if you interpret:

- R by the equality $=$
- $\blacksquare$ by the integer 0
- μ by the function $x \mapsto x + 1$

It holds that $0 = 0$ and $\forall x, x = x \Rightarrow x + 1 = x + 1$ but we don't have $2 = 1$.

Semantic proofs of the independence of Euclid's 5th postulate

Using Poincaré disk model: straight lines consist of all segments of circles contained within that disk that are orthogonal to the boundary of the disk, plus all diameters of the disk.



Outline

Tarski's axioms

- 11 axioms
- two predicates (β $A\,B\,C$, $AB \equiv CD$)
- no definition inside the axiom system

Part 1

Six axioms without existential quantification:

Congruence Pseudo-Transitivity

$$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$$

Congruence Symmetry $AB \equiv BA$

Congruence Identity $AB \equiv CC \Rightarrow A = B$

Between identity $\beta\ A\ B\ A \Rightarrow A = B$

Five segments
$$AB \equiv A'B' \wedge BC \equiv B'C' \wedge \\ AD \equiv A'D' \wedge BD \equiv B'D' \wedge \\ \beta\ A\ B\ C \wedge \beta\ A'\ B'\ C' \wedge A \neq B \Rightarrow CD \equiv C'D'$$
:

Side-Angle-Side expressed without angles.

Upper dimension $P \neq Q \wedge AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \Rightarrow$ $Col\ ABC$

Part 2

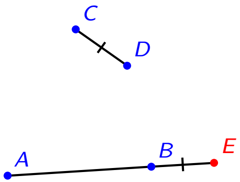
Five axioms with existential quantification:

1. Lower dimension
2. Segment construction
3. Pasch
4. Parallel postulate
5. Continuity: Dedekind cuts or line-circle continuity

Lower Dimension

$$\exists ABC, \neg Col(A, B, C)$$

Segment construction axiom

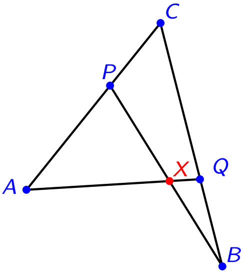


$$\exists E, \beta\ A\ B\ E \wedge BE \equiv CD$$

Pasch's axiom

Allows to formalize some gaps in Euclid's Elements.
We have the inner form :

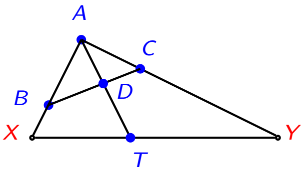
$$\beta\ A\ P\ C \wedge \beta\ B\ Q\ C \Rightarrow \exists X, \beta\ P\ X\ B \wedge \beta\ Q\ X\ A$$



Moritz Pasch
(1843-1930)

Parallel postulate

$$\exists XY, \beta\ A\ D\ T \wedge \beta\ B\ D\ C \wedge A \neq D \Rightarrow \beta\ A\ B\ X \wedge \beta\ A\ C\ Y \wedge \beta\ X\ T\ Y$$



- This statement is equivalent to Euclid 5th postulate.
- Comes from an incorrect proof of Euclid 5th by Legendre.

Adrien-Marie Legendre (1752-1833) (watercolor caricature by Julien Léopold Boilly)

Main idea

Study the maximum distance between the points in the axioms with existential quantification:

- **Lower dim** Initial Constant.
- **Inner Pasch** The distance is conserved.
- **Segment Construction** The distance is at most doubled.
- **Line Circle Continuity** The distance is preserved.
- **Euclid** We can build points arbitrarily far.

The proof

- Skolemize the axiom system: replace existential quantification with function symbols.
- Apply Herbrand's theorem.

Herbrand's theorem

Herbrand's theorem says that under some assumptions (the theory is first-order and does not contains existential), if the theory proves an existential theorem $\exists y\, \phi(a, y)$, with ϕ quantifier-free, then there exist finitely many terms $t_1, \ldots, t_n$ such that the theory proves

$$\phi(a, t_1(a)) \vee \phi(a, t_2(a)) \ldots \vee \ldots \phi(a, t_n(a)).$$

Example in geometry

Dropping or erecting a perpendicular.

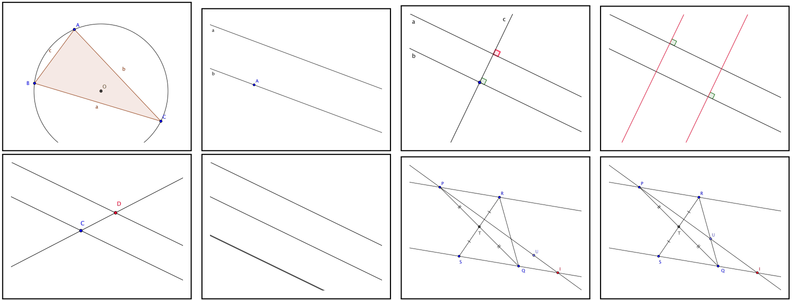
Extension to continuity

- Replace Dedekind continuity by line-circle continuity + polynomial of odd degress have zeros.
- Roots of polynomials can be bounded in terms of their coefficients.

Some Other Parallel Postulates

with Pierre Boutry

```
Theorem parallel_postulates:
 decidability_of_intersection ->
 ((triangle_circumscription <-> tarski_parallel_postulate) /\
 (playfair                  <-> tarski_parallel_postulate) /\
 (par_perp_perp_property    <-> tarski_parallel_postulate) /\
 (par_perp_2_par_property   <-> tarski_parallel_postulate) /\
 (proclus                   <-> tarski_parallel_postulate) /\
 (transitivity_of_par       <-> tarski_parallel_postulate) /\
 (strong_parallel_postulate <-> tarski_parallel_postulate) /\
 (euclid_5                  <-> tarski_parallel_postulate)).
```



Next talk by Charly Gries about other equivalences.

Bibliography I