Automation of geometry using Coq

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TPHOLs 2004, Park City, Utah

Outline

Motivations

- Geometry is one of the most successful areas of automated theorem proving.
- Proof assistants need *automation*, formalizing geometry is a tedious task.

- The use of a proof assistant provides a way to combine geometrical proofs with larger proofs (involving induction for instance).
- The verification of the proofs by the Coq kernel provides a high level of confidence.

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- Coordinate free (but not number free).
- Produces *readable* proofs.

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1. Find a point which is not used to build any other point.
   - The theorem must be stated *constructively*.
2. *Eliminate* every occurrence of this point from the goal.
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3. Repeat until the goal contains only *free* points.
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The goal must be :

- stated constructively (as a sequence of constructions),
- using only two geometric quantities :
  1. the signed area of a triangle ($S_{ABC} = S_{BCA} = -S_{BAC}$)
  2. the ratio of two oriented distances $\frac{\overline{AB}}{\overline{CD}}$ where $AB \parallel CD$
- combined using arithmetic expressions (+,-,*,/).

Using these two quantities :

Geometric notions	Formalization
A,B and C are collinear	$S_{ABC} = 0$
$AB \parallel CD$	$S_{ABC} = S_{ABD}$
I is the midpoint of AB	$\frac{\overline{AB}}{\overline{AI}} = 2 \wedge S_{ABI} = 0$

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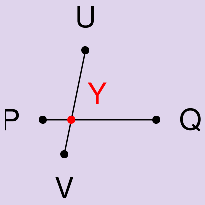
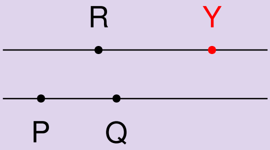
The basic constructions

Point on line.

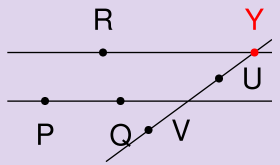
Point on intersection.

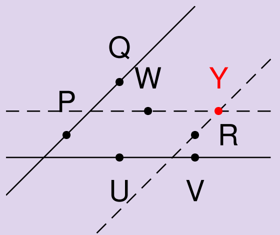


Point on parallel line.



The complex constructions





Elimination lemmas

We have to get rid of :

- ratios of oriented distances,
- signed areas

One example :

If Y is the intersection of (PQ) and (UV) then :

$$\text{For every } A \text{ and } B,\ S_{ABY} = \frac{S_{PUV} * S_{ABQ} + S_{QVU} * S_{ABP}}{(S_{PUQV})}$$

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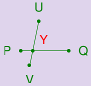
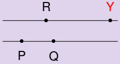
Construction	Elimination formulas	
	$\mathcal{S}_{ABY} =$	If $AY \parallel CD$ then $\frac{\overline{AY}}{\overline{CD}} =$
	$\lambda \mathcal{S}_{ABQ} + (1-\lambda)\mathcal{S}_{ABP}$	$\begin{cases} \frac{\frac{\overline{AP}}{\overline{PQ}}+\lambda}{\frac{\overline{CD}}{\overline{PQ}}} & \text{if } A \in PQ \\ \frac{\mathcal{S}_{APQ}}{\mathcal{S}_{CPDQ}} & \text{otherwise}^a. \end{cases}$
	$\frac{\mathcal{S}_{PUV}\mathcal{S}_{ABQ}+\mathcal{S}_{QVU}\mathcal{S}_{ABP}}{\mathcal{S}_{PUQV}}$	$\begin{cases} \frac{\mathcal{S}_{AUV}}{\mathcal{S}_{CUDV}} & \text{if } A \notin UV \\ \frac{\mathcal{S}_{APQ}}{\mathcal{S}_{CPDQ}} & \text{otherwise.} \end{cases}$
	$\mathcal{S}_{ABR} + \lambda \mathcal{S}_{APBQ}$	$\begin{cases} \frac{\frac{\overline{AR}}{\overline{PQ}}+\lambda}{\frac{\overline{CD}}{\overline{PQ}}} & \text{if } A \in RY \\ \frac{\mathcal{S}_{APRQ}}{\mathcal{S}_{CPDQ}} & \text{otherwise.} \end{cases}$

[a] $\mathcal{S}_{ABCD}$ is a notation for $\mathcal{S}_{ABC} + \mathcal{S}_{ACD}$.

Eliminating free points

Choose three non collinear points O,U and V

$$S_{ABY} = \begin{vmatrix} S_{OUA} & S_{OVA} & S_{UVA} \\ S_{OUB} & S_{OVB} & S_{UVB} \\ S_{OUY} & S_{OVY} & S_{UVY} \end{vmatrix}$$

The implementation is done :

- using LTac (the tactic language of Coq),
- the reflection mechanism (some sub-tactics are written using Coq itself).

We have to :

1. describe the axiomatic,
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Dimension axioms

lower bound $\exists A, B, C | \mathcal{S}_{ABC} \neq 0$

upper bound $\mathcal{S}_{ABC} = \mathcal{S}_{ABD} + \mathcal{S}_{ADC} + \mathcal{S}_{DBC}$

Construction axioms

existence $(\forall A, B : Point, r : F), \exists P : Point \mid (\mathcal{S}_{ABP} = 0) \wedge \overline{AP} = r\overline{AB}$

unicity $\forall A, B, P, P' : Point, r : F\ A \neq B \rightarrow$
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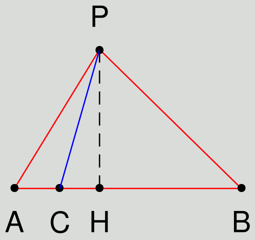
Proportions axiom

$$A \neq C \rightarrow \neg(\mathcal{S}_{PAC} = 0) \rightarrow (\mathcal{S}_{ABC} = 0) \rightarrow \frac{\overline{AB}}{\overline{AC}} = \frac{\mathcal{S}_{PAB}}{\mathcal{S}_{PAC}}$$



We need to prove :

- some simplification lemmas,
- the construction lemmas,
- the elimination lemmas,
- ...

Approximately 6000 lines of Coq.

Some tactics:

initialization translates the goal into the language.

simplification performs trivial simplifications.

unification rewrites all occurrences of a geometric quantity into the same expression.

elimination eliminates a point from a goal.

free point elimination treat the goal in order to keep only *independent* variables.

conclusion mainly apply the standard Coq tactic `Field` (dealing with equalities on fields).

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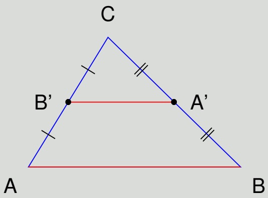
Midpoint theorem.

```
forall A B C A' B' : Point,
midpoint A' B C ->
midpoint B' A C ->
parallel A' B' A B.
```



geoinit.

```
H : on_line_d A' B C (1 / 2)
H0 : on_line_d B' A C (1 / 2)
============================
 S A' A B' + S A' B' B = 0
```

eliminate B'.

```
H : on_line_d A' B C (1 / 2)
============================
 1/2 * S A' A C + (1-1/2) * S A' A A +
 (1/2 * S B A' C + (1-1/2) * S B A' A) = 0
```

basic_simpl.

```
H : on_line_d A' B C (1 / 2)
============================
 1/2 * S A' A C +
     (1/2 * S B A' C + 1/2 * S B A' A) = 0
```

eliminate A'.

```
============================
 1/2*(1/2 * S A C C + (1-1/2) * S A C B) +
(1/2*(1/2 * S C B C + (1-1/2) * S C B B) +
 1/2*(1/2 * S A B C + (1-1/2) * S A B B))= 0
```

basic_simpl.

```
============================
 1/2*(1/2* S A C B) + 1/2*(1/2* S A B C) = 0
```

unify_signed_areas.

```
============================
 1/2*(1/2* S A C B)+1/2*(1/2* - S A C B) = 0
```

field_and_conclude.

```
Proof completed.
```

Some examples :

- Ceva
- Menelaus
- Pascal
- Desargues
- Centroid
- Midpoint
- Gauss-Line
- . . .

Some problems :

- For example :
  - Nerhing
  - Pappus
  - . . .
- The `Field` tactic is not very efficient.
- We need to perform more simplifications.
- No counter example is provided.

23 examples are proved within 160 seconds.

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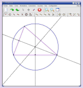

- Integration of this development with Frédérique Guilhot's work on high school geometry.

[a] `https://gna.org/projects/geocaml`

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- Implementation using other theorem provers ?
- Pedagogical applications.

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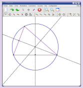

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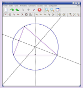

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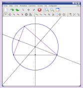

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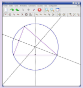

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Bibliography

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